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# UNIFORMLY CONVERGENT DIFFERENCE SCHEME FOR A SEMILINEAR REACTION-DIFFUSION PROBLEM

SAMIR KARASULJIĆ[1], ENES DUVNJAKOVIĆ AND HELENA ZARIN

ABSTRACT. In this work we consider the singularly perturbed one-dimensional semilinear reaction-diffusion problem

$$\varepsilon^2 y''(x) = f(x,y), \quad x \in (0,1), \quad y(0)=0,\ y(1)=0,$$

where $f$ is a nonlinear function. Here the second-order derivative is multiplied by a small positive parameter and consequently, the solution of the problem has boundary layers. A new difference scheme is constructed on a modified Shishkin mesh with $\mathcal{O}(N)$ points for this problem. We prove existence and uniqueness of a discrete solution on such a mesh and show that it is accurate to the order of $N^{-2}\ln^2 N$ in the discrete maximum norm. We present numerical results that verify this rate of convergence.

## 1. INTRODUCTION

We consider the semilinear singularly perturbed problem

$$\varepsilon^2 y''(x) = f(x,y) \ \text{ on } \ (0,1)\,, \tag{1.1}$$

$$y(0)=0,\ y(1)=0, \tag{1.2}$$

where $0<\varepsilon<1$. We assume that the nonlinear function $f$ is continuously differentiable, i.e. for $k \geq 2$, $f \in C^k([0,1]\times\mathbb{R})$, and that it has a strictly positive derivative with respect to $y$

$$\frac{\partial f}{\partial y} = f_y \geq m > 0 \ \text{ on } \ [0,1]\times\mathbb{R}\ (m = const). \tag{1.3}$$

A solution of $(1.1)-(1.2)$ usually exhibits sharp boundary layers at the endpoints of $(0,1)$, when the parameter $\varepsilon$ is near zero. When classical numerical methods are applied to $(1.1)-(1.2)$, one does not obtain $\varepsilon$–uniform results on the entire interval $(0,1)$, because of which we shall use nonstandard discretization of $(1.1)-(1.2)$.

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[1] *corresponding author*

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Many authors have considered the problem $(1.1)-(1.2)$ under various hypotheses on $f$, see e.g. Herceg and Miloradović [7] , Uzelac and Surla [14] , Vulanović [15] , [16] , Sun and Stynes [13] etc.

Uniformly convergent methods with respect to $\varepsilon$ for the problem $(1.1)-(1.2)$ under condition (1.3) have also been examined. Vulanović [15] applies a central difference scheme to the problem $(1.1)-(1.3)$ and proves second-order uniform convergence on a specially graded mesh of Bakhvalov type. D'Annunzio [3] uses a simple central difference scheme on a special locally quasi-equidistant mesh to solve a more general problem than $(1.1)-(1.3)$. The last result was significantly improved by Sun and Stynes in [13] using the mesh of Shishkin type.

Our paper is devoted to the construction of approximations on a Shishkin-type mesh. Our aim is to construct a difference scheme with coefficients which behave similar to the solution of the starting problem. It is well-known that in modelling of the boundary layer of the exact solution of the problem $(1.1)-(1.3)$, it is used suitable exponential functions with a perturbation parameter $\varepsilon$. We intend to get a scheme for calculation of the numerical solution with coefficients which acting on the same or similar way as mentioned exponential functions. A motivation for constructing this kind of difference scheme is getting as good numerical results as possible.

Discretization in this paper is based on the paper Boglaev [2]. Unlike our previous work [5] where we only constructed a difference scheme for the boundary value problem $(1.1)-(1.3)$ and performed a numerical test, in this work we also prove the existence and uniqueness of the numerical solution. Further, we show $\varepsilon$–uniform convergence of the numerical solution to the exact solution on a suitable layer-adapted mesh. We also verify the rate and order of convergence on a numerical example.

**Remark 1.1.** *Throughout the paper we denote by $C$, sometimes subscripted, a generic positive constant that may take different values in different formulas, but is always independent with respect to $N$ and $\varepsilon$.*

## 2. Construction of the nonlinear difference scheme

In this section we construct a difference scheme which generates a system of nonlinear equations and solving this system produces the values of the numerical solution at the mesh points. The scheme will be constructed based on the results in solving the linear boundary value problem and Green's function for a suitable differential operator. The method was first introduced Boglaev [2].

Let us now consider the differential equation (1.1) in an equivalent form

$$L_\varepsilon y(x) := \varepsilon^2 y''(x) - \gamma y(x) = \psi(x, y(x)) \quad \text{on} \quad [0,1],$$

where

$$\psi(x,y) = f(x,y) - \gamma y,$$

and $\gamma \geq m$ is a chosen constant.

On an arbitrary grid

$$0 = x_0 < x_1 < x_2 < \ldots < x_N = 1,$$

consider the following boundary value problems

$$(2.1)\qquad \begin{array}{c} L_\varepsilon u_i(x) := 0 \text{ on } (x_i, x_{i+1}),\\ u_i(x_i) = 1,\ u_i(x_{i+1}) = 0,\\ (i = 0, 1, ..., N-1), \end{array} \quad \text{and} \quad \begin{array}{c} L_\varepsilon u_i(x) := 0 \text{ on } (x_i, x_{i+1}),\\ u_i(x_i) = 0,\ u_i(x_{i+1}) = 1,\\ (i = 0, 1, ..., N-1). \end{array}$$

We denote the solutions of problems (2.1) by $u_i^I(x)$, $u_i^{II}(x)$, $(i = 0, 1, 2, ..., N-1)$, respectively. Functions $u_i^I$ and $u_i^{II}$ are known from [11] , i.e.

$$(2.2)\qquad u_i^I(x) = \frac{\sinh\left(\beta\left(x_{i+1} - x\right)\right)}{\sinh\left(\beta h_i\right)},\ u_i^{II}(x) = \frac{\sinh\left(\beta\left(x - x_i\right)\right)}{\sinh\left(\beta h_i\right)},\ x \in [x_i, x_{i+1}], \\ (i = 0, 1, 2, ..., N-1),$$

where $\beta = \dfrac{\sqrt{\gamma}}{\varepsilon}$, $h_i = x_{i+1} - x_i$.

Consider a new boundary value problem

$$(2.3)\qquad \begin{array}{c} L_\varepsilon y_i(x) = \psi(x, y_i(x)) \text{ on } (x_i, x_{i+1}),\\ y_i(x_i) = y(x_i),\ y_i(x_{i+1}) = y(x_{i+1}),\\ (i = 0, 1, 2, ..., N-1). \end{array}$$

It is clear that $y_i(x) \equiv y(x)$ on $[x_i, x_{i+1}]$, $(i = 0, 1, 2, ..., N-1)$. The solution of (2.3) is given by

$$y_i(x) = C_1 u_i^I(x) + C_2 u_i^{II}(x) + \int_{x_i}^{x_{i+1}} G_i(x, s)\psi(s, y(s))\,\mathrm{d}\,s, \quad x \in [x_i, x_{i+1}],$$

where $G_i(x, s)$ is the Green's function associated with the operator $L_\varepsilon$ on the interval $[x_i, x_{i+1}]$.

The function $G_i(x, s)$ in this case has the following form

$$G_i(x, s) = \frac{1}{\varepsilon^2 w_i(s)} \begin{cases} u_i^{II}(x) u_i^I(s), & x_i \le x \le s \le x_{i+1},\\ u_i^I(x) u_i^{II}(s), & x_i \le s \le x \le x_{i+1}, \end{cases}$$

where

$$\begin{aligned} w_i(s) =& u_i^{II}(s)\left(u_i^I\right)'(s) - u_i^I(s)\left(u_i^{II}\right)'(s)\\ =& \tfrac{\sinh(\beta(s - x_i))}{\sinh(\beta h_i)} \cdot \left(\tfrac{\sinh(\beta(x_{i+1} - s))}{\sinh(\beta h_i)}\right)' - \tfrac{\sinh(\beta(x_{i+1} - s))}{\sinh(\beta h_i)} \cdot \left(\tfrac{\sinh(\beta(s - x_i))}{\sinh(\beta h_i)}\right)'\\ =& \frac{-\beta}{\sinh(\beta h_i)} \ne 0, \quad s \in [x_i, x_{i+1}], \end{aligned}$$

because the solutions $u_i^I$ and $u_i^{II}$ are linearly independent.

From the boundary conditions in (2.3), we have that $C_1 = y(x_i) =: y_i$, $C_2 = y(x_{i+1}) =: y_{i+1}$, $(i = 0, 1, 2, \ldots, N-1)$. Hence, the solution $y_i(x)$ of (2.3) on the interval $[x_i, x_{i+1}]$ has the following form

$$(2.4)\qquad y_i(x) = y_i u_i^I(x) + y_{i+1} u_i^{II}(x) + \int_{x_i}^{x_{i+1}} G_i(x, s)\psi(s, y(s))\,\mathrm{d}\,s.$$

The boundary value problem

$$\begin{array}{c} L_\varepsilon y(x) := \psi(x, y) \quad \text{on} \quad (0, 1),\\ y(0) = y(1) = 0, \end{array}$$

has a unique continuously differentiable solution $y \in C^{k+2}(0,1)$. Since $y_i(x) \equiv y(x)$ on $[x_i, x_{i+1}]$, $(i = 0, 1, 2, ..., N-1)$ we have that

$$y_i'(x_i) = y_{i-1}'(x_i),\ (i = 1, 2, ..., N-1). \tag{2.5}$$

Now, differentiating (2.4) and using (2.5), we get

$$\begin{aligned} &y_{i-1}\left(u_{i-1}^I\right)'(x_i) + y_i\left[\left(u_{i-1}^{II}\right)'(x_i) - \left(u_i^I\right)'(x_i)\right] + y_{i+1}\left[-\left(u_i^{II}\right)'(x_i)\right] = \\ &= \frac{\partial}{\partial x}\left[\int_{x_i}^{x_{i+1}} G_i(x,s)\psi(s,y(s))\,\mathrm{d}\,s - \int_{x_{i-1}}^{x_i} G_{i-1}(x,s)\psi(s,y(s))\,\mathrm{d}\,s\right]_{x=x_i}. \end{aligned} \tag{2.6}$$

Define

$$a_i := -\left(u_{i-1}^I\right)'(x_i),\ c_i := \left(u_{i-1}^{II}\right)'(x_i) - \left(u_i^I\right)'(x_i),\ b_i := \left(u_i^{II}\right)'(x_i).$$

Using (2.2) we have that

$$a_i = \frac{\beta}{\sinh(\beta h_{i-1})},\ b_i = \frac{\beta}{\sinh(\beta h_i)} \quad \text{and} \quad c_i = \frac{\beta}{\tanh(\beta h_{i-1})} + \frac{\beta}{\tanh(\beta h_i)}.$$

Now (2.6) takes the following form

$$\begin{aligned} &-a_i y_{i-1} + c_i y_i - b_i y_{i+1} = \\ &\int_{x_i}^{x_{i+1}} \frac{\partial}{\partial x}\left(G_i(x,s)\right)\Big|_{x=x_i} \psi(s,y(s))\,\mathrm{d}\,s - \int_{x_{i-1}}^{x_i} \frac{\partial}{\partial x}\left(G_{i-1}(x,s)\right)\Big|_{x=x_i} \psi(s,y(s))\,\mathrm{d}\,s. \end{aligned} \tag{2.7}$$

After finding the derivatives on the right hand side (2.7), we get that

$$\begin{aligned} a_i y_{i-1} - c_i y_i + b_i y_{i+1} &= \frac{1}{\varepsilon^2}\left[\int\limits_{x_{i-1}}^{x_i} u_{i-1}^{II}(s)\psi(s,y(s))\,\mathrm{d}\,s + \int\limits_{x_i}^{x_{i+1}} u_i^I(s)\psi(s,y(s))\,\mathrm{d}\,s\right], \\ y_0 = 0,\ \ y_N &= 0,\ \ (i = 1, 2, ..., N-1). \end{aligned} \tag{2.8}$$

Clearly, we cannot explicitly compute the integrals in (2.8) in general. Therefore, we approximate the function $\psi(x, y(x))$ on the interval $[x_{i-1}, x_i]$ by

$$\overline{\psi}_{i-1} = \psi\left(\tfrac{x_{i-1}+x_i}{2}, \tfrac{\overline{y}_{i-1}+\overline{y}_i}{2}\right),$$

where $\overline{y}_i$ are approximate values of the solution $y$ of the problem (1.1) – (1.2) at points $x_i$.

Finally, from (2.8) we get the following difference scheme

$$a_i\overline{y}_{i-1} - c_i\overline{y}_i + b_i\overline{y}_{i+1} = \frac{1}{\varepsilon^2}\left[\overline{\psi}_{i-1}\int_{x_{i-1}}^{x_i} u_{i-1}^{II}(s)\,\mathrm{d}\,s + \overline{\psi}_i\int_{x_i}^{x_{i+1}} u_i^I(s)\,\mathrm{d}\,s\right],$$
$$(i = 1, 2, ..., N-1).$$

From (2.2), we have

$$\int_{x_{i-1}}^{x_i} u_{i-1}^{II}(s)ds = \frac{1}{\beta}\cdot\frac{\cosh(\beta h_{i-1})}{\sinh(\beta h_{i-1})} - \frac{1}{\beta}\cdot\frac{1}{\sinh(\beta h_{i-1})},$$
$$\int_{x_i}^{x_{i+1}} u_i^I(s)ds = \frac{1}{\beta}\cdot\frac{\cosh(\beta h_i)}{\sinh(\beta h_i)} - \frac{1}{\beta}\cdot\frac{1}{\sinh(\beta h_i)}.$$

Hence, our difference scheme has the following form

$$a_i\overline{y}_{i-1} - c_i\overline{y}_i + b_i\overline{y}_{i+1} = \frac{1}{\gamma}\overline{\psi}_{i-1}(d_i - a_i) + \frac{1}{\gamma}\overline{\psi}_i(d_{i+1} - a_{i+1}),$$

where $d_i = \dfrac{\beta}{\tanh(\beta h_{i-1})}$.

After some computation, we get

$$\text{(2.9)}\quad \frac{a_i + d_i}{2}\overline{y}_{i-1} - \left(\frac{a_i + d_i}{2} + \frac{a_{i+1} + d_{i+1}}{2}\right)\overline{y}_i + \frac{a_{i+1} + d_{i+1}}{2}\overline{y}_{i+1} = \frac{\triangle d_i}{\gamma}\overline{f}_{i-1} + \frac{\triangle d_{i+1}}{\gamma}\overline{f}_i,$$

where $\triangle d_i = d_i - a_i$ and $\overline{f}_i = \overline{\psi}_i + \gamma(\overline{y}_i + \overline{y}_{i+1})/2$, $(i = 1, 2, ..., N-1)$.

Using (2.9) let us introduce the discrete problem of the problem (1.1) – (1.3),

$$F\overline{y} = \left((F\overline{y})_0, (F\overline{y})_1, \ldots, (F\overline{y})_N\right)^T = 0,$$

where

$$\begin{aligned}
(F\overline{y})_0 &:= \overline{y}_0,\\
(F\overline{y})_i &:= \tfrac{a_i+d_i}{2}\overline{y}_{i-1} - \left(\tfrac{a_i+d_i}{2} + \tfrac{a_{i+1}+d_{i+1}}{2}\right)\overline{y}_i + \tfrac{a_{i+1}+d_{i+1}}{2}\overline{y}_{i+1}\\
&\quad - \tfrac{\triangle d_i}{\gamma}\overline{f}_{i-1} - \tfrac{\triangle d_{i+1}}{\gamma}\overline{f}_i, (i = 1, 2, \ldots, N-1),\\
(F\overline{y})_N &:= \overline{y}_N,
\end{aligned}$$

and its equivalent normalized form

$$\text{(2.10)}\qquad \tilde{F}\overline{y} = \left(\left(\tilde{F}\overline{y}\right)_0, \left(\tilde{F}\overline{y}\right)_1, \ldots, \left(\tilde{F}\overline{y}\right)_N\right)^T = 0,$$

where

$$\begin{aligned}
(\tilde{F}\overline{y})_0 &:= \overline{y}_0,\\
(\tilde{F}\overline{y})_i &:= \frac{\gamma}{\triangle d_i + \triangle d_{i+1}}(F\overline{y})_i, (i = 1, 2, \ldots, N-1),\\
(\tilde{F}\overline{y})_N &:= \overline{y}_N.
\end{aligned}$$

Here we use the maximum norm

$$\|u\|_\infty = \max_{0\leqslant i\leqslant N} |u_i|,$$

for any vector $u = (u_0, u_1, \ldots, u_n)^T \in \mathbb{R}^{N+1}$ and the corresponding matrix norm.

**Theorem 2.1.** *The discrete problem* (2.10) *for* $\gamma \geq f_y$, *has the unique solution* $\overline{y} = (\overline{y}_0, \overline{y}_1, \overline{y}_2, \ldots, \overline{y}_{N-1}, \overline{y}_N)^T$, *with* $\overline{y}_0 = \overline{y}_N = 0$. *Moreover, the following stability inequality holds*

$$\|w - v\|_\infty \leqslant \frac{1}{m}\left\|\tilde{F}w - \tilde{F}v\right\|_\infty,$$

*for any vectors* $v = (v_0, v_1, \ldots, v_N)^T \in \mathbb{R}^{N+1}$, $w = (w_0, w_1, \ldots, w_N)^T \in \mathbb{R}^{N+1}$.

*Proof.* We use a technique from [7] and [16] , and the proof of existence of the solution of $\tilde{F}(\overline{y})=0$ is based on the proof of the following relation: $\left\|\tilde{F}'(\overline{y})^{-1}\right\|_\infty \leq C$, where $\tilde{F}'(\overline{y})$ is the Fréchet derivative of $\tilde{F}$.

The Fréchet derivative $H := \tilde{F}'(\overline{y})$ of the operator defined in (2.10) is a tridiagonal matrix. Let $H=[h_{ij}]$. The non-zero elements of this tridiagonal matrix are

$$\begin{aligned} h_{0,0} =&1,\ h_{N,N}=1,\\ h_{i,i} =&-\tfrac{\gamma}{\triangle d_i+\triangle d_{i+1}}\left[\tfrac{a_i+d_i}{2}+\tfrac{a_{i+1}+d_{i+1}}{2}+\tfrac{\triangle d_i}{2}\cdot\tfrac{f_{i-1,\overline{y}_i}}{\gamma}+\tfrac{\triangle d_{i+1}}{2}\cdot\tfrac{f_{i,\overline{y}_i}}{\gamma}\right]<0,\\ h_{i,i-1} =&\tfrac{\gamma}{\triangle d_i+\triangle d_{i+1}}\left[a_i+\tfrac{\triangle d_i}{2}\left(1-\tfrac{f_{i-1,\overline{y}_{i-1}}}{\gamma}\right)\right]>0,\\ h_{i,i+1} =&\tfrac{\gamma}{\triangle d_i+\triangle d_{i+1}}\left[a_{i+1}+\tfrac{\triangle d_{i+1}}{2}\left(1-\tfrac{f_{i,\overline{y}_{i+1}}}{\gamma}\right)\right]>0, \end{aligned}$$

where $f_{j,\overline{y}_k}=\frac{\partial}{\partial \overline{y}_k}\overline{f}_j$, $j\in\{i-1,i\}$, $k\in\{i-1,i,i+1\}$. Hence $H$ is an $L$–matrix.

Since

$$\begin{aligned} |h_{i,i}|-|h_{i,i-1}|-|h_{i-1,i}| =&\tfrac{\gamma}{\triangle d_i+\triangle d_{i+1}}\left[\tfrac{\triangle d_i}{\gamma}\cdot\tfrac{f_{i-1,\overline{y}_i}+f_{i-1,\overline{y}_{i-1}}}{2}+\tfrac{\triangle d_{i+1}}{\gamma}\cdot\tfrac{f_{i,\overline{y}_i}+f_{i,\overline{y}_{i+1}}}{2}\right]\\ \geqslant&\tfrac{\gamma}{\triangle d_i+\triangle d_{i+1}}\left[\tfrac{\triangle d_i}{\gamma}\cdot\tfrac{m+m}{2}+\tfrac{\triangle d_{i+1}}{\gamma}\cdot\tfrac{m+m}{2}\right]\\ =&m, \end{aligned}$$

the matrix $H$ is also an $M$–matrix and

$$\left\|H^{-1}\right\|_\infty \leq \frac{1}{m}. \tag{2.11}$$

Now, by Hadamard's Theorem (5.3.10 from [10] we can conclude that $\tilde{F}$ defined in (2.10), is a homeomorphism. Since $\mathbb{R}^{N+1}$ is nonempty set, there exists a solution of the problem (2.10) and regarding that 0 is the only image of F we come to the conclusion that $\overline{y}$ is the only solution of the problem (2.10).

The second part of the proof is based on the part of the proof of Theorem 3 from [6] . We have that

$$\tilde{F}w-\tilde{F}v=\left(\tilde{F}'u\right)(w-v)$$

for some $u=(u_0,u_1,\dots,u_N)^T\in\mathbb{R}^{N+1}$. Therefore

$$w-v=(\tilde{F}'u)^{-1}\left(\tilde{F}w-\tilde{F}v\right)$$

and finally because of (2.11) we have that

$$\|w-v\|_\infty=\left\|(\tilde{F}'u)^{-1}(\tilde{F}w-\tilde{F}v)\right\|_\infty\leqslant\frac{1}{m}\left\|\tilde{F}w-\tilde{F}v\right\|_\infty.$$

□

## 3. Construction of the mesh

The solution changes rapidly near $x = 0$ and $x = 1$. Hence the mesh has to be refined there. Various meshes have been proposed in the literature. The most frequently analysed are the exponentially graded mesh of Bakhvalov [1] and piecewise uniform mesh of Shishkin [12].

Here we shall use the smoothed Shishkin mesh from [8, 9] . Let $N+1$ be the number of mesh points, $q \in (0, 1/2)$ and $\sigma > 0$ the mesh parameters. Define the Shishkin-mesh transition point by

$$\lambda := \min\left\{ \frac{\sigma\varepsilon}{\sqrt{m}} \ln N, q \right\}.$$

Let us chose $\sigma = 2$.

**Remark 3.1.** *For the sake of simplicity in representation, we assume that* $\lambda = 2\varepsilon(\sqrt{m})^{-1}\ln N$, *as otherwise the problem can be analysed in the classical way. We shall also assume that* $qN$ *is an integer. This is easily achieved by choosing* $q = 1/4$ *and* $N$ *divisible by 4 for example.*

The mesh $\triangle : x_0 < x_1 < ... < x_N$ is generated by $x_i = \varphi(i/N)$ with the mesh generating function

$$\varphi(t) := \begin{cases} \frac{\lambda}{q} t & t \in [0, q], \\ p(t-q)^3 + \frac{\lambda}{q} t & t \in [q, 1/2], \\ 1 - \varphi(1-t) & t \in [1/2, 1], \end{cases}$$

where $p$ is chosen so that $\varphi(1/2) = 1/2$, i.e. $p = \frac{1}{2}(1 - \frac{\lambda}{q})(\frac{1}{2} - q)^{-3}$. Note that $\varphi \in C^1[0, 1]$ with $\|\varphi'\|_\infty, \|\varphi''\|_\infty \le C$. Therefore the mesh sizes $h_i = x_{i+1} - x_i$, $i = 0, 1, 2, ..., N-1$ satisfy (see [9] for details)

(3.1)

$$h_i = \int_{i/N}^{(i+1)/N} \varphi'(t)\,\mathrm{d}\,t \le CN^{-1}, \quad |h_{i+1} - h_i| = \left| \int_{i/N}^{(i+1)/N} \int_t^{t+1/N} \varphi''(s)\,\mathrm{d}\,s \right| \le \frac{C}{N^2}.$$

## 4. The error estimating of the nonlinear difference scheme

We will prove theorem on uniform convergence of the difference scheme (2.9) on the part of the mesh which corresponds to $[0, 1/2]$, while the proof on $[1/2, 1]$ can be analogously derived.

Namely, in the analysis of the value of the error the functions $e^{-\frac{x}{\varepsilon}\sqrt{m}}$ and $e^{-\frac{1-x}{\varepsilon}\sqrt{m}}$ appear. For these functions we have that $e^{-\frac{x}{\varepsilon}\sqrt{m}} \geqslant e^{-\frac{1-x}{\varepsilon}\sqrt{m}}$, $\forall x \in [0, 1/2]$ and $e^{-\frac{x}{\varepsilon}\sqrt{m}} \leqslant e^{-\frac{1-x}{\varepsilon}\sqrt{m}}$, $\forall x \in [1/2, 1]$. In the boundary layer in the neighbourhood of $x = 0$, we have that $e^{-\frac{x}{\varepsilon}\sqrt{m}} >> e^{-\frac{1-x}{\varepsilon}\sqrt{m}}$, while in the boundary layer in the neighbourhood of $x = 1$ we have that $e^{-\frac{x}{\varepsilon}\sqrt{m}} << e^{-\frac{1-x}{\varepsilon}\sqrt{m}}$. Based on the above, it is enough to prove the theorem on the part of the mesh which corresponds to $[0, 1/2]$ with the exclusion of the function $e^{-\frac{1-x}{\varepsilon}\sqrt{m}}$, or on $[1/2, 1]$ but with the exclusion of the function $e^{-\frac{x}{\varepsilon}\sqrt{m}}$. Note that we need to take care of the fact that in the first case $h_{i-1} \leqslant h_i$, and in the second case $h_{i-1} \geqslant h_i$.

The proof uses the decomposition of the solution $y$ of the problem $(1.1)-(1.2)$ to the layer component $s$ and a regular component $r$, given in the following assertion.

**Theorem 4.1.** [15] *The solution $y$ to problem $(1.1)-(1.2)$ can be represented in the following way:*

$$y = r + s,$$

*where for $j = 0, 1, ..., k+2$ and $x \in [0,1]$ we have that*

$$\left|r^{(j)}(x)\right| \leqslant C, \tag{4.1}$$

*and*

$$\left|s^{(j)}(x)\right| \leqslant C\varepsilon^{-j}\left(e^{-\frac{x}{\varepsilon}\sqrt{m}} + e^{-\frac{1-x}{\varepsilon}\sqrt{m}}\right).$$

On equidistant part of the mesh, that is for $x_i, x_{i\pm1} \in [0,\lambda]$ and $i = 1, 2, \ldots, N/4-1$, we will use Taylor expansions for the function $y$

$$\begin{aligned} y_{i-1} - y_i &= -y_i' h_{i-1} + \frac{y_i''}{2}h_{i-1}^2 - \frac{y_i'''}{6}h_{i-1}^3 + \frac{y^{(iv)}(\zeta_i^-)}{24}h_{i-1}^4, \\ y_i - y_{i+1} &= -y_i' h_i - \frac{y_i''}{2}h_i^2 - \frac{y_i'''}{6}h_i^3 - \frac{y^{(iv)}(\zeta_i^+)}{24}h_i^4, \end{aligned} \tag{4.2}$$

while for function $f$ we will use Taylor expansions

$$\begin{aligned} f_{i-1} =& f\left(\tfrac{x_{i-1}+x_i}{2}, \tfrac{y_{i-1}+y_i}{2}\right) \\ =& \varepsilon^2 y_i'' - \frac{1}{2}\varepsilon^2 y_i''' h_{i-1} + \frac{1}{2}f_y(x_i,y_i)\left(-y_i' h_{i-1} + \frac{y_i''}{2}h_{i-1}^2 - \frac{y_i'''}{6}h_{i-1}^3\right. \\ &\left. + \frac{y^{(iv)}(\zeta_i^-)}{24}h_{i-1}^4\right) + \frac{1}{8}f_{yy}(\xi_i^-,\eta_i^-)(y_i - y_{i-1})^2 + \frac{1}{8}f_{xx}(\xi_i^-,\eta_i^-)h_{i-1}^2 \\ &+ \frac{1}{4}f_{xy}(\xi_i^-,\eta_i^-)(y_i - y_{i-1})h_{i-1}, \\ f_i =& f\left(\tfrac{x_i+x_{i+1}}{2}, \tfrac{y_i+y_{i+1}}{2}\right) \\ =& \varepsilon^2 y_i'' + \frac{1}{2}\varepsilon^2 y_i''' h_i + \frac{1}{2}f_y(x_i,y_i)\left(y_i' h_i + \frac{y_i''}{2}h_i^2 + \frac{y_i'''}{6}h_i^3 + \frac{y^{(iv)}(\zeta_i^+)}{24}h_i^4\right) \\ &+ \frac{1}{8}f_{yy}(\xi_i^+,\eta_i^+)(y_{i+1} - y_i)^2 + \frac{1}{8}f_{xx}(\xi_i^+,\eta_i^+)h_i^2 + \frac{1}{4}f_{xy}(\xi_i^+,\eta_i^+)(y_{i+1} - y_i)h_i, \end{aligned} \tag{4.3}$$

where $y_i = y(x_i)$, $\xi_i^- \in ((x_{i-1}+x_i)/2, x_i)$, $\zeta_i^- \in (x_{i-1}, x_i), \xi_i^+ \in (x_i, (x_i + x_{i+1})/2)$, $\zeta_i^+ \in (x_i, x_{i+1})$, $\eta_i^- \in ((y_{i-1}+y_i)/2, y_i)$ and $\eta_i^+ \in (y_i, (y_i + y_{i+1})/2)$. For $x_i, x_{i\pm1} \in \left[x_{N/4-1}, \lambda\right] \cup [\lambda, 1/2]$, i.e. $i = N/4, \ldots, N/2-1$, we will use Taylor expansions for the function $y$

$$\begin{aligned} y_{i-1} - y_i &= -y_i' h_{i-1} + \frac{y''(\mu_i^-)}{2}h_{i-1}^2, \quad \mu_i^- \in (x_{i-1}, x_i), \\ y_i - y_{i+1} &= -y_i' h_i - \frac{y''(\mu_i^+)}{2}h_i^2, \quad \mu_i^+ \in (x_i, x_{i+1}). \end{aligned} \tag{4.4}$$

Let us start with the following three lemmas that will be further used in the proof of the uniform convergence on the part of the mesh from § 3 which corresponds to $\left[x_{N/4-1}, 1/2\right]$, $x_{N/4} = \lambda$.

**Lemma 4.1.** *Assume that* $\varepsilon \leqslant \frac{C}{N}$. *In the part of the modified Shishkin mesh from § 3 when* $x_i, x_{i\pm1} \in \left[x_{N/4-1}, \lambda\right] \cup [\lambda, 1/2]$, *we have the following estimate:*

$$\left| \frac{-\frac{\cosh(\beta h_{i-1})-1}{\gamma \sinh(\beta h_{i-1})} f_{i-1} - \frac{\cosh(\beta h_i)-1}{\gamma \sinh(\beta h_i)} f_i}{\frac{\cosh(\beta h_{i-1})-1}{\gamma \sinh(\beta h_{i-1})} + \frac{\cosh(\beta h_i)-1}{\gamma \sinh(\beta h_i)}} \right| \leqslant \frac{C}{N^2},\ (i = N/4, ..., N/2-1).$$

Proof of the lemma is given in Appendix 7.1.

**Lemma 4.2.** *Assume that* $\varepsilon \leqslant \frac{C}{N}$. *In the part of the modified Shishkin mesh from § 3 when* $x_i, x_{i\pm1} \in \left[x_{N/4-1}, \lambda\right] \cup [\lambda, 1/2]$, *we have the following estimate*

$$\left| \frac{\frac{\cosh(\beta h_{i-1})-1}{2\sinh(\beta h_{i-1})}(y_{i-1} - y_i) - \frac{\cosh(\beta h_i)-1}{2\sinh(\beta h_i)}(y_i - y_{i+1})}{\frac{\cosh(\beta h_{i-1})-1}{\gamma \sinh(\beta h_{i-1})} + \frac{\cosh(\beta h_i)-1}{\gamma \sinh(\beta h_i)}} \right| \leqslant \frac{C}{N^2},\ (i = N/4, ..., N/2-1).$$

Proof of the lemma is given in Appendix 7.2.

**Lemma 4.3.** *Assume that* $\varepsilon \leqslant \frac{C}{N}$. *In the part of the modified Shishkin mesh from § 3 when* $x_i, x_{i\pm1} \in \left[x_{N/4-1}, \lambda\right] \cup [\lambda, 1/2]$, *we have the following estimate*

$$\frac{1}{\frac{\cosh(\beta h_{i-1})-1}{\gamma \sinh(\beta h_{i-1})} + \frac{\cosh(\beta h_i)-1}{\gamma \sinh(\beta h_i)}} \left| \frac{y_{i-1} - y_i}{\sinh(\beta h_{i-1})} - \frac{y_i - y_{i+1}}{\sinh(\beta h_i)} \right| \leqslant \frac{C}{N^2},\ i = N/4, ..., N/2-1.$$

Proof of the lemma is given in Appendix 7.3.

The proof of the theorem on $\varepsilon-$ uniform convergence is based on the relation $\|y - \overline{y}\|_\infty \leqslant C \left\| \tilde{F}y - \tilde{F}\overline{y} \right\|_\infty$.

Since $\tilde{F}\overline{y} = 0$, it only remains to estimate $\left\| \tilde{F}y \right\|_\infty$. Now we can state the main theorem on $\varepsilon-$uniform convergence of our difference scheme and the specially chosen layer-adapted mesh.

**Theorem 4.2.** *The difference scheme* (2.9) *on the mesh from Section 3 is uniformly convergent with respect to* $\varepsilon$, *and*

$$\max_{0 \le i \le N} |y(x_i) - \overline{y}_i| \le C \frac{\ln^2 N}{N^2},$$

*where* $y(x)$ *is the solution of the problem* $(1.1)-(1.2)$, $\overline{y}$ *is the corresponding numerical solution of* (2.9), *and* $C > 0$ *is a constant independent with respect to* $N$ *and* $\varepsilon$.

*Proof.* Suppose first that $x_i \in [0, \lambda], i = 1, \ldots, N/4-1$. On this part of the mesh, we have that $h_{i-1} = h_i \le C\frac{\varepsilon \ln N}{N}$. The scheme (2.9) for the function $y$ can be represented as

$$(Fy)_i = \frac{\cosh(\beta h_i)+1}{2}\left[y_{i-1} - y_i - (y_i - y_{i+1})\right] - \frac{\cosh(\beta h_i)-1}{\gamma}(f_{i-1} + f_i), \quad (i = 1, ..., N/4-1). \tag{4.5}$$

Now, putting (4.2), (4.3) and

$$\cosh(\beta h_i) = 1 + \frac{(\beta h_i)^2}{2} + \mathcal{O}\left(\ln^4 N/N^4\right),$$

into (4.5), we get that

$$\begin{aligned}(Fy)_i =& \left[\frac{\beta^2 h_i^2}{4} + \mathcal{O}(\ln^4 N/N^4)\right] y_i'' h_i^2 \\ &+ \left(1 + \frac{\beta^2 h_i^2}{4} + \mathcal{O}(\ln^4 N/N^4)\right) \frac{y^{(iv)}(\zeta_i^-)h_i^4 + y^{(iv)}(\zeta_i^+)h_i^4}{24} \\ &- \frac{\mathcal{O}(\ln^4 N/N^4)}{\gamma}\left[2\varepsilon^2 y_i'' + \frac{1}{2} f_y(x_i, y_i)\left(y_i'' h_i^2\right.\right. \\ &\left.\left.+\frac{y^{(iv)}(\zeta_i^-)h_i^4 + y^{(iv)}(\zeta_i^+)h_i^4}{24}\right) + SD_i\right] \\ &- \frac{\beta^2 h_i^2}{2\gamma}\left[\frac{1}{2} f_y(x_i, y_i)\left(y_i'' h_i^2 + \frac{y^{(iv)}(\zeta_i^-)h_i^4 + y^{(iv)}(\zeta_i^+)h_i^4}{24}\right) + SD_i\right],\end{aligned}$$

where

$$\begin{aligned}SD_i =& \frac{1}{8} f_{yy}(\xi_i^-, \eta_i^-)(y_i - y_{i-1})^2 + \frac{1}{8} f_{xx}(\xi_i^-, \eta_i^-)h_i^2 + \frac{1}{4} f_{xy}(\xi_i^-, \eta_i^-)(y_i - y_{i-1})h_i \\ &+ \frac{1}{8} f_{yy}(\xi_i^+, \eta_i^+)(y_{i+1} - y_i)^2 + \frac{1}{8} f_{xx}(\xi_i^+, \eta_i^+)h_i^2 + \frac{1}{4} f_{xy}(\xi_i^+, \eta_i^+)(y_{i+1} - y_i)h_i.\end{aligned}$$

For the sake of normalization, dividing (4.5) with $\frac{\cosh(\beta h_i)-1}{\gamma}$ we get

$$2|(\tilde{F}y)_i| = \left|\frac{(Fy)_i}{\frac{\cosh(\beta h_i)-1}{\gamma}}\right| \leq C\frac{\ln^2 N}{N^2}. \tag{4.6}$$

Now, suppose that $x_i \in [x_{N/4-1}, \lambda] \cup [\lambda, 1/2]$ for $i = N/4, \ldots, N/2-1$. The scheme (2.9) for the function $y$ can be written as

$$\begin{aligned}(Fy)_i =& \frac{\cosh(\beta h_{i-1}) - 1}{2\sinh(\beta h_{i-1})}(y_{i-1} - y_i) - \frac{\cosh(\beta h_i) - 1}{2\sinh(\beta h_i)}(y_i - y_{i+1}) \\ &+ \frac{y_{i-1} - y_i}{\sinh(\beta h_{i-1})} - \frac{y_i - y_{i+1}}{\sinh(\beta h_i)} \\ &- \frac{\cosh(\beta h_{i-1}) - 1}{\gamma\sinh(\beta h_{i-1})} f_{i-1} - \frac{\cosh(\beta h_i) - 1}{\gamma\sinh(\beta h_i)} f_i, \ (i = N/4, ..., N/2-1).\end{aligned}$$

From the inequality

$$\begin{aligned}&\left|\frac{(Fy)_i}{\frac{\cosh(\beta h_{i-1})-1}{\gamma\sinh(\beta h_{i-1})} + \frac{\cosh(\beta h_i)-1}{\gamma\sinh(\beta h_i)}}\right| \leqslant \frac{1}{\frac{\cosh(\beta h_{i-1})-1}{\gamma\sinh(\beta h_{i-1})} + \frac{\cosh(\beta h_i)-1}{\gamma\sinh(\beta h_i)}} \\ &\cdot \left(\left|\frac{\cosh(\beta h_{i-1})-1}{2\sinh(\beta h_{i-1})}(y_{i-1} - y_i) - \frac{\cosh(\beta h_i)-1}{2\sinh(\beta h_i)}(y_i - y_{i+1})\right|\right. \\ &\left.+ \left|\frac{y_{i-1}-y_i}{\sinh(\beta h_{i-1})} - \frac{y_i - y_{i+1}}{\sinh(\beta h_i)}\right| + \left|-\frac{\cosh(\beta h_{i-1})-1}{\gamma\sinh(\beta h_{i-1})} f_{i-1} - \frac{\cosh(\beta h_i)-1}{\gamma\sinh(\beta h_i)} f_i\right|\right),\end{aligned}$$

based on Lemma 1–Lemma 3, we have that

$$|(\tilde{F}y)_i| = \left|\frac{(Fy)_i}{\frac{\cosh(\beta h_{i-1})-1}{\gamma\sinh(\beta h_{i-1})} + \frac{\cosh(\beta h_i)-1}{\gamma\sinh(\beta h_i)}}\right| \leqslant \frac{C}{N^2}. \tag{4.7}$$

According to (4.6) and (4.7), the proof is complete. □

## 5. The numerical results

In this section we present numerical results to confirm the uniform accuracy of the scheme (2.9).

Consider the following problem from [14]

$$\varepsilon^2 y'' = (1+y)(1+(1+y)^2) \quad \text{on} \quad (0,1)\,, \tag{5.1}$$

$$y(0) = y(1) = 0, \tag{5.2}$$

whose exact solution is unknown. The nonlinear system of equations is solved by Newton's method with initial guess $y_0 = -1$ that represents the reduced solution. The value of the constant $\gamma = 4$ has been chosen so that the condition $\gamma \geq f_y(x,y)$, $\forall(x,y) \in [0,1] \times [y_L, y_U] \subset [0,1] \times \mathbb{R}$ is fulfilled, where $y_L$ and $y_U$ are lower and upper solutions of the test problem (5.1) – (5.2) and their values are $y_L = -1$ and $y_U = 0$. Because of the fact that the exact solution is unknown, we define the computed error $E_N$ and the computed rate of convergence Ord in the usual way (double-mesh method, see [4, 13, 14]

$$E_N = \max_{0\leq i\leq N} \left|\tilde{y}^{2N}(x_i) - \overline{y}^N(x_i)\right|, \qquad \text{Ord} = \frac{\ln E_N - \ln E_{2N}}{\ln \frac{2k}{k+1}},$$

where $N = 2^k$, $k = 6, 7, ..., 11$, $\overline{y}^N(x_i)$ is the numerical solution on a mesh with $N$ subintervals, and $\tilde{y}^{2N}(x_i)$ is the numerical solution on a mesh with $2N$ subintervals and the transition point altered slightly to $\tilde{\lambda}_\varepsilon = \min\left\{\frac{1}{4}, \frac{2\varepsilon}{\sqrt{m}} \ln \frac{N}{2}\right\}$.

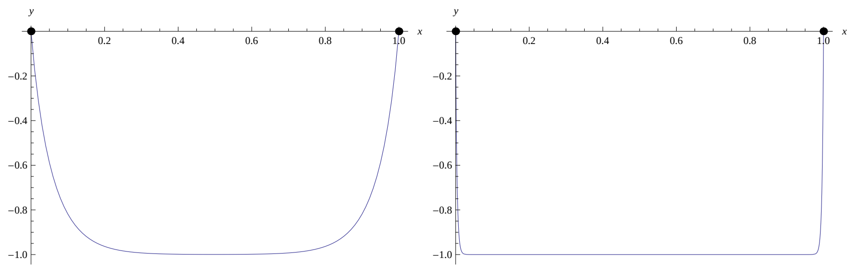


Figure 1. Graphics of approximate solutions for values $\varepsilon = 2^{-3}$, $\varepsilon = 2^{-10}$.

In Figure 1 we display the computed solution of (5.1)–(5.2) for two values of the parameter $\varepsilon$. For different values of the perturbation parameter $\varepsilon$, in Table 1 we present the results of numerical experiments that clearly confirm the robustness of the method as well as that theoretical and experimental results match.

| $N$ | $E_n$ | Ord | $E_n$ | Ord | $E_n$ | Ord |
|---|---|---|---|---|---|---|
| $2^6$ | $4.5606e-04$ | 2.42 | $3.1000e-03$ | 1.99 | $7.9175e-03$ | 1.99 |
| $2^7$ | $1.2375e-04$ | 2.04 | $1.0597e-03$ | 2.00 | $2.7093e-03$ | 2.00 |
| $2^8$ | $3.9496e-05$ | 2.01 | $3.4620e-04$ | 2.00 | $8.8552e-04$ | 2.00 |
| $2^9$ | $1.2398e-05$ | 2.00 | $1.0956e-04$ | 2.00 | $2.8030e-04$ | 2.00 |
| $2^{10}$ | $3.8199e-06$ | 2.00 | $3.3815e-05$ | 2.00 | $8.6565e-05$ | 2.00 |
| $2^{11}$ | $1.1562e-06$ | 2.00 | $1.0229e-05$ | 2.00 | $2.6187e-05$ | 2.00 |
| $2^{12}$ | $3.4400e-07$ | 2.00 | $3.0434e-06$ | 2.00 | $7.7911e-06$ | 2.00 |
| $2^{13}$ | $1.0093e-07$ | – | $8.9294e-07$ | – | $2.2860e-06$ | – |
| $\varepsilon$ | $2^{-3}$ | | $2^{-5}$ | | $2^{-10}$ | |

| $N$ | $E_n$ | Ord | $E_n$ | Ord | $E_n$ | Ord |
|---|---|---|---|---|---|---|
| $2^6$ | $7.9175e-03$ | 1.99 | $7.9175e-03$ | 1.99 | $7.9175e-03$ | 1.99 |
| $2^7$ | $2.7093e-03$ | 2.00 | $2.7093e-03$ | 1.99 | $2.7093e-03$ | 1.99 |
| $2^8$ | $8.8552e-04$ | 2.00 | $8.8552e-04$ | 2.00 | $8.8552e-04$ | 2.00 |
| $2^9$ | $2.8030e-04$ | 2.00 | $2.8030e-04$ | 2.00 | $2.8030e-04$ | 2.00 |
| $2^{10}$ | $8.6565e-05$ | 2.00 | $8.6565e-05$ | 2.00 | $8.6565e-05$ | 2.00 |
| $2^{11}$ | $2.6187e-05$ | 2.00 | $2.6187e-05$ | 2.00 | $2.6187e-05$ | 2.00 |
| $2^{12}$ | $7.7911e-06$ | 2.00 | $7.7911e-06$ | 2.00 | $7.7911e-06$ | 2.00 |
| $2^{13}$ | $2.2860e-06$ | – | $2.2860e-06$ | – | $2.2860e-06$ | – |
| $\varepsilon$ | $2^{-15}$ | | $2^{-25}$ | | $2^{-30}$ | |

| $N$ | $E_n$ | Ord | $E_n$ | Ord | $E_n$ | Ord |
|---|---|---|---|---|---|---|
| $2^6$ | $7.9175e-03$ | 1.99 | $7.9176e-03$ | 1.99 | $7.9175e-03$ | 1.98 |
| $2^7$ | $2.7093e-03$ | 1.99 | $2.7095e-03$ | 2.00 | $2.7223e-03$ | 2.01 |
| $2^8$ | $8.8552e-04$ | 2.00 | $8.8553e-04$ | 2.00 | $8.5552e-04$ | 2.00 |
| $2^9$ | $2.8030e-04$ | 2.00 | $2.8030e-04$ | 2.00 | $2.8030e-04$ | 2.00 |
| $2^{10}$ | $8.6565e-05$ | 2.00 | $8.6565e-05$ | 2.00 | $8.6571e-05$ | 1.99 |
| $2^{11}$ | $2.6187e-05$ | 2.00 | $2.6189e-05$ | 2.00 | $2.6329e-05$ | 1.99 |
| $2^{12}$ | $7.7911e-06$ | 2.00 | $7.7915e-06$ | 2.00 | $7.8633e-06$ | 1.99 |
| $2^{13}$ | $2.2860e-06$ | – | $2.2864e-06$ | – | $2.3213e-06$ | – |
| $\varepsilon$ | $2^{-35}$ | | $2^{-40}$ | | $2^{-45}$ | |

TABLE 1. Errors $E_N$ and convergence rates Ord for approximate solutions.

## 6. Discussion

In this paper we present a discretization of a one-dimensional semilinear reaction-diffusion problem, with suitable assumptions that ensure the existence and uniqueness of the continuous problem. We prove the existence and uniqueness of the numerical solution, the $\varepsilon$–uniform convergence using a suitable layer-adaptive mesh and finally we perform a numerical experiment which agrees with theoretical results.

The presented method should be expandable to discretization of higher dimensional boundary value problems without major problems. Namely, hyperbolic functions appear in the difference scheme coefficients, so one has to choose the discretization in which these hyperbolic functions remain functions of one variable, which is not difficult to do. In this case we could separate the terms in which the same variables appear and the analysis would be reduced to the one presented in this paper. In the case of the discretization in which hyperbolic functions of several variables appear, the analysis would be more difficult. Clearly, the above discussion is only related to suitable Shishkin-type meshes. Using Bakhvalov-type meshes the analysis of one-dimensional boundary value problems becomes substantially more difficult, as is the analysis of the dicretization of higher dimensional boundary value problems.

## 7. Appendix

### 7.1. Proof of the Lemma 4.1.

*Proof.* Due to $\varepsilon^2 y''(x_i) = f(x_i, y(x_i))$, Theorem 4.1 for both components $r$ and $s$ in part of the mesh $[x_{N/4-1}, \lambda] \cup [\lambda, 1/2]$ and the assumption $\varepsilon \leqslant \frac{C}{N}$, we have that

$$\left| \frac{-\frac{\cosh(\beta h_{i-1})-1}{\gamma \sinh(\beta h_{i-1})} f_{i-1} - \frac{\cosh(\beta h_i)-1}{\gamma \sinh(\beta h_i)} f_i}{\frac{\cosh(\beta h_{i-1})-1}{\gamma \sinh(\beta h_{i-1})} + \frac{\cosh(\beta h_i)-1}{\gamma \sinh(\beta h_i)}} \right| \leqslant \frac{\frac{\cosh(\beta h_{i-1})-1}{\gamma \sinh(\beta h_{i-1})}}{\frac{\cosh(\beta h_{i-1})-1}{\gamma \sinh(\beta h_{i-1})} + \frac{\cosh(\beta h_i)-1}{\gamma \sinh(\beta h_i)}} |f_{i-1}| + \frac{\frac{\cosh(\beta h_i)-1}{\gamma \sinh(\beta h_i)}}{\frac{\cosh(\beta h_{i-1})-1}{\gamma \sinh(\beta h_{i-1})} + \frac{\cosh(\beta h_i)-1}{\gamma \sinh(\beta h_i)}} |f_i| \leqslant |f_{i-1}| + |f_i| \leqslant \frac{C}{N^2}.$$

□

### 7.2. Proof of the Lemma 4.2.

*Proof.* Let us use again the decomposition from Theorem 4.1 and assumption $\varepsilon \leqslant \frac{C}{N}$.

For the layer component $s$ we have that

$$\left| \frac{\frac{\cosh(\beta h_{i-1})-1}{2\sinh(\beta h_{i-1})}(s_{i-1} - s_i) - \frac{\cosh(\beta h_i)-1}{2\sinh(\beta h_i)}(s_i - s_{i+1})}{\frac{\cosh(\beta h_{i-1})-1}{\gamma \sinh(\beta h_{i-1})} + \frac{\cosh(\beta h_i)-1}{\gamma \sinh(\beta h_i)}} \right| \leqslant \frac{\gamma}{2}\left(|s_{i-1} - s_i| + |s_i - s_{i+1}|\right).$$

Using (4.2), $e^{-\frac{x}{\varepsilon}\sqrt{m}} \geqslant e^{-\frac{1-x}{\varepsilon}\sqrt{m}}$, $\forall x \in [0, 1/2]$, the monotonicity of the function $e^{-\frac{x}{\varepsilon}\sqrt{m}}$ and putting $x_{N/4-1} = \frac{2\varepsilon \ln N}{\sqrt{m}} \cdot \frac{N/4-1}{N/4}$ into $e^{-\frac{x}{\varepsilon}\sqrt{m}}$, we get that

$$\begin{aligned} |s_{i-1} - s_i| + |s_i - s_{i+1}| &\leqslant 4|s_{i-1}| \leqslant C_1 e^{-2\ln N \cdot \frac{N-4}{N}} = C_1 e^{-2\ln N} e^{\frac{8\ln N}{N}} \\ &\leqslant \frac{C_1}{N^2}\left(1 + \frac{8\ln N}{N} + \cdots\right) \qquad (7.1) \\ &\leqslant \frac{C}{N^2}. \end{aligned}$$

For the regular component $r$, due to $\frac{\cosh x - 1}{\sinh x} = \tanh \frac{x}{2}$, we have that

$$\left| \frac{\frac{\cosh(\beta h_{i-1})-1}{2\sinh(\beta h_{i-1})}(r_{i-1} - r_i) - \frac{\cosh(\beta h_i)-1}{2\sinh(\beta h_i)}(r_i - r_{i+1})}{\frac{\cosh(\beta h_{i-1})-1}{\gamma \sinh(\beta h_{i-1})} + \frac{\cosh(\beta h_i)-1}{\gamma \sinh(\beta h_i)}} \right| = \frac{\gamma}{2} \cdot \frac{1}{\tanh\frac{\beta h_{i-1}}{2} + \tanh\frac{\beta h_i}{2}} \left|\tanh\frac{\beta h_{i-1}}{2}(r_{i-1} - r_i) - \tanh\frac{\beta h_i}{2}(r_i - r_{i+1})\right|.$$

Using Taylor expansions (4.4) for $r_{i-1}$ and $r_{i+1}$, we get that

$$
\begin{aligned}
&\frac{\gamma}{2}\cdot\frac{1}{\tanh\frac{\beta h_{i-1}}{2}+\tanh\frac{\beta h_i}{2}}\left|\tanh\tfrac{\beta h_{i-1}}{2}(r_{i-1}-r_i)-\tanh\tfrac{\beta h_i}{2}(r_i-r_{i+1})\right|\\
&=\tfrac{\gamma}{2}\cdot\tfrac{1}{\tanh\frac{\beta h_{i-1}}{2}+\tanh\frac{\beta h_i}{2}}\left|\tanh\tfrac{\beta h_{i-1}}{2}\left(-r_i'h_{i-1}+\tfrac{r''(\mu_i^-)}{2}h_{i-1}^2\right)\right.\\
&\quad\left.+\tanh\tfrac{\beta h_i}{2}\left(r_i'h_i+\tfrac{r''(\mu_i^+)}{2}h_i^2\right)\right|\\
&\leqslant C\left(h_{i-1}^2+h_i^2+\tfrac{|r_i'|}{\tanh\frac{\beta h_{i-1}}{2}+\tanh\frac{\beta h_i}{2}}\left|\tanh\tfrac{\beta h_{i-1}}{2}h_{i-1}-\tanh\tfrac{\beta h_i}{2}h_i\right|\right).
\end{aligned}
\tag{7.2}
$$

Because of the monotonicity of the function $\tanh x$ and (4.1), we get the following estimate

$$
\frac{|r_i'|}{\tanh\frac{\beta h_{i-1}}{2}+\tanh\frac{\beta h_i}{2}}\left|\tanh\tfrac{\beta h_{i-1}}{2}h_{i-1}-\tanh\tfrac{\beta h_i}{2}h_i\right|\leqslant C\left|\frac{\tanh\frac{\beta h_{i-1}}{2}}{\tanh\frac{\beta h_i}{2}}h_{i-1}-h_i\right|.
$$

Now, let us use the second inequality from (3.1) for $h_i-h_{i-1}$ in order to get

$$
\begin{aligned}
&\left|\frac{\tanh\frac{\beta h_{i-1}}{2}}{\tanh\frac{\beta h_i}{2}}h_{i-1}-h_i\right|=\left|\frac{\tanh\frac{\beta h_{i-1}}{2}}{\tanh\frac{\beta h_i}{2}}h_{i-1}-\left(h_{i-1}+\frac{C}{N^2}\right)\right|\\
&=\left|\frac{\tanh\frac{\beta h_{i-1}}{2}}{\tanh\frac{\beta h_i}{2}}-1\right|h_{i-1}+\frac{C}{N^2}=\frac{\tanh\frac{\beta h_i}{2}-\tanh\frac{\beta h_{i-1}}{2}}{\tanh\frac{\beta h_i}{2}}h_{i-1}+\frac{C}{N^2}\\
&=\frac{\frac{e^{\beta h_i}-1}{e^{\beta h_i}+1}-\frac{e^{\beta h_{i-1}}-1}{e^{\beta h_{i-1}}+1}}{\frac{e^{\beta h_i}-1}{e^{\beta h_i}+1}}h_{i-1}+\frac{C}{N^2}=2\frac{e^{\beta h_i}-e^{\beta h_{i-1}}}{(e^{\beta h_i}-1)(e^{\beta h_{i-1}}+1)}h_{i-1}+\frac{C}{N^2}\\
&=2\frac{\beta(h_i-h_{i-1})+\frac{\beta^2(h_i^2-h_{i-1}^2)}{2!}+\frac{\beta^3(h_i^3-h_{i-1}^3)}{3!}+\dots}{(e^{\beta h_i}-1)(e^{\beta h_{i-1}}+1)}h_{i-1}+\frac{C}{N^2}\\
&=2\frac{\sum\limits_{n=1}^{+\infty}\frac{\beta^n(h_i^n-h_{i-1}^n)}{n!}}{(e^{\beta h_i}-1)(e^{\beta h_{i-1}}+1)}h_{i-1}+\frac{C}{N^2}.
\end{aligned}
\tag{7.3}
$$

From the identity

$$
a^n-b^n=(a-b)(a^{n-1}+a^{n-2}b+\dots+ab^{n-2}+b^{n-1}),\ n\in\mathbb{N},
\tag{7.4}
$$

and $h_{i-1}\leqslant h_i$, $i=1,2,\dots,N/2-1$, we get the majorization

$$
\begin{aligned}
h_i^n-h_{i-1}^n&=(h_i-h_{i-1})(h_i^{n-1}+h_i^{n-2}h_{i-1}+\dots+h_ih_{i-1}^{n-2}+h_{i-1}^{n-1})\\
&\leqslant n(h_i-h_{i-1})h_i^{n-1}.
\end{aligned}
\tag{7.5}
$$

Moreover, for the last relation from (7.3), due to (7.5), we have

$$
\begin{aligned}
&2\frac{\sum_{n=1}^{+\infty}\frac{\beta^n(h_i^n-h_{i-1}^n)}{n!}}{(e^{\beta h_i}-1)(e^{\beta h_{i-1}}+1)}h_{i-1}+\frac{C}{N^2}\leqslant 2\frac{\beta(h_i-h_{i-1})\sum_{n=0}^{+\infty}\frac{(\beta h_i)^n}{n!}}{(e^{\beta h_i}-1)(e^{\beta h_{i-1}}+1)}h_{i-1}+\frac{C}{N^2}\\
&=2\frac{\beta(h_i-h_{i-1})e^{\beta h_i}}{(e^{\beta h_i}-1)(e^{\beta h_{i-1}}+1)}h_{i-1}+\frac{C}{N^2}\\
&=2\frac{\beta(h_i-h_{i-1})(e^{\beta h_i}-1+1)}{(e^{\beta h_i}-1)(e^{\beta h_{i-1}}+1)}h_{i-1}+\frac{C}{N^2}\\
&\leqslant 2(h_i-h_{i-1})\frac{e^{\beta h_i}-1}{e^{\beta h_i}-1}\cdot\frac{\beta h_{i-1}}{e^{\beta h_{i-1}}+1}+2(h_i-h_{i-1})\frac{\beta h_{i-1}}{(e^{\beta h_i}-1)(e^{\beta h_{i-1}}+1)}+\frac{C}{N^2}\\
&\leqslant C\left(h_i-h_{i-1}+\frac{1}{N^2}\right).
\end{aligned}
\tag{7.6}
$$

Now, collecting (3.1), (7.1), (7.2), (7.3) and (7.6), the statement of the lemma is therefore proven. □

### 7.3. **Proof of the Lemma 4.3.**

*Proof.* We are using again the decomposition from Theorem 4.1 and expansions (4.4).

For the regular component $r$, we have that

$$
\begin{aligned}
&\frac{1}{\frac{\cosh(\beta h_{i-1})-1}{\gamma\sinh(\beta h_{i-1})}+\frac{\cosh(\beta h_i)-1}{\gamma\sinh(\beta h_i)}}\left|\frac{r_{i-1}-r_i}{\sinh(\beta h_{i-1})}-\frac{r_i-r_{i+1}}{\sinh(\beta h_i)}\right|\\
&\leqslant\frac{\gamma}{\frac{\cosh(\beta h_i)-1}{\sinh(\beta h_i)}}\left|\frac{r_{i-1}-r_i}{\sinh(\beta h_{i-1})}-\frac{r_i-r_{i+1}}{\sinh(\beta h_i)}\right|\\
&=\gamma\left|\frac{\sinh(\beta h_i)(r_{i-1}-r_i)-\sinh(\beta h_{i-1})(r_i-r_{i+1})}{(\cosh(\beta h_i)-1)\sinh(\beta h_{i-1})}\right|\\
&=\gamma\left|\frac{r_i'\left(\sum_{n=0}^{+\infty}\frac{(\beta h_{i-1})^{2n+1}}{(2n+1)!}h_i-\sum_{n=0}^{+\infty}\frac{(\beta h_i)^{2n+1}}{(2n+1)!}h_{i-1}\right)}{(\cosh(\beta h_i)-1)\sinh(\beta h_{i-1})}\right.\\
&\quad\left.+\gamma\,\frac{\frac{r''(\mu_i^+)}{2}\sum_{n=0}^{+\infty}\frac{(\beta h_{i-1})^{2n+1}}{(2n+1)!}h_i^2+\frac{r''(\mu_i^-)}{2}\sum_{n=0}^{+\infty}\frac{(\beta h_i)^{2n+1}}{(2n+1)!}h_{i-1}^2}{(\cosh(\beta h_i)-1)\sinh(\beta h_{i-1})}\right|.
\end{aligned}
\tag{7.7}
$$

Let us first estimate the expressions from (7.7) using the first derivatives. Now we have that

$$
\begin{aligned}
&\gamma\left|\frac{r_i'\left(\sum\limits_{n=0}^{+\infty}\frac{(\beta h_{i-1})^{2n+1}}{(2n+1)!}h_i-\sum\limits_{n=0}^{+\infty}\frac{(\beta h_i)^{2n+1}}{(2n+1)!}h_{i-1}\right)}{(\cosh(\beta h_i)-1)\sinh(\beta h_{i-1})}\right|\\
&=\gamma\left|r_i'\frac{\left(\beta h_{i-1}+\frac{\beta^3h_{i-1}^3}{3!}+\dots\right)h_i-\left(\beta h_i+\frac{\beta^3h_i^3}{3!}+\dots\right)h_{i-1}}{(\cosh(\beta h_i)-1)\sinh(\beta h_{i-1})}\right|\\
&=\gamma\left|r_i'\frac{\beta h_{i-1}h_i\sum\limits_{n=1}^{+\infty}\frac{\beta^{2n}(h_i^{2n}-h_{i-1}^{2n})}{(2n+1)!}}{\sum\limits_{n=1}^{+\infty}\frac{(\beta h_i)^{2n}}{(2n)!}\sinh(\beta h_{i-1})}\right|.
\end{aligned}
\tag{7.8}
$$

The identity (7.4) yields

$$
\begin{aligned}
&\frac{\beta^{2n}(h_i^{2n}-h_{i-1}^{2n})}{(2n+1)!}=\\
&\frac{\beta^{2n}(h_i^2-h_{i-1}^2)(h_i^{2(n-1)}+h_i^{2(n-2)}h_{i-1}^2+\dots+h_i^2h_{i-1}^{2(n-2)}+h_{i-1}^{2(n-1)})}{(2n+1)!}\\
&\leqslant\frac{\beta^{2n}(h_i^2-h_{i-1}^2)nh_i^{2(n-1)}}{(2n+1)!}<\frac{\beta^{2n}(h_i^2-h_{i-1}^2)h_i^{2(n-1)}}{(2n)!},\ \forall n\in\mathbb{N}.
\end{aligned}
\tag{7.9}
$$

If we use the last expression from (7.9) into (7.8), together with (4.1), we get

$$
\begin{aligned}
&\gamma\left|r_i'\frac{\beta h_{i-1}h_i\sum\limits_{n=1}^{+\infty}\frac{\beta^{2n}(h_i^{2n}-h_{i-1}^{2n})}{(2n+1)!}}{\sum\limits_{n=1}^{+\infty}\frac{(\beta h_i)^{2n}}{(2n)!}\sinh(\beta h_{i-1})}\right|\leqslant\gamma\left|r_i'\frac{\beta h_{i-1}h_i\sum\limits_{n=1}^{+\infty}\frac{\beta^{2n}(h_i^2-h_{i-1}^2)h_i^{2(n-1)}}{(2n)!}}{\sum\limits_{n=1}^{+\infty}\frac{(\beta h_i)^{2n}}{(2n)!}\sinh(\beta h_{i-1})}\right|\\
&\leqslant 2\gamma\left|r_i'\frac{\beta h_{i-1}}{\sinh(\beta h_{i-1})}\cdot h_i^2(h_i-h_{i-1})\cdot\frac{\sum\limits_{n=1}^{+\infty}\frac{\beta^{2n}h_i^{2(n-1)}}{(2n)!}}{\sum\limits_{n=1}^{+\infty}\frac{(\beta h_i)^{2n}}{(2n)!}}\right|\\
&\leqslant C(h_i-h_{i-1}).
\end{aligned}
\tag{7.10}
$$

For the terms from (7.7) with second derivatives we have

$$\left|\frac{\frac{r''(\mu_i^+)}{2}\sum\limits_{n=0}^{+\infty}\frac{(\beta h_{i-1})^{2n+1}}{(2n+1)!}h_i^2+\frac{r''(\mu_i^-)}{2}\sum\limits_{n=0}^{+\infty}\frac{(\beta h_i)^{2n+1}}{(2n+1)!}h_{i-1}^2}{(\cosh(\beta h_i)-1)\sinh(\beta h_{i-1})}\right| \tag{7.11}$$

$$\leqslant\left|\frac{\frac{r''(\mu_i^+)}{2}\sum\limits_{n=0}^{+\infty}\frac{(\beta h_{i-1})^{2n+1}}{(2n+1)!}h_i^2}{(\cosh(\beta h_i)-1)\sinh(\beta h_{i-1})}\right|+\left|\frac{\frac{r''(\mu_i^-)}{2}\sum\limits_{n=0}^{+\infty}\frac{(\beta h_i)^{2n+1}}{(2n+1)!}h_{i-1}^2}{(\cosh(\beta h_i)-1)\sinh(\beta h_{i-1})}\right|.$$

Again, considering (4.1), for the first summand from the last expression from (7.11), we have that

$$\left|\frac{\frac{r''(\mu_i^+)}{2}\sum\limits_{n=0}^{+\infty}\frac{(\beta h_{i-1})^{2n+1}}{(2n+1)!}h_i^2}{(\cosh(\beta h_i)-1)\sinh(\beta h_{i-1})}\right|=\left|\frac{\frac{r''(\mu_i^+)}{2}\sinh(\beta h_{i-1})h_i^2}{(\cosh(\beta h_i)-1)\sinh(\beta h_{i-1})}\right|=\left|\frac{\frac{r''(\mu_i^+)h_i^2}{2}}{\cosh(\beta h_i)-1}\right| \tag{7.12}$$

$$\leqslant\left|\frac{r''(\mu_i^+)h_i^2}{\beta^2h_i^2}\right|\leqslant C\varepsilon^2,$$

while the second summand can be estimated using

$$\left|\frac{\frac{r''(\mu_i^-)}{2}\sum\limits_{n=0}^{+\infty}\frac{(\beta h_i)^{2n+1}}{(2n+1)!}h_{i-1}^2}{(\cosh(\beta h_i)-1)\sinh(\beta h_{i-1})}\right|=\left|\frac{r''(\mu_i^-)}{2}\cdot\frac{\beta h_i\sum\limits_{n=0}^{+\infty}\frac{(\beta h_i)^{2n}}{(2n+1)!}h_{i-1}^2}{\left(\sum\limits_{n=0}^{+\infty}\frac{(\beta h_i)^{2n}}{(2n)!}-1\right)\sinh(\beta h_{i-1})}\right|$$

$$\leqslant\left|\frac{r''(\mu_i^-)}{2}\cdot\frac{\beta h_ih_{i-1}^2}{\sum\limits_{n=1}^{+\infty}\frac{(\beta h_i)^{2n}}{(2n)!}\sinh(\beta h_{i-1})}\right|+\left|\frac{r''(\mu_i^-)}{2}\cdot\frac{\beta h_{i-1}^2h_i\sum\limits_{n=1}^{+\infty}\frac{(\beta h_i)^{2n}}{(2n+1)!}}{\sum\limits_{n=1}^{+\infty}\frac{(\beta h_i)^{2n}}{(2n)!}\sinh(\beta h_{i-1})}\right| \tag{7.13}$$

$$\leqslant\left|r''(\mu_i^-)\cdot\frac{\beta h_ih_{i-1}^2}{(\beta h_i)^2\cdot\beta h_{i-1}}\right|+\left|\frac{r''(\mu_i^-)}{2}\cdot\frac{\beta h_{i-1}^2h_i}{\beta h_{i-1}}\cdot\frac{\sum\limits_{n=1}^{+\infty}\frac{(\beta h_i)^{2n}}{(2n+1)!}}{\sum\limits_{n=1}^{+\infty}\frac{(\beta h_i)^{2n}}{(2n)!}}\right|$$

$$\leqslant C(\varepsilon^2+h_{i-1}h_i).$$

For the layer component $s$, first we have that

$$
\begin{aligned}
&\frac{1}{\frac{\cosh(\beta h_{i-1})-1}{\gamma\sinh(\beta h_{i-1})}+\frac{\cosh(\beta h_i)-1}{\gamma\sinh(\beta h_i)}}\left|\frac{s_{i-1}-s_i}{\sinh(\beta h_{i-1})}-\frac{s_i-s_{i+1}}{\sinh(\beta h_i)}\right| \\
&\leqslant \gamma\left|\frac{\sinh(\beta h_i)(s_{i-1}-s_i)-\sinh(\beta h_{i-1})(s_i-s_{i+1})}{(\cosh(\beta h_i)-1)\sinh(\beta h_{i-1})}\right| \\
&=\gamma\left|\frac{\sum\limits_{n=0}^{+\infty}\frac{(\beta h_i)^{2n+1}}{(2n+1)!}(s_{i-1}-s_i)-\sum\limits_{n=0}^{+\infty}\frac{(\beta h_{i-1})^{2n+1}}{(2n+1)!}(s_i-s_{i+1})}{\sum\limits_{n=1}^{+\infty}\frac{(\beta h_i)^{2n}}{(2n)!}\sinh(\beta h_{i-1})}\right| \\
&\leqslant \gamma\left|\frac{\beta h_i(s_{i-1}-s_i)-\beta h_{i-1}(s_i-s_{i+1})}{\sum\limits_{n=1}^{+\infty}\frac{(\beta h_i)^{2n}}{(2n)!}\sinh(\beta h_{i-1})}\right| \\
&\quad+\gamma\left|\frac{\beta h_i\sum\limits_{n=1}^{+\infty}\frac{(\beta h_i)^{2n}}{(2n+1)!}(s_{i-1}-s_i)-\beta h_{i-1}\sum\limits_{n=1}^{+\infty}\frac{(\beta h_{i-1})^{2n}}{(2n+1)!}(s_i-s_{i+1})}{\sum\limits_{n=1}^{+\infty}\frac{(\beta h_i)^{2n}}{(2n)!}\sinh(\beta h_{i-1})}\right|.
\end{aligned}
\tag{7.14}
$$

The first summand can be bounded with

$$
\begin{aligned}
&\gamma\left|\frac{\beta h_i(s_{i-1}-s_i)-\beta h_{i-1}(s_i-s_{i+1})}{\sum\limits_{n=1}^{+\infty}\frac{(\beta h_i)^{2n}}{(2n)!}\sinh(\beta h_{i-1})}\right| \\
&\leqslant \gamma\left|\frac{\beta h_i\left[-s_i'h_{i-1}+\frac{s''(\mu_i^-)}{2}h_{i-1}^2\right]-\beta h_{i-1}\left[-\left(s_i'h_i+\frac{s''(\mu_i^+)}{2}h_i^2\right)\right]}{\frac{\beta^2h_i^2}{2}\cdot\beta h_{i-1}}\right| \\
&\leqslant \varepsilon^2\left|s''(\mu_i^-)\frac{h_{i-1}}{h_i}+s''(\mu_i^+)\right|\leqslant C\varepsilon^2\left(\frac{e^{-\frac{\mu_i^-}{\varepsilon}\sqrt{m}}}{\varepsilon^2}\cdot\frac{h_{i-1}}{h_i}+\frac{e^{-\frac{\mu_i^+}{\varepsilon}\sqrt{m}}}{\varepsilon^2}\right)\leqslant\frac{C}{N^2}.
\end{aligned}
\tag{7.15}
$$

For the second summand in (7.14) we get

$$\gamma\left|\frac{\beta h_i\sum\limits_{n=1}^{+\infty}\frac{(\beta h_i)^{2n}}{(2n+1)!}(s_{i-1}-s_i)-\beta h_{i-1}\sum\limits_{n=1}^{+\infty}\frac{(\beta h_{i-1})^{2n}}{(2n+1)!}(s_i-s_{i+1})}{\sum\limits_{n=1}^{+\infty}\frac{(\beta h_i)^{2n}}{(2n)!}\sinh(\beta h_{i-1})}\right|$$
$$\leqslant\gamma\frac{\beta h_i}{\sinh(\beta h_{i-1})}\cdot\frac{\sum\limits_{n=1}^{+\infty}\frac{(\beta h_i)^{2n}}{(2n+1)!}}{\sum\limits_{n=1}^{+\infty}\frac{(\beta h_i)^{2n}}{(2n)!}}|s_{i-1}-s_i|+\gamma\frac{\beta h_{i-1}}{\beta h_{i-1}}\cdot\frac{\sum\limits_{n=1}^{+\infty}\frac{(\beta h_{i-1})^{2n}}{(2n+1)!}}{\sum\limits_{n=1}^{+\infty}\frac{(\beta h_i)^{2n}}{(2n)!}}|s_i-s_{i+1}|$$

and

$$\gamma\frac{\sum\limits_{n=1}^{+\infty}\frac{(\beta h_{i-1})^{2n}}{(2n+1)!}}{\sum\limits_{n=1}^{+\infty}\frac{(\beta h_i)^{2n}}{(2n)!}}|s_i-s_{i+1}|\leqslant\frac{C}{N^2}. \tag{7.16}$$

In the expression

$$\gamma\frac{\beta h_i}{\sinh(\beta h_{i-1})}\cdot\frac{\sum\limits_{n=1}^{+\infty}\frac{(\beta h_i)^{2n}}{(2n+1)!}}{\sum\limits_{n=1}^{+\infty}\frac{(\beta h_i)^{2n}}{(2n)!}}|s_{i-1}-s_i|, \tag{7.17}$$

there is a ratio $\frac{\beta h_i}{\sinh(\beta h_{i-1})}$. Though inequality $\frac{\beta h_i}{\sinh(\beta h_{i-1})}\leqslant\frac{h_i}{h_{i-1}}$ holds true, the quotient $\frac{h_i}{h_{i-1}}$ is not bounded for $x_i=\lambda$ and $\varepsilon\to 0$. This is why we are going to estimate the expression (7.17) separately on the transition part and on the nonequidistant part of the mesh.

In the case $i=N/4$, we can write

$$\gamma\left|\frac{\beta h_i\sum\limits_{n=1}^{+\infty}\dfrac{(\beta h_i)^{2n}}{(2n+1)!}(s_{i-1}-s_i)}{\sum\limits_{n=1}^{+\infty}\dfrac{(\beta h_i)^{2n}}{(2n)!}\sinh(\beta h_{i-1})}\right|=\gamma\left|\frac{\sinh(\beta h_i)-\beta h_i}{\cosh(\beta h_i)-1}\cdot\frac{s_{i-1}-s_i}{\sinh(\beta h_{i-1})}\right|,$$

since $\sum_{n=1}^{+\infty} \frac{x^{2n}}{(2n)!} = \cosh x - 1$ and $\sum_{n=1}^{+\infty} \frac{x^{2n+1}}{(2n+1)!} = \sinh x - x$, $\forall x \in \mathbb{R}$. The function $r(x) = \frac{\sinh x - x}{\cosh x - 1}$ takes values from the interval $(0, 1)$ when $x > 0$. Thus

$$(7.18)\quad \gamma \left| \frac{\beta h_i \sum\limits_{n=1}^{+\infty} \frac{(\beta h_i)^{2n}}{(2n+1)!}(s_{i-1} - s_i)}{\sum\limits_{n=1}^{+\infty} \frac{(\beta h_i)^{2n}}{(2n)!} \sinh(\beta h_{i-1})} \right| = \gamma \left| \frac{\sinh(\beta h_i) - \beta h_i}{\cosh(\beta h_i) - 1} \cdot \frac{s_{i-1} - s_i}{\sinh(\beta h_{i-1})} \right|$$

$$\leqslant \gamma \frac{|s_{i-1} - s_i|}{\sinh(\beta h_{i-1})} \leqslant C \frac{\frac{\ln N}{N^3}}{\frac{\ln N}{N}} = \frac{C}{N^2}.$$

When $i = N/4 + 1, \ldots, N/2 - 1$, we can use $\frac{\sum_{n=1}^{+\infty} \frac{x^{2n}}{(2n+1)!}}{\sum_{n=1}^{+\infty} \frac{x^{2n}}{(2n)!}} = \frac{\sinh x - x}{x(\cosh x - 1)} = p(x)$ and $0 < p(x) < \frac{1}{3}$ for $x > 0$. Therefore

$$(7.19)\quad \gamma \left| \frac{\beta h_i \sum\limits_{n=1}^{+\infty} \frac{(\beta h_i)^{2n}}{(2n+1)!}(s_{i-1} - s_i)}{\sum\limits_{n=1}^{+\infty} \frac{(\beta h_i)^{2n}}{(2n)!} \sinh(\beta h_{i-1})} \right| \leqslant \gamma \frac{\beta h_i}{\beta h_{i-1}} \cdot \frac{\sum\limits_{n=1}^{+\infty} \frac{(\beta h_i)^{2n}}{(2n+1)!}}{\sum\limits_{n=1}^{+\infty} \frac{(\beta h_i)^{2n}}{(2n)!}} |s_{i-1} - s_i| \leqslant \frac{C}{N^2}.$$

Using (3.1), (7.10), (7.12), (7.13), (7.15), (7.16), (7.18) and (7.19) completes the proof of the lemma. ☐


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Department of Mathematics
Faculty of sciences, University of Tuzla
Univerzitetska 4, 75000, Bosnia and Herzegovina
*E-mail address*: samir.karasuljic@untz.ba

Department of Mathematics
Faculty of sciences, University of Tuzla
Univerzitetska 4, 75000, Bosnia and Herzegovina
*E-mail address*: enes.duvnjakovic@untz.ba

Department of Mathematics and Informatics
Faculty of sciences, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Serbia
*E-mail address*: helena.zarin@dmi.uns.ac.rs