CURVATURE PROPERTIES OF RIEMANNIAN MANIFOLDS WITH CIRCULANT STRUCTURES

DIMITAR RAZPOPOV AND GEORGI DZHELEPOV

ABSTRACT. We study a Riemannian manifold $M$ equipped with a circulant structure $Q$, which is an isometry with respect to the metric. We consider two types of such manifolds: a 3-dimensional manifold $M$ where the third power of $Q$ is the identity, and a 4-dimensional manifold $M$ where the fourth power of $Q$ is the identity. In a single tangent space of $M$ we have a special tetrahedron constructed by vectors of a $Q$-basis. The aim of the present paper is to find relations among the sectional curvatures of the 2-planes associated with the four faces of this tetrahedron and its cross sections passing through the medians and the edges of these faces.

1. INTRODUCTION

The study of pseudo-Riemannian manifolds with additional structures in differential geometry is of great interest to many mathematicians. Substantial results are associated with the sectional curvatures of some characteristic 2-planes, determined in every tangent space of the manifold (for instance [1], [5], [7], [8], [9], [11]).

In the present paper, we continue our research on the manifolds with additional structures, introduced in [6] and [10]. We consider a Riemannian manifold $M$ equipped with a circulant structure $Q$, which is an isometry
with respect to the metric $g$. We study two classes of such manifolds determined by special properties of the curvature tensor. We find expressions of the curvatures of special 2-planes formed by vectors in a tangent space $T_pM$, $p \in M$.

First, we consider a 3-dimensional manifold $(M, g, Q)$ where the third power of $Q$ is the identity. In a single tangent space of $(M, g, Q)$ we have a special tetrahedron constructed by vectors of a $Q$-basis of $T_pM$. We find a relation among the sectional curvatures generated by the four faces of the tetrahedron and its cross sections passing through the medians and the edges of these faces. Farther, we consider a 4-dimensional manifold $(M, g, Q)$ where the fourth power of $Q$ is the identity. We find a relation among the sectional curvatures of the faces and some cross sections of a tetrahedron constructed by vectors of a $Q$-basis of $T_pM$. Let us note that the obtained results for $(M, g, Q)$ in the case when $n = 4$ are not similar to the results at $n = 3$.

2. Preliminaries

We consider a $n$-dimensional Riemannian manifold $M$ with a metric $g$, equipped with an endomorphism $Q$ in $T_pM$, such that $Q^n = \text{id}$, $Q \neq \pm \text{id}$. Moreover, we suppose that $Q$ is a circulant structure, i.e. the matrix of the components of $Q$ in some basis is circulant. We assume that $g$ is positive definite metric and $Q$ is compatible with $g$ such that

$$g(Qx, Qy) = g(x, y).$$

Here and anywhere in this work $x, y, z, u$ will stand for arbitrary elements of the algebra of the smooth vector fields on $M$ or vectors in $T_pM$.

We denote by $(M, g, Q)$ the manifold $M$ equipped with the metric $g$ and the structure $Q$.

Let $\nabla$ be the Riemannian connection of the metric $g$ on $(M, g, Q)$. The curvature tensor $R$ of $\nabla$ is determined by

$$R(x, y)z = \nabla_x \nabla_y z - \nabla_y \nabla_x z - [x, y]z.$$  

The tensor of type $(0, 4)$ associated with $R$ is defined by the identity

$$R(x, y, z, u) = g(R(x, y)z, u).$$
We say that a manifold \((M, g, Q)\) is in class \(\mathcal{L}_0\) if the structure \(Q\) is parallel with respect to \(g\), i.e., \(\nabla Q = 0\).

We say that a manifold \((M, g, Q)\) is in class \(\mathcal{L}_1\) if
\[
(2.2) \quad R(x, y, Qz, Qu) = R(x, y, z, u).
\]

We say that a manifold \((M, g, Q)\) is in class \(\mathcal{L}_2\) if
\[
(2.3) \quad R(Qx, Qy, Qz, Qu) = R(x, y, z, u).
\]

In [3] and [10] it is shown that \(\mathcal{L}_0 \subset \mathcal{L}_1 \subset \mathcal{L}_2\) are valid for cases \(n = 3\) and \(n = 4\) respectively.

Further, we will find the sectional curvatures of special 2-planes of \(T_pM\) when \((M, g, Q)\) is a 3-dimensional manifold and also when \((M, g, Q)\) is a 4-dimensional manifold. For this purpose, bearing in mind the well-known linear properties of the curvature tensor \(R\), we obtain the following identity
\[
(2.4) \quad R(Qx - x, Qx - Q^2x, Qx - x, Qx - Q^2x) =
\]
\[
R(Qx, Q^2x, Qx, Q^2x) + R(x, Qx, x, Qx) + R(x, Q^2x, x, Q^2x) + 2R(x, Qx, Qx, Q^2x) - 2R(x, Qx, x, Q^2x) - 2R(x, Qx, x, Q^2x).
\]

If \(\{x, y\}\) is a non-degenerate 2-plane spanned by vectors \(x, y \in T_pM\), then its sectional curvature is
\[
(2.5) \quad \mu(x, y) = \frac{R(x, y, x, y)}{g(x, x)g(y, y) - g^2(x, y)}.
\]

3. Curvature properties of a 3-dimensional \((M, g, Q)\)

First, we recall facts from [4] and [6], which are necessary for our consideration.

Let \((M, g, Q)\) be a 3-dimensional Riemannian manifold and let the local coordinates of \(Q\) be given by the circulant matrix
\[
(Q_i^j) = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}
\]

Hence \(Q\) satisfies
\[
(3.1) \quad Q^3 = \text{id}.
\]
We suppose that $g$ is a positive definite metric and the property (2.1) holds.

A basis of type $\{x, Qx, Q^2x\}$ of $T_pM$ is called a Q-basis. In this case we say that the vector $x$ induces a Q-basis of $T_pM$.

The angles between the vectors of a Q-basis are

\begin{equation}
\angle(x, Qx) = \angle(Qx, Q^2x) = \angle(x, Q^2x) = \varphi,
\end{equation}

where $\varphi \in (0, \frac{2\pi}{3})$. Evidently, an orthogonal Q-basis exists ([6]).

**Theorem 3.1.** [4] Let $(M, g, Q)$ satisfy (2.3). If a vector $x$ induces a Q-basis, then for the sectional curvatures of the basic 2-planes we have

$$\mu(x, Qx) = \mu(x, Q^2x) = \mu(Qx, Q^2x).$$

Due to Theorem 3.1, $(M, g, Q) \in \mathcal{L}_2$ has only one basic sectional curvature $\mu(x, Qx)$. It depends only on $\varphi = \angle(x, Qx)$ and we denote it by $\mu(\varphi)$ (see [4]).

Further in this section, we consider a tetrahedron whose faces are constructed by the 2-planes $\{x, Qx\}$, $\{Qx, Q^2x\}$ and $\{x, Q^2x\}$. The base of the tetrahedron is constructed by the 2-plane $\alpha = \{Qx - x, Qx - Q^2x\}$.

Without loss of generality we suppose that $g(x, x) = 1$. Hence, using (2.1) and (3.2), we calculate

\begin{equation}
\begin{align*}
g(x, Qx) &= \cos \varphi, \\
g(Qx - x, Qx - Q^2x) &= 1 - \cos \varphi, \\
g(Qx - x, Qx - x) &= g(Qx - Q^2x, Qx - Q^2x) = 2 - 2\cos \varphi,
\end{align*}
\end{equation}

which implies that the base of the terahedron is an equilateral triangle.

In the next theorem, we obtain an expression of the sectional curvature of $\alpha$ by the sectional curvature of $\{x, Qx\}$ and by the sectional curvature of $\beta = \{Q^2x, Qx + x\}$. The 2-plane $\beta$ determines a cross section of the tetrahedron.

**Theorem 3.2.** Let $(M, g, Q)$ belong to $\mathcal{L}_2$. Then the curvature of the 2-plane $\alpha = \{Qx - x, Qx - Q^2x\}$ is

\begin{equation}
\mu(\alpha) = \frac{3(1 + \cos \varphi)}{1 - \cos \varphi} \mu(\varphi) - \frac{2(1 + 2\cos \varphi)}{1 - \cos \varphi} \mu(\beta),
\end{equation}

where $\varphi = \angle(x, Qx)$, $\beta = \{Q^2x, Qx + x\}$.

**Proof.** The conditions (2.3) and (3.1) imply ([3]):

\begin{equation}
R_1 = R(x, Qx, x, Qx) = R(x, Q^2x, x, Q^2x) = R(Qx, Q^2x, Qx, Q^2x),
\end{equation}
\[ R_2 = R(x, Qx, x, Q^2x) = R(x, Q^2x, Qx, Q^2x) = R(x, Qx, Q^2x, Qx). \]

Then from (2.4) we get

\[ R(Qx - x, Qx - Q^2x, Qx - x, Qx - Q^2x) = 3R_1 - 6R_2. \]

On the other hand, taking into account (3.5) and (3.6), we calculate

\[ R(Q^2x, Qx + x, Q^2x, Qx + x) = 2(R_1 + R_2). \]

Together with (3.7), (3.8) yields

\[ R(Qx - x, Qx - Q^2x, Qx - x, Qx - Q^2x) = 9R_1 - 3R(Q^2x, Qx + x, Q^2x, Qx + x). \]

Now, using (2.1) and (3.2), we find

\[ g(Qx + x, Qx + x) = 2 + 2\cos\varphi, \quad g(Q^2x, Qx + x) = 2\cos\varphi. \]

We apply equalities (3.3), (3.9) and (3.10) in (2.5), and obtain (3.4). \qed

In our previous work, we obtain the following relation among the sectional curvatures of 2-planes of the type \( \{x, Qx\} \), whose basis vectors \( x \) and \( Qx \) determine angles \( \varphi, \frac{\pi}{2} \) and \( \frac{\pi}{3} \), respectively.

**Theorem 3.3.** [4] Let \((M, g, Q)\) satisfy (2.3). If a vector \( x \) induces a \( Q \)-basis, then

\[ \mu(\varphi) = \frac{1 - 2\cos\varphi}{1 + \cos\varphi} \mu\left(\frac{\pi}{2}\right) + \frac{3\cos\varphi}{1 + \cos\varphi} \mu\left(\frac{\pi}{3}\right), \]

where \( \varphi = \angle(x, Qx) \).

From Theorem 3.2 and Theorem 3.3 we establish the following

**Proposition 3.1.** Let \((M, g, Q)\) belong to \( \mathcal{L}_2 \). Then the curvatures of the 2-planes \( \alpha = \{Qx - x, Qx - Q^2x\} \) and \( \beta = \{Q^2x, x + Qx\} \) satisfy

\[ \mu(\alpha) = \frac{3}{1 - \cos\varphi} \left((1 - 2\cos\varphi)\mu\left(\frac{\pi}{2}\right) + 3\cos\varphi\mu\left(\frac{\pi}{3}\right)\right) - \frac{2(1 + 2\cos\varphi)}{1 - \cos\varphi} \mu(\beta). \]

**Corollary 3.1.** If \((M, g, Q)\) belongs to \( \mathcal{L}_2 \) and \( \varphi = \frac{\pi}{2} \), then

\[ \mu(\alpha) = 3\mu\left(\frac{\pi}{2}\right) - 2\mu(\beta). \]

Further, for a manifold \((M, g, Q) \in \mathcal{L}_1\) we find an expression of \( \mu(\alpha) \) by \( \mu(\varphi) \). Also we get \( \mu(\beta) \).
Theorem 3.4. Let \((M, g, Q)\) belong to \(\mathcal{L}_1\). Then the curvatures of the 2-planes 
\[
\alpha = \{Qx - x, Qx - Q^2x\} \quad \text{and} \quad \beta = \{Q^2x, x + Qx\}
\]
are
\[
\mu(\alpha) = 3 \cot^2 \frac{\varphi}{2} \mu(\varphi), \quad \mu(\beta) = 0,
\]
where \(\varphi = \angle(x, Qx)\).

Proof. From (2.2), (3.5) and (3.6) we get \(R_1 = -R_2\). Thus, equalities (3.7) and (3.8) become
\[
R(Qx - x, Qx - Q^2x, Qx - x, Qx - Q^2x) = 9R_1, \\
R(Q^2x, Qx + x, Q^2x, Qx + x) = 0.
\]
(3.12)
Now, applying (3.3), (3.10) and (3.12) in (2.5), we obtain (3.11). \(\square\)

Corollary 3.2. Let \((M, g, Q)\) belong to \(\mathcal{L}_1\). Then

i) the inequality \(\mu(\alpha) > \mu(\varphi)\) holds;

ii) \(\mu(\alpha) = 3\mu(\pi/2)\).

Proof. i) The cotangent function is decreasing in the interval \((0, \pi)\). Therefore, bearing in mind the condition \(\varphi \in (0, \pi/3)\), we get \(\cot \frac{\varphi}{2} > \sqrt{3}\). Hence, because of the first equality of (3.11), we have that \(\mu(\alpha) > \mu(\varphi)\) for every \(\varphi \in (0, \pi/3)\).

ii) If we put \(\varphi = \pi/2\) into (3.11), then the proof follows. \(\square\)

4. Curvature properties of a 4-dimensional \((M, g, Q)\)

In the beginning of this section we recall some basic facts for a 4-dimensional \((M, g, Q)\), known from [2] and [10].

Let \((M, g, Q)\) be a 4-dimensional Riemannian manifold and let the local coordinates of \(Q\) be given by the circulant matrix
\[
(Q^i_j) = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{pmatrix}.
\]
Hence \(Q\) satisfies
\[
Q^4 = \text{id}, \quad Q^2 \neq \pm \text{id}.
\]
We assume that $g$ is a positive definite metric and the property (2.1) is valid.

A basis of type $\{x, Qx, Q^2x, Q^3x\}$ of $T_pM$ is called a $Q$-basis. In this case we say that the vector $x$ induces a $Q$-basis of $T_pM$. The angles between the vectors of a $Q$-basis are as follows

$$
\angle(x, Qx) = \angle(Qx, Q^2x) = \angle(x, Q^3x) = \angle(Q^2x, Q^3x) = \varphi,$$

$$
\angle(x, Q^2x) = \angle(Qx, Q^3x) = \theta,
$$

where $\varphi \in (0, \pi), \theta \in (0, \pi)$.

In [10], it is proved the inequality $3 - 4 \cos \varphi + \cos \theta < 0$ and the existence of an orthogonal $Q$-basis.

**Theorem 4.1.** [2] Let $(M, g, Q)$ belong to $L_2$. If a vector $x$ induces a $Q$-basis, then for the sectional curvatures of the basic 2-planes we have

$$
\mu(x, Qx) = \mu(Qx, Q^2x) = \mu(Q^2x, Q^3x) = \mu(Q^3x, x),
$$

$$
\mu(x, Q^2x) = \mu(Qx, Q^3x).
$$

Due to Theorem 4.1 there are only two basic sectional curvatures. They are $\mu(x, Qx)$ and $\mu(x, Q^2x)$. The sectional curvature $\mu(x, Qx)$ depends on $\varphi = \angle(x, Qx)$. We denote $\mu(\varphi) = \mu(x, Qx)$.

Let $x$ induce a $Q$-basis of $T_pM$. Then the vectors $x$, $Qx$ and $Q^2x$ determine a tetrahedron, whose faces are constructed by the 2-planes $\{x, Qx\}$, $\{Qx, Q^2x\}$ and $\{x, Q^2x\}$. The base of the tetrahedron is constructed by the 2-plane $\alpha = \{Qx - x, Qx - Q^2x\}$.

Without loss of generality we suppose $g(x, x) = 1$. Thus, it follows from (2.1) and (4.2) that:

$$
g(x, Qx) = \cos \varphi, \quad g(x, Q^2x) = \cos \theta,
$$

$$
g(Qx - x, Qx - Q^2x) = 1 - 2 \cos \varphi + \cos \theta,
$$

$$
g(Qx - x, Qx - x) = g(Qx - Q^2x, Qx - Q^2x) = 2 - 2 \cos \varphi.
$$

The latter equalities show that the base of the tetrahedron is an isosceles triangle. In the following theorems, we obtain an expression of the sectional curvature of $\alpha$ by the sectional curvatures of $\{x, Qx\}$ and $\{x, Q^2x\}$, and also by the sectional curvatures of $\gamma = \{Q^2x, x + Qx\}$, $\delta = \{x, Qx + Q^2x\}$ and $\sigma = \{Qx, x + Q^2x\}$. The 2-planes $\gamma$, $\delta$ and $\sigma$ determine cross sections of the tetrahedron.
Theorem 4.2. Let \((M, g, Q)\) belong to \(\mathcal{L}_2\). Then the curvature of the 2-plane \(\alpha = \{Qx - x, Qx - Q^2x\}\) is

\[
\mu(\alpha) = \frac{1}{(1 - \cos \theta)(3 - 4 \cos \varphi + \cos \theta)} \left( 6(1 - \cos^2 \varphi)\mu(\varphi) + 3(1 - \cos^2 \theta)\mu(\beta) - 2(1 + \cos \theta - 2 \cos^2 \varphi)\mu(\sigma) - (2 + 2 \cos \varphi - (\cos \varphi + \cos \theta)^2)(\mu(\gamma) + \mu(\delta)) \right),
\]

where \(\beta = \{x, Q^2x\}, \gamma = \{Q^2x, Qx + x\}, \delta = \{x, Qx + Q^2x\}, \sigma = \{Qx, x + Q^2x\}\).

Proof. We denote

\[
R_1 = R(x, Qx, x, Qx), \quad R_2 = R(x, Q^2x, x, Q^2x),
\]

\[
R_3 = R(x, Qx, Q^2x, x), \quad R_4 = R(x, Qx, Qx, Q^2x),
\]

\[
R_5 = R(x, Q^2x, Q^2x, x).
\]

Then, from (2.3), (2.4) and (4.1), we get

\[
R(Qx - x, Qx - Q^2x, Qx - x, Qx - Q^2x) = 2R_1 + 2R_3 + R_2 + 2R_4 + 2R_5,
\]

On the other hand, using (4.5), we calculate

\[
R(Q^2x, Qx + x, Q^2x, x) = R_1 + R_2 - 2R_5,
\]

\[
R(x, Qx + Q^2x, x, Qx + Q^2x) = R_1 + R_2 - 2R_3,
\]

\[
R(Qx, x + Q^2x, Qx, x + Q^2x) = 2R_1 - 2R_4.
\]

Applying (4.7) in (4.6) we find

\[
R(Qx - x, Qx - Q^2x, Qx - x, Qx - Q^2x) = 6R_1 + 3R_2
\]

\[
- R(Q^2x, Qx + x, Q^2x, x)
\]

\[
- R(x, Qx + Q^2x, x, Qx + Q^2x)
\]

\[
- R(Qx, x + Q^2x, Qx, x + Q^2x).
\]

From (2.1), (2.2) and (4.3) we have

\[
g(Qx + x, Qx + x) = g(Qx + Q^2x, Qx + Q^2x) = 2 + 2 \cos \varphi,
\]

\[
g(x, Qx + Q^2x) = g(Q^2x, Qx + x) = \cos \theta + \cos \varphi,
\]

\[
g(x + Q^2x, x + Q^2x) = 2 + 2 \cos \theta, \quad g(Qx, x + Q^2x) = 2 \cos \varphi.
\]

Therefore, (2.5), (4.3), (4.8) and (4.9) imply (4.4).
Corollary 4.1. Let \((M, g, Q)\) belong to \(L_2\). If \(\varphi = \theta\), then
\[
\mu(\alpha) = \frac{1}{3(1 - \cos \varphi)} \left( (1 + \cos \varphi)(6\mu(\varphi) + 3\mu(\beta)) - 2(1 + 2\cos \varphi)(\mu(\gamma) + \mu(\delta) + \mu(\sigma)) \right).
\]
In particular, if \(\varphi = \theta = \frac{\pi}{2}\), then
\[
\mu(\alpha) = \frac{1}{3} \left( 6\mu\left(\frac{\pi}{2}\right) + 3\mu(\beta) - 2\mu(\gamma) - 2\mu(\delta) - 2\mu(\sigma) \right).
\]

Now, for a manifold \((M, g, Q) \in L_1\) we find expressions of \(\mu(\alpha), \mu(\beta), \mu(\sigma), \mu(\gamma)\) and \(\mu(\delta)\) by \(\varphi, \theta\) and \(\mu(\varphi)\).

Theorem 4.3. Let \((M, g, Q)\) belong to \(L_1\). Then the curvatures of the 2-planes \(\alpha = \{Qx - x, Qx - Q^2x\}, \beta = \{x, Q^2x\}, \gamma = \{Q^2x, Qx + x\}, \delta = \{x, Qx + Q^2x\}\) and \(\sigma = \{Qx, x + Q^2x\}\) are
\[
\begin{align*}
\mu(\alpha) &= \frac{4(1 + \cos \varphi)}{3 - 4\cos \varphi + \cos \theta} \mu(\varphi), \quad \mu(\beta) = \mu(\sigma) = 0, \\
\mu(\gamma) &= \frac{1 - \cos^2 \varphi}{2 + 2\cos \varphi + (\cos \varphi - \cos \theta)^2} \mu(\varphi).
\end{align*}
\]
\[(4.10)\]

Proof. By using (2.2) and (4.5) we get that \(R_1 = R_4\) and \(R_2 = R_3 = R_5 = 0\). Thus, from (4.6) and (4.7), we have
\[
\begin{align*}
R(Qx - x, Qx - Q^2x, Qx - x, Qx - Q^2x) &= 4R_1, \\
R(Qx, x + Q^2x, Qx, x + Q^2x) &= 0, \\
R(x, Qx + Q^2x, x, Qx + Q^2x) &= R(Q^2x, Qx + x, Q^2x, Qx + x) = R_1.
\end{align*}
\]
We apply the latter equalities, (4.3) and (4.9) in (2.5) and obtain (4.10).
\[\square\]

Finally, due to Theorem 4.3, we state the following

Corollary 4.2. Let \((M, g, Q)\) belong to \(L_1\). If \(\varphi = \theta\), then
\[
\begin{align*}
\mu(\alpha) &= \frac{4}{3} \cot^2 \frac{\varphi}{2} \mu(\varphi), \quad \mu(\gamma) = \mu(\delta) = \frac{1 - \cos^2 \varphi}{2 + 2\cos \varphi} \mu(\varphi), \\
\mu(\beta) = \mu(\sigma) &= \frac{1}{2} \mu(\pi).
\end{align*}
\]
In particular, if \(\varphi = \theta = \frac{\pi}{2}\), then
\[
\begin{align*}
\mu(\alpha) &= \frac{4}{3} \mu\left(\frac{\pi}{2}\right), \quad \mu(\gamma) = \mu(\delta) = \frac{1}{2} \mu\left(\frac{\pi}{2}\right).
\end{align*}
\]
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DEPARTMENT OF MATHEMATICS AND INFORMATICS
AGRICULTURAL UNIVERSITY OF PLOVDIV
12 MENDELEEV BLVD, 4000 PLOVDIV, BULGARIA
E-mail address: razpopov@au-plovdiv.bg

DEPARTMENT OF MATHEMATICS AND INFORMATICS
AGRICULTURAL UNIVERSITY OF PLOVDIV
12 MENDELEEV BLVD, 4000 PLOVDIV, BULGARIA
E-mail address: dzhelepov@au-plovdiv.bg