SOME PROPERTIES OF FUNCTIONS CONCERNED WITH OZAKI AND NUNOKAWA RESULT

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ABSTRACT. For analytic functions $f(z)$ with $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$ and $f^{(n)}(0) \neq 0$ in the open unit disk $U$, a new class $\mathcal{A}_n(\lambda)$ concerned with Ozaki and Nunokawa result in 1972 is introduced. The object of the present paper is to discuss some properties of functions $f(z)$ in the class $\mathcal{A}_n(\lambda)$.

1. INTRODUCTION

Let $\mathcal{A}_n$ be the class of functions $f(z)$ of the form

\begin{equation}
    f(z) = z + \sum_{k=n}^{\infty} a_k z^k \quad (n = 2, 3, 4, \cdots)
\end{equation}

with $a_n \neq 0$ which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

For $f(z) \in \mathcal{A}_n$, we define the subclass $\mathcal{A}_n(\lambda)$ of $\mathcal{A}_n$ consisting of $f(z)$ which satisfy

\begin{equation}
    \left| \frac{z^2 f'(z)}{f(z)^2} - 1 \right| < \lambda \quad (z \in U)
\end{equation}

for some real $\lambda > 0$. We note that $f(z) \in \mathcal{A}_2(1)$ is univalent in $U$ by Ozaki and Nunokawa [5].

The class $\mathcal{A}_2(\lambda)$ was defined and discussed by Obradović and Ponnusamy [4]. The class $\mathcal{A}_3(\lambda)$ with $0 < \lambda < 1$ was considered by Singh [7].

Recently, Shimoda, Nakamura and Owa [6] discussed some radius problems for $f(z) \in \mathcal{A}_2(\lambda)$ with $a_3 - a_2^2 = 0$.

In order to discuss our problems for $f(z) \in \mathcal{A}_n(\lambda)$, we have to recall here the following lemmas.

**Lemma 1.1.** Let the function $w(z)$ defined by

\begin{equation}
    w(z) = b_n z^n + b_{n+1} z^{n+1} + b_{n+2} z^{n+2} + \cdots \quad (b_n \neq 0)
\end{equation}

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be analytic in \( \mathbb{U} \). If \(|w(z)|\) attains its maximum value at a point \( z_0 \in \mathbb{U} \) on the circle \(|z| = r\), then there exists a real number \( k \geq n \) such that

\[
(1.4) \quad z_0w'(z_0) = kw(z_0)
\]

and

\[
(1.5) \quad \text{Re} \left( \frac{z_0w''(z_0)}{w'(z_0)} \right) \geq k - 1.
\]

This lemma was given by Jack [2] (also, due to Miller and Mocanu [3]).

**Lemma 1.2.** Let the function \( p(z) \) given by

\[
(1.6) \quad p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots
\]

be analytic in \( \mathbb{U} \). If \( p(z) \) satisfies \( \text{Re}(p(z)) > 0 \) \( (z \in \mathbb{U}) \), then

\[
(1.7) \quad |c_n| \leq 2 \quad (n = 1, 2, 3, \cdots).
\]

The equality in (1.7) is attained for \( p(z) \) given by

\[
(1.8) \quad p(z) = \frac{1 + z}{1 - z}.
\]

This lemma is called as Carathéodory’s lemma [1].

2. Some properties for the class \( \mathcal{A}_n(\lambda) \)

First result of our problems is contained in:

**Theorem 2.1.** If \( f(z) \in \mathcal{A}_n(n - 2) \) with \( n \geq 3 \), then

\[
(2.1) \quad \left| \frac{z}{f(z)} - 1 \right| < 1 \quad (z \in \mathbb{U})
\]

and

\[
(2.2) \quad \text{Re} \left( \frac{f(z)}{z} \right) > \frac{1}{2} \quad (z \in \mathbb{U}).
\]

**Proof.** For \( \mathcal{A}_n(n - 2) \), we define the function \( w(z) \) by

\[
(2.3) \quad \left| \frac{z^2 f'(z)}{f(z)^2} - 1 \right| = |w(z) - zw'(z)| < n - 2 \quad (z \in \mathbb{U}).
\]

Then \( w(z) \) is analytic in \( \mathbb{U} \) and \( w(0) = 0 \).

Further, we see that

\[
(2.4) \quad \max_{|z| = r} |w(z)| = |w(z_0)| = 1.
\]

Then, Lemma 1.1 gives us that

\[
(2.5) \quad z_0w'(z_0) = kw(z_0) \quad (k \geq n - 1).
\]
It follows from (2.3), (2.4), and (2.5) that
\[
\left| \frac{z_0^2 f'(z_0)}{f(z_0)^2} - 1 \right| = \left| w(z_0) \left( 1 - \frac{z_0 w'(z_0)}{w(z_0)} \right) \right| = |k - 1| \geq n - 2.
\]
This contradicts that \( f(z) \in A_n(n - 2) \). This means that there is no \( z_0 \in \mathbb{U} \) such that \( |w(z_0)| = 1 \). Therefore, we conclude that \( |w(z)| < 1 \) (\( z \in \mathbb{U} \)), and prove that equation (2.1) Furthermore, it is easy to see that (2.1) gives equation (2.2), proving the theorem. \( \square \)

Next our result is:

**Theorem 2.2.** If \( f(z) \in A_n \left( \frac{n - 3}{4} \right) \) with \( n \geq 4 \), then

\[
|\frac{f(z)}{z} - 1| < 1 \quad (z \in \mathbb{U}).
\]

**Proof.** Let us define the function \( w(z) \) by
\[
w(z) = \frac{f(z)}{z} - 1,
\]
for \( f(z) \in A_n \left( \frac{n - 3}{4} \right) \). It follows that:

\[
|\frac{z^2 f'(z)}{f(z)^2} - 1| = \left| \frac{1}{w(z) + 1} \left( \frac{zw'(z)}{w(z) + 1} - w(z) \right) \right| < \frac{n - 3}{4} \quad (z \in \mathbb{U}).
\]

Since, \( w(z) \) is analytic in \( \mathbb{U} \) and \( w(0) = 0 \), we suppose that there exists a point \( z_0 \in \mathbb{U} \) such that
\[
\max_{|z| = r} |w(z)| = |w(z_0)| = 1.
\]
Then, we can write \( w(z_0) = e^{i\theta} \) and
\[
z_0 w'(z_0) = kw(z_0) \quad (k \geq n - 1).
\]
Therefore, we have that
\[
|\frac{z_0^2 f'(z_0)}{f(z_0)^2} - 1| = \left| \frac{w(z_0)}{w(z_0) + 1} \left( \frac{k}{w(z_0) + 1} - 1 \right) \right| \geq \frac{|k - 1 - w(z_0)|}{|w(z_0) + 1|^2} \geq \frac{n - 3}{4},
\]
which contradicts our condition (2.7). This shows that there is no \( z_0 \in \mathbb{U} \) such that \( |w(z_0)| = 1 \) for \( z_0 \in \mathbb{U} \). Thus we have that \( |w(z)| < 1 \) for all \( z \in \mathbb{U} \), proving (2.6). \( \square \)
3. COEFFICIENT INEQUALITIES FOR THE CLASS $A_2(\lambda)$

In this section we consider some coefficient inequalities for $f(z)$ in the class $A_2(\lambda)$.

**Theorem 3.1.** If $f(z) \in A_2(\lambda)$ with $0 < \lambda < 1$, then

$$(3.1) \quad |a_3 - a_2^2| \leq 2\lambda,$$

$$(3.2) \quad |a_4 - 2a_2 a_3 + a_2^3| \leq \lambda,$$

and

$$(3.3) \quad |a_5 - 2a_2 a_4 + 3a_2^2 a_3 - a_3^2 - a_2^4| \leq \frac{2}{3}\lambda.$$  

**Proof.** Let us define the function $p(z)$ by

$$p(z) = \frac{z^2 f'(z)}{f(z)^2} \cdot \left(1 - \frac{1}{\lambda}\right) \quad (z \in \mathbb{U}).$$

This gives us that $p(z)$ is analytic in $\mathbb{U}$, $\text{Re}p(z) > 0$ ($z \in \mathbb{U}$), and

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots.$$  

Noting that

$$\frac{z^2 f'(z)}{f(z)^2} \cdot 1 + \frac{1}{\lambda} = \frac{1}{\lambda} (a_3 - a_2^2) z^2 + 2(a_4 - 2a_2 a_3 + a_2^3) z^3 + 3(a_5 - 2a_2 a_4 + 3a_2^2 a_3 - a_3^2 - a_2^4) z^4 + \cdots,$$

we have that

$$a_3 - a_2^2 = \lambda c_2,$$

$$2(a_4 - 2a_2 a_3 + a_2^3) = \lambda c_3,$$

and

$$3(a_5 - 2a_2 a_4 + 3a_2^2 a_3 - a_3^2 - a_2^4) = \lambda c_4.$$  

Therefore, applying Lemma 1.2, we prove (3.1), (3.2) and (3.3). 

If $a_2 = 0$ in Theorem 2.2, then we have:

**Corollary 3.1.** If $f(z) \in A_3(\lambda)$ with $0 < \lambda < 1$, then

$$|a_3| \leq 2\lambda,$$

$$|a_4| \leq \lambda,$$

and

$$|a_5| \leq \frac{8}{3}\lambda.$$
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