NEW CLASS FOR CERTAIN ANALYTIC FUNCTIONS CONCERNED WITH THE STRIP DOMAINS

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ABSTRACT. By considering a certain univalent function in the open unit disk $U$ which maps $U$ onto the strip domain $W$ with $\alpha < w < \beta$ ($\alpha < 1$, $\beta > 1$), some new class for certain analytic functions is discussed.

1. INTRODUCTION

Let $A$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $U = \{ z \in \mathbb{C} : |z| < 1 \}$. The subclass of $A$ consisting of all univalent functions $f(z)$ in $U$ is denoted by $S$.

A function $f(z) \in A$ is said to be starlike of order $\alpha$ in $U$ if it satisfies

$$\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in U)$$

for some real number $\alpha$ with $0 \leq \alpha < 1$. This class is denoted by $S^*(\alpha)$ and $S^*(0) = S^*$. The class $S^*(\alpha)$ was introduced by Robertson [2]. It is well-known that $S^*(\alpha) \subset S^* \subset S$.

Furthermore, let $M(\beta)$ be the class of functions $f(z) \in A$ which satisfy

$$\text{Re} \left( \frac{zf'(z)}{f(z)} \right) < \beta \quad (z \in U)$$

for some real number $\beta$ with $\beta > 1$. The class $M(\beta)$ was investigated by Uralegaddi, Ganigi and Sarangi [4].

Let $p(z)$ and $q(z)$ be analytic in $U$. Then the function $p(z)$ is said to be subordinate to $q(z)$ in $U$, written by

$$(1.1) \quad p(z) \prec q(z) \quad (z \in U),$$

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if there exists a function \( w(z) \) which is analytic in \( \mathbb{U} \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) \( (z \in \mathbb{U}) \), and such that \( p(z) = q(w(z)) \) \( (z \in \mathbb{U}) \). From the definition of the subordinations, it is easy to show that the subordination (1.1) implies that

\[
p(0) = q(0) \quad \text{and} \quad p(\mathbb{U}) \subset q(\mathbb{U}).
\]

In particular, if \( q(z) \) is univalent in \( \mathbb{U} \), then the subordination (1.1) is equivalent to the condition (1.2).

**Remark 1.1.** Let \( p(z) \) and \( q(z) \) be analytic in \( \mathbb{U} \). Then the subordination (1.1) implies that

\[
|p'(0)| \leq |q'(0)|,
\]

and \( |p'(0)| = |q'(0)| \) if and only if \( p(z) = q(e^{i\theta}z) \) for some real number \( \theta \) with \( 0 \leq \theta < 2\pi \) (cf. [1]).

Motivated by the classes \( S^*(\alpha) \) and \( M(\beta) \), we define new class for certain analytic functions. Let \( S(\alpha, \beta) \) denote the class of functions \( f(z) \in \mathcal{A} \) which satisfy the inequality

\[
(1.3) \quad \alpha < \text{Re} \left( \frac{zf''(z)}{f(z)} \right) < \beta \quad (z \in \mathbb{U})
\]

for some real number \( \alpha \) \( (\alpha < 1) \) and some real number \( \beta \) \( (\beta > 1) \). Note that \( \alpha \) is not necessary to be positive in the class \( S(\alpha, \beta) \).

**Remark 1.2.** Let \( f(z) \in S(\alpha, \beta) \). If \( \alpha \geq 0 \), then \( f(z) \) is starlike in \( \mathbb{U} \), which implies that \( f(z) \) is univalent in \( \mathbb{U} \).

In order to discuss our new class \( S(\alpha, \beta) \), we need to consider a certain univalent function in \( \mathbb{U} \) which maps \( \mathbb{U} \) onto the strip domain \( w \) with \( \alpha < \text{Re} \ w < \beta \).

**Theorem 1.1.** Let \( \alpha \) and \( \beta \) be real numbers with \( \alpha < 1 \) and \( \beta > 1 \). Then the function \( S_{\alpha, \beta}(z) \) defined by

\[
(1.4) \quad S_{\alpha, \beta}(z) = 1 + \frac{\beta - \alpha}{\pi} \log \left( \frac{1 - e^{i\pi(1-\alpha)} z}{1 - e^{-i\pi(1-\alpha)} z} \right) \quad (z \in \mathbb{U})
\]

is analytic and univalent in \( \mathbb{U} \) with \( S_{\alpha, \beta}(0) = 1 \). In addition, \( S_{\alpha, \beta}(z) \) maps \( \mathbb{U} \) onto the strip domain \( w \) with \( \alpha < \text{Re} \ w < \beta \).

**Proof.** Note that the function \( S_{\alpha, \beta}(z) \) defined by (1.4) can be written as follows:

\[
(1.5) \quad S_{\alpha, \beta}(z) = \frac{\alpha + \beta}{2} + \frac{\beta - \alpha}{\pi} \log \left( \frac{ze^{-i\pi(1-\alpha)} + (-\bar{z})e^{i\pi(1-\alpha)} e^{-i\pi(1-\alpha)} z}{1 - e^{-i\pi(1-\alpha)} z} \right).
\]

From the equality (1.5), a simple check gives us that \( S_{\alpha, \beta}(z) \) maps \( \mathbb{U} \) onto the strip domain \( w \) with \( \alpha < \text{Re} \ w < \beta \). Moreover, it is easy to see that the function \( S_{\alpha, \beta}(z) \) is analytic and univalent in \( \mathbb{U} \) with \( S_{\alpha, \beta}(0) = 1 \).

**Remark 1.3.**

\[
S_{\alpha, \beta}(z) = 1 + \frac{\beta - \alpha}{\pi} \log \left( \frac{1 - e^{i\pi(1-\alpha)} z}{1 - e^{-i\pi(1-\alpha)} z} \right) = 1 + \sum_{n=1}^{\infty} B_n z^n,
\]
where

\[ B_n = \frac{2(\beta - \alpha)}{n\pi} \sin \frac{n\pi(1 - \alpha)}{\beta - \alpha} \quad (n = 1, 2, \cdots). \]

Specially, we note that the coefficient \( B_n \) defined by (1.6) is the real number. Since

\[
\lim_{\beta \to +\infty} B_n = \lim_{\beta \to +\infty} \left\{ 2(1 - \alpha) \frac{\sin \frac{n\pi(1 - \alpha)}{\beta - \alpha}}{\frac{n\pi(1 - \alpha)}{\beta - \alpha}} \right\} = 2(1 - \alpha),
\]

a simple check gives us that

\[
S_{\alpha, \beta}(z) = 1 + \sum_{n=1}^{\infty} 2(1 - \alpha)z^n = \frac{1 + (1 - 2\alpha)z}{1 - z} \quad (\beta \to +\infty),
\]

which implies that \( S_{\alpha, \beta}(z) \) \((\beta \to +\infty)\) maps \( \mathbb{U} \) onto the right half-plane \( w \) with \( \text{Re} \, w > \alpha \).

On the other hand, it is easy to see that

\[
\lim_{\alpha \to -\infty} B_n = \lim_{\alpha \to -\infty} \left\{ \frac{2(\beta - \alpha)}{n\pi} \sin \left( \frac{n\pi(1 - \beta)}{\beta - \alpha} + n\pi \right) \right\}
\]

\[
= \lim_{\alpha \to -\infty} \left\{ \frac{2(\beta - \alpha)}{n\pi} (-1)^n \sin \frac{n\pi(1 - \beta)}{\beta - \alpha} \right\}
\]

\[
= \lim_{\alpha \to -\infty} \left\{ 2(\beta - 1)(-1)^n \frac{\sin \frac{n\pi(1 - \beta)}{\beta - \alpha}}{\frac{n\pi(1 - \beta)}{\beta - \alpha}} \right\} = 2(\beta - 1)(-1)^{n-1}.
\]

Therefore, we find that

\[
S_{\alpha, \beta}(z) = 1 + \sum_{n=1}^{\infty} 2(\beta - 1)(-1)^{n-1}z^n = \frac{1 - (1 - 2\beta)z}{1 + z} \quad (\alpha \to -\infty),
\]

which implies that \( S_{\alpha, \beta}(z) \) \((\alpha \to -\infty)\) maps \( \mathbb{U} \) onto the left half-plane \( w \) with \( \text{Re} \, w < \beta \).

We give some example for \( f(z) \in S(\alpha, \beta) \) as follows.

**Example 1.1.** Let us consider the function \( f(z) \) given by

\[ f(z) = z \exp \left\{ \frac{\beta - \alpha}{\pi} i \int_0^z \frac{1}{t} \log \left( \frac{1 - e^{i \frac{n\pi(1 - \alpha)}{\beta - \alpha} t}}{1 - e^{-i \frac{n\pi(1 - \alpha)}{\beta - \alpha} t}} \right) \, dt \right\}
\]

\[ = z + \frac{2(\beta - \alpha)}{\pi} \frac{\pi(1 - \alpha)}{\beta - \alpha} z^2 + \cdots \quad (z \in \mathbb{U})
\]

with \( \alpha < 1 \) and \( \beta > 1 \). Then we have

\[
\frac{zf'(z)}{f(z)} = 1 + \frac{\beta - \alpha}{\pi} i \log \left( \frac{1 - e^{i \frac{n\pi(1 - \alpha)}{\beta - \alpha} z}}{1 - e^{-i \frac{n\pi(1 - \alpha)}{\beta - \alpha} z}} \right) = S_{\alpha, \beta}(z) \quad (z \in \mathbb{U}).
\]

According to Theorem 1.1, it is clear that the function \( f(z) \) given by (1.7) satisfies the inequality (1.3), which implies that \( f(z) \in S(\alpha, \beta) \).

Applying the function \( S_{\alpha, \beta}(z) \) defined by (1.4), we give a necessary and sufficient condition for \( f(z) \in A \) to belong to the class \( S(\alpha, \beta) \).
Lemma 1.1. Let \( f(z) \in \mathcal{A} \). Then \( f(z) \in S(\alpha, \beta) \) if and only if

\[
\frac{zf'(z)}{f(z)} < 1 + \frac{\beta - \alpha}{\pi} i \log \left( \frac{1 - e^{i \frac{\pi}{\beta-\alpha}} z}{1 - e^{-i \frac{\pi}{\beta-\alpha}} z} \right) = S_{\alpha, \beta}(z) \quad (z \in \mathbb{U}),
\]

where \( \alpha < 1 \) and \( \beta > 1 \).

By using the subordination (1.8), we discussed some properties for \( f(z) \in S(\alpha, \beta) \).

2. SOME RESULTS

Rogosinski [3] proved the coefficient estimates for subordinate functions.

Lemma 2.1. Let \( q(z) = \sum_{n=1}^{\infty} B_n z^n \) be analytic and univalent in \( \mathbb{U} \), and suppose that \( q(z) \) maps \( \mathbb{U} \) onto a convex domain. If \( p(z) = \sum_{n=1}^{\infty} A_n z^n \) is analytic in \( \mathbb{U} \) and satisfies the following subordination

\[
p(z) < q(z) \quad (z \in \mathbb{U}),
\]

\[
|A_n| \leq |B_1| \quad (n = 1, 2, \cdots).
\]

Applying Lemma 2.1, we deduced some coefficient estimate for \( f(z) \in S(\alpha, \beta) \) below.

Theorem 2.1. If the function \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S(\alpha, \beta) \), then

\[
|a_n| \leq \prod_{k=2}^{n} \frac{k - 2 + \frac{2(\beta - \alpha)}{\pi} \sin \frac{\pi}{\beta - \alpha}}{k - 1} \quad (n = 2, 3, \cdots).
\]

Proof. According to the assertion of Lemma 1.1 the function \( f(z) \) satisfies the subordination (1.8). Let us define the function \( p(z) \) by

\[
p(z) = \frac{zf'(z)}{f(z)} \quad (z \in \mathbb{U}).
\]

Then, the subordination (1.8) can be written as follows:

\[
p(z) < S_{\alpha, \beta}(z) \quad (z \in \mathbb{U}),
\]

where \( S_{\alpha, \beta}(z) \) is defined by (1.4). Note that the function \( S_{\alpha, \beta}(z) \) is convex in \( \mathbb{U} \), and has the following form

\[
S_{\alpha, \beta}(z) = 1 + \sum_{n=1}^{\infty} B_n z^n,
\]

where \( B_n \) is defined by (1.6). If we let

\[
p(z) = 1 + \sum_{n=1}^{\infty} A_n z^n,
\]

then by Lemma 2.1, we see that the subordination (2.2) implies that

\[
|A_n| \leq B_1 \quad (n = 1, 2, \cdots),
\]
where

\begin{equation}
B_1 = \frac{2(\beta - \alpha)}{\pi} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha}.
\end{equation}

Now, the equality (2.1) implies that

\[zf'(z) = p(z)f(z).\]

Then, the coefficients of \( z^n \) in both sides lead to

\begin{equation}
(n - 1)a_n = A_{n-1} + A_{n-2}a_2 + \cdots + A_1a_{n-1} \quad (n = 2, 3, \cdots).
\end{equation}

A simple calculation combined with the inequality (2.3) yields that

\[|a_n| = \frac{1}{n - 1} |A_{n-1} + A_{n-2}a_2 + \cdots + A_1a_{n-1}| \]

\[\leq \frac{1}{n - 1} (|A_{n-1}| + |A_{n-2}||a_2| + \cdots + |A_1||a_{n-1}|)\]

\[\leq \frac{B_1}{n - 1} \sum_{k=2}^{n} |a_{k-1}| \quad (|a_1| = 1), \]

where \( B_1 \) is given in (2.4). To prove the assertion of the theorem, we need show that

\begin{equation}
X_n = \frac{B_1}{n - 1} \sum_{k=2}^{n} |a_{k-1}| \leq \prod_{k=2}^{n} \frac{k - 2 + B_1}{k - 1} \quad (n = 2, 3, \cdots).
\end{equation}

We now use the mathematical induction for the proof of the theorem. Since

\[X_2 = B_1|a_1| = B_1 \quad (|a_1| = 1),\]

it is clear that the assertion is holds true for \( n = 2 \).

We assume that the proposition is true for \( n = m \). Then, some calculation gives us that

\[X_{m+1} = \frac{B_1}{(m + 1) - 1} \sum_{k=2}^{m+1} |a_{k-1}| = \frac{B_1}{m} \left( \sum_{k=2}^{m} |a_{k-1}| + |a_m| \right) \]

\[\leq \frac{B_1}{m} \left( 1 + \frac{B_1}{m - 1} \right) \sum_{k=2}^{m} |a_{k-1}| = \frac{m - 1 + B_1}{m} \frac{B_1}{m - 1} \sum_{k=2}^{m} |a_{k-1}| \]

\[\leq \frac{m - 1 + B_1}{m} \prod_{k=2}^{m} \frac{k - 2 + B_1}{k - 1} = \prod_{k=2}^{m+1} \frac{k - 2 + B_1}{k - 1}, \]

which implies that the inequality (2.6) is true for \( n = m + 1 \).

By the mathematical induction, we prove that

\[|a_n| \leq \prod_{k=2}^{n} \frac{k - 2 + B_1}{k - 1} \quad (n = 2, 3, \cdots),\]

where \( B_1 \) is given in (2.4). This completes the proof of Theorem 2.1.
We next give sharp bounds on the second and third coefficients for \( f(z) \in S(\alpha, \beta) \). To obtain some sharp coefficient estimates, we need the following lemma due to Rogosinski [3].

**Lemma 2.2.** Let \( p(z) = \sum_{n=1}^{\infty} A_n z^n \) and \( q(z) = \sum_{n=1}^{\infty} B_n z^n \) be analytic in \( \mathbb{U} \). If \( p(z) \prec q(z) \) (\( z \in \mathbb{U} \)), then

\[
\sum_{k=1}^{m} |A_k|^2 \leq \sum_{k=1}^{m} |B_k|^2 \quad (m = 1, 2, \ldots).
\]

By Remark 1.1 and Lemma 2.2, we obtained

**Theorem 2.2.** If the function \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S(\alpha, \beta) \), then

\[
|a_2| \leq \frac{2(\beta - \alpha)}{\pi} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha},
\]

and

\[
|a_3| \leq \frac{\beta - \alpha}{\pi} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha} \left( \cos \frac{\pi(1 - \alpha)}{\beta - \alpha} + \frac{2(\beta - \alpha)}{\pi} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha} \right).
\]

Moreover, the equality holds in either inequality if and only if

\[
f(z) = z \exp \left\{ \int_{0}^{z} \frac{S_{\alpha, \beta}(e^{i\theta}t) - 1}{t} \, dt \right\},
\]

for some real number \( \theta \) (\( 0 \leq \theta < 2\pi \)), where \( S_{\alpha, \beta}(z) \) is defined by (1.4).

**Proof.** According to the proof of Theorem 2.1, the function \( f(z) \) satisfies the subordination (2.2). Also, it follows from the equality (2.5) that

\[
a_2 = A_1
\]

and

\[
2a_3 = A_2 + A_1 a_2 = A_2 + A_1^2.
\]

Then by Remark 1.1, we have

\[
|p'(0)| \leq |S'_{\alpha, \beta}(0)|,
\]

with equality if and only if \( p(z) = S_{\alpha, \beta}(e^{i\theta}z) \) for some real number \( \theta \) (\( 0 \leq \theta < 2\pi \)). This implies that

\[
|a_2| = |A_1| = |p'(0)| \leq |S'_{\alpha, \beta}(0)| = B_1 = \frac{2(\beta - \alpha)}{\pi} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha}
\]

with equality if and only if \( f(z) \) is given in (2.9).

We next prove the sharp bound on the third coefficient for \( f(z) \in S(\alpha, \beta) \). According to Lemma 2.2, the subordination (2.2) implies that

\[
|A_1|^2 + |A_2|^2 \leq B_1^2 + B_2^2.
\]
Thus, we find that

\[ 2|a_3| = |A_2 + A_1^2| \leq |A_2| + |A_1|^2 \]

\[ = \frac{2B_2|A_2|}{2B_2} + |A_1|^2 \leq \frac{B_2^2 + |A_2|^2}{2B_2} + |A_1|^2 \]

\[ = \frac{B_2^2 + (|A_1|^2 + |A_2|^2) + (2B_2 - 1)|A_1|^2}{2B_2} \]

\[ \leq \frac{B_2^2 + (B_1^2 + B_2^2) + (2B_2 - 1)B_1^2}{2B_2} = B_2 + B_1^2 \]

\[ = \frac{2(\beta - \alpha)}{\pi} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha} \left( \cos \frac{\pi(1 - \alpha)}{\beta - \alpha} + \frac{2(\beta - \alpha)}{\pi} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha} \right). \]

Note that we have employed the equality (2.10) here, so equality holds if and only if \( f(z) \) is given in (2.9). This completes the proof of the theorem. \( \square \)

**Remark 2.1.** According to Theorem 1.1 and Example 1.1, the extremal function \( f(z) \in S(\alpha, \beta) \) is given by (1.7), and its rotation. Note that the function \( f(z) \) given by (1.7) has the following form:

\[ f(z) = z \exp \left\{ \int_0^z \frac{S_{\alpha, \beta}(t) - 1}{t} \, dt \right\} = z \exp \left\{ \sum_{k=1}^{\infty} \frac{B_k}{k} z^k \right\} = z + \sum_{n=2}^{\infty} b_n z^n, \]

where

\[ b_n = \sum_{k=1}^{n-1} \frac{1}{k!} \left\{ \sum_{\substack{j=1 \atop m_j=n-1}}^{k} \left( \prod_{j=1}^{k} \frac{B_{m_j}}{m_j} \right) \right\}. \]

From this fact, we expect the sharp bounds on the coefficients for \( f(z) \in S(\alpha, \beta) \) as follows:

(2.11)

\[ |a_n| \leq \sum_{k=1}^{n-1} \frac{1}{k!} \left\{ \sum_{\substack{j=1 \atop m_j=n-1}}^{k} \left( \prod_{j=1}^{k} \frac{2(\beta - \alpha)}{m_j^2 \pi} \sin \frac{m_j \pi(1-\alpha)}{\beta - \alpha} \right) \right\} \quad (n = 2, 3, \cdots). \]

Actually, letting \( n = 2 \) in (2.11), we have the coefficient inequality (2.7). Also, taking \( n = 3 \) in (2.11), we find the coefficient inequality (2.8).
REFERENCES


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