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A NUMERICAL STUDY OF ITO EQUATION AND SAWADA-KOTERA EQUATION BOTH OF TIME-FRACTIONAL TYPE

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ABSTRACT. We consider the Ito equation and the Sawada-Kotera equation both of time-fractional type in this paper. The approximate solutions of these equations are calculated in the form of series obtained by q-Homotopy Analysis Method (q-HAM). The presence of fraction-factor in this method gives it an edge over other existing analytical methods for nonlinear differential equations. Comparisons are made with Modified Adomian decomposition method MADM, homotopy perturbation method HPM and the exact solutions. Numerical results are obtained using Mathematica 8.

1. INTRODUCTION

The generalized KdV equation is an essential model for several physical phenomena including waves in nonlinear LC circuit with mutual inductance between neighboring inductors and shallow-water waves near critical value of surface tension [11]. The need for analytical solution to this class of model arises due to the absence of general solution, though the exact solution of the fifth order KdV equation was found for the special case of solitary waves in [17].

Generally, for the past three decades, fractional calculus has been considered with great importance due to its various applications in physics, fluid flow, control theory of dynamical systems, chemical physics, electrical networks, and so on. The quest of getting accurate methods for solving resulted nonlinear model involving fractional order is of utmost concern of many researchers in this field today.

Various methods have been put to use successfully to obtain analytical solutions such as Adomian Decomposition Method (ADM) [1, 16], Variational Iteration Method (VIM) [8, 13], Homotopy Perturbation Method (HPM) [7, 12, 15]. One of the powerful analytical approach to solving nonlinear differential equations is Homotopy Analysis Method (HAM) [2, 9]. Recently, a modified HAM called q-Homotopy Analysis Method was introduced in [5]. It was proven that the presence of fraction factor in this method enables a fast convergence better than the usual HAM which then makes is more reliable. The Sawada-Kotera Equation was considered in [4] using HAM, in [2] using modified ADM and in [3] using HPM.

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To the best of our knowledge, no attempt has been made regarding analytical solutions of time-fractional Sawada-Kotera Equation and time-fractional Ito Equation using q-Homotopy Analysis Method. In this paper, we consider these equations subject to some appropriate initial conditions. We compare the results obtained by our method with the results obtained using MADM in [2] and HPM in [3], when $\alpha = 1$ to affirm the reliability of the method including numerical values.

2. Preliminaries

This section is devoted to necessary tools for the actualization of the aim of this paper including definitions and some known results. This work adopts Caputo's definition to some concepts of fractional derivatives which is a modification of the Riemann-Liouville's definition and has the advantage of dealing properly with initial value problems. The initial conditions are given in terms of the field variables and their integer order which is the case in many physical processes.

Definition 2.1. *A real function l is said to be in the space C_μ, $\mu \in \mathbb{R}$, $x > 0$, if there exists a real number $p(> \mu)$ such that*

$$l(x) = x^p l_1(x),$$

where $l_1 \in C[0, \infty)$ and it is said to be in the space C_μ^m iff $l^{(m)} \in C_\mu$, $m \in \mathbb{N}$.

Definition 2.2. *The Riemann-Liouville's (RL) fractional integral operator of order $\alpha \geq 0$, of a function $f \in L^1(a, b)$ is given as*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \ \ t > 0, \ \ \alpha > 0,$$

where Γ is the Gamma function and $I^0 f(t) = f(t)$.

Definition 2.3. *The Riemann-Liouville's (RL) fractional derivative of order $0 < \alpha < 1$, of a function f is*

$$D_{0_+}^\alpha f(t) = D I_{0^+}^{1-\alpha} f(t).$$

provided the right-hand side exists where $D = d/dt$.

Definition 2.4. *The fractional derivative in the Caputo's sense is defined as* [14],

$$^C D^\alpha f(t) = I^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau,$$

where $n - 1 < \alpha \leq n$, $n \in \mathbb{N}$, $t > 0$.

Caputo's fractional derivative also has a useful property [6]

$$I^{\alpha C} D^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{k^k}{k!},$$

where $n - 1 < \alpha \leq n$.

Lemma 2.1. *Let $\alpha \geq 0$, $\beta \geq 0$ and $f \in CL(a, b)$. Then*

$$I_a^\alpha I^\beta f(t) = I_a^{\alpha+\beta} f(t),$$

for all $t \in (a, b]$.

Lemma 2.2. *Let $t \in (a,b]$. Then*

$$\left[I_a^{\alpha}(t-a)^{\beta}\right](t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}(t-a)^{\beta+\alpha}, \qquad \alpha \geqslant 0, \quad \beta > 0.$$

Remark 2.1. *From the definitions given above, we observed that the Riemann-Liouville fractional derivative of a constant function is not equal to zero while that of Caputo fractional derivative of constant function is zero.*

3. Method of Solution (q-HAM)

We consider the following differential equation of the form

$$N\left[D_t^{\beta}u(x,t)\right] - f(x,t) = 0$$

where N is a nonlinear operator, D_t^{β} denote the Caputo fractional derivative, (x,t) are independent variables, $f(x,t)$ is a known function and $u(x,t)$ is an unknown function.

$$(1-nq)L\left(\phi(x,t;q) - u_0(x,t)\right) = qhH(x,t)\left(N[D_t^{\beta}\phi(x,t;q)] - f(x,t)\right), \tag{3.1}$$

where $n \geqslant 1$, $q \in \left[0, \frac{1}{n}\right]$ denotes the so-called embedded parameter, L ia an auxiliary linear operator with the property $L[f] = 0$ when $f = 0$, $h \neq 0$ is an auxiliary parameter, $H(x,t)$ is a non-zero auxiliary function.

It is clearly seen that when $q = 0$ and $q = \frac{1}{n}$, equation (3.1) becomes

$$\phi(x,t;0) = u_0(x,t) \quad \textit{and} \quad \phi(x,t;\frac{1}{n}) = u(x,t)$$

respectively. So, as q increases from 0 to $\frac{1}{n}$, the solution $\phi(x,t;q)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$.

If $u_0(x,t)$, L, h, $H(x,t)$ are chosen appropriately, solution $\phi(x,t;q)$ of equation(3.1) exists for $q \in \left[0, \frac{1}{n}\right]$.

Taylor series expansion of $\phi(x,t;q)$ gives

$$\phi(x,t;r) = u_0(x,t) + \sum_{m=1}^{\infty} \phi_m(x,t)q^m. \tag{3.2}$$

where

$$\phi_m(x,t) = \frac{1}{m!}\frac{\partial^m F(x,t;q)}{\partial^m r}|_{q=o}.$$

If we assume that the auxiliary linear operator L, the initial guess u_0, the auxiliary parameter h and $H(x,t)$ are properly chosen such that the series (3.2) converges at $q = \frac{1}{n}$, then we have

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)\left(\frac{1}{n}\right)^m.$$

Define

$$u_n(x,t) = \{u_0(x,t), u_1(x,t), \cdots, u_n(x,t)\}.$$

Differentiating equation (3.1) m-times with respect to the (embedding) parameter q, then evaluating at $q = 0$ and finally dividing them by $m!$, we have the so called mth-order deformation equation (Lioa [9, 10]) as

$$L\left[u_m(x,t) - \chi_m^* u_{m-1}(x,t)\right] = hH(x,t)\breve{R}_m\left(\vec{u}_{m-1}\right). \tag{3.3}$$

with initial conditions:

$$u_m^{(k)}(x,0) = 0, \qquad k = 0,1,2,...,m-1.$$

where

$$\breve{R}_m\left(\vec{u}_{m-1}\right) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\left(N[D_t^\beta \phi(x,t;q)] - f(x,t)\right)}{\partial q^{m-1}}|_{q=0}$$

and

$$\chi_m^* = \begin{cases} 0 & m \leqslant 1 \\ \\ n & otherwise, \end{cases}$$

Remark 3.1. *It should be emphasized that $u_m(x,t)$ for $m \geqslant 1$, is governed by the linear operator* (3.3) *with the linear boundary conditions that come from the original problem. The existence of the factor $\left(\frac{1}{n}\right)^m$ gives more chances for better convergence, faster that the solution obtained by the standard HAM. Of course, when $n = 1$, we are in the case of the standard HAM.*

4. Applications

Consider the generalized fifth-order KdV equation of time-fractional type

$$D_t^\beta u + au^2 u_x + bu_x u_{xx} + cuu_{xxx} + du_{xxxxx} = 0 \qquad t > 0,\ 0 < \alpha \leqslant 1 \tag{4.1}$$

with the initial condition

$$u(x,0) = f(x;k;\lambda).$$

To apply q-HAM, we choose the linear operator

$$L[\phi(x,t;q)] = D_t^\beta \phi(x,t;q)$$

with property that $L[c_1] = 0$, c_1 is constant.

We use initial approximation $u_0(x,t) = u(x,0)$. We can then define the nonlinear operator as

$$\begin{aligned} N[\phi(x,t;q)] &= D_t^\beta \phi_t(x,t;q) + a\left(\phi(x,t;q)\right)^2 \phi_x(x,t;q) + b\phi_x(x,t;q)\phi_{xx}(x,t;q) \\ &\quad + c\phi(x,t;q)\phi_{xxx}(x,t;q) + d\phi_{xxxxx}(x,t;q). \end{aligned}$$

We construct the zeroth order deformation equation

$$(1-nq)L\left[\phi(x,t;q) - u_0(x,t)\right] = qhH(x,t)N\left[\phi(x,t;q)\right].$$

We choose $H(x,t) = 1$ to obtain the mth-order deformation equation to be

$$L\left[u_m(x,t) - \chi_m^* u_{m-1}(x,t)\right] = h\breve{R}_m\left(\vec{u}_{m-1}\right).$$

with initial condition for $m \geqslant 1$, $u_m(x,0) = 0$,

$$\chi_m^* = \begin{cases} 0 & m \leqslant 1 \\ n & otherwise, \end{cases}$$

and

$$\begin{aligned} \breve{R}_m\left(\vec{u}_{m-1}\right) &= D_t^{\beta} u_{(m-1)} + a \sum_{k=0}^{m-1} \sum_{j=0}^{k} u_j u_{k-j} u_{(m-1-k)x} + b \sum_{k=0}^{m-1} u_{kx} u_{(m-1-k)xx} \\ &+ c \sum_{k=0}^{m-1} u_k u_{(m-1-k)xxx} + d u_{(m-1)xxxxx}. \end{aligned} \tag{4.2}$$

So, the solution to the equation (4.1) for $m \geqslant 1$ becomes

$$u_m(x,t) = \chi_m^* u_{m-1} + h I^{\alpha}\left[\breve{R}_m\left(\vec{u}_{m-1}\right)\right]. \tag{4.3}$$

Then the series solution expression by q-HAM can be written in the form

$$u(x,t;n;h) \cong U_M(x,t;n;h) = \sum_{j=0}^{M} u_i(x,t;n;h)\left(\frac{1}{n}\right)^i \tag{4.4}$$

Equation (4.4) is an appropriate solution to the problem (4.1) in terms of convergence parameter h and n.

Remark 4.1. *The fraction factor $\left(\frac{1}{n}\right)^m$ highly increase the convergence chances than that of HAM.*

4.1. **The Ito time-fractional Equation.** Taking $a = 2$, $b = 6$, $c = 3.$ and $d = 1$, we obtain the Ito time-fractional equation

$$D_t^{\beta} u + 2u^2 u_x + 6u_x u_{xx} + 3u u_{xxx} + u_{xxxxx} = 0 \qquad t > 0,\ 0 < \beta \leqslant 1 \tag{4.5}$$

with the initial condition

$$u(x,0) = 20k^2 - 30k^2 \tanh^2(kx),\ taking\ \lambda = 0. \tag{4.6}$$

The exact solution of (4.5) together with condition (4.6) for $\beta = 1$ is given as

$$u(x,t) = 20k^2 - 30k^2 \tanh^2(kx - 96k^4).$$

This problem has been solved in [2] using Modified Adomian Decomposition Method(MADM).

We use initial approximation

$$u_0(x,t) = u(x,0) = 20k^2 - 30k^2 \tanh^2(kx).$$

Using equations (4.2) and (4.3), taking $a = 2$, $b = 6$, $c = 3.$ and $d = 1$, we therefore obtain components of the solution using q-HAM successively as follows

$$\begin{aligned} u_1(x,t) &= -570hk^7 \tanh(kx) sech^2(kx) \frac{t^\alpha}{\Gamma(\alpha+1)} \\ u_2(x,t) &= \frac{2880hk^7 t^\beta sech^4(kx)\left[192hk^5 t^{2\alpha}(-2+\cosh(2kx))\right]}{\Gamma(\beta+1)\Gamma(2\beta+1)} \\ &\quad - \frac{2880hk^7 t^\beta sech^4(kx)\left[(h+n)\sinh(2kx)\Gamma(2\beta+1)\right]}{\Gamma(\beta+1)\Gamma(2\beta+1)}. \end{aligned}$$

In the same way, $u_m(x,t)$ for $m = 3, 4, \cdots$ can be obtained using Mathematica-8.

Then the series solution expression by q-HAM can be written in the form

$$u(x,t;n;h) \cong U_M(x,t;n;h) = \sum_{j=0}^{M} u_j(x,t;n;h) \left(\frac{1}{n}\right)^j \tag{4.7}$$

Equation (4.7) is an appropriate solution to the problem (4.5) in terms of convergence parameter h and n.

4.2. **The time-fractional Sawada-Kotera Equation.** Taking $a = 45$, $b = 15$, $c = 15.$ and $d = 1$, we obtain the Sawada-Kotera time-fractional equation

$$D_t^\beta u + 45u^2 u_x + 15u_x u_{xx} + 15uu_{xxx} + u_{xxxxx} = 0 \qquad t > 0,\ 0 < \beta \leqslant 1 \tag{4.8}$$

with the initial condition

$$u(x,0) = 2k^2 sech^2\left[k(x-\lambda)\right]. \tag{4.9}$$

The exact solution of (4.8) together with condition (4.9) for $\beta = 1$ is given as

$$u(x,t) = 2k^2 sech^2\left[k(x - 16k^4 t - \lambda)\right].$$

We use initial approximation

$$u_0(x,t) = u(x,0) = 2k^2 sech^2\left[k(x-\lambda)\right].$$

Using equations (4.2) and (4.3), taking $a = 45$, $15 = 6$, $c = 15.$ and $d = 1$, we therefore obtain components of the solution using q-HAM successively as follows

$$\begin{aligned} u_1(x,t) &= -64hk^7 \tanh[k(x-\lambda)] sech^2[k(x-\lambda)] \frac{t^\beta}{\Gamma(\beta+1)} \\ u_2(x,t) &= -64h(n+h)k^7 \tanh[k(x-\lambda)] sech^2[k(x-\lambda)] \frac{t^\beta}{\Gamma(\beta+1)} \\ &\quad + \frac{1024h^2 t^{2\beta} k^1 2(cosh[2k(x-\lambda)]-2) sech^4[k(x-\lambda)]}{\Gamma(2\alpha+1)}. \end{aligned}$$

In the same way, $u_m(x,t)$ for $m = 3, 4, \cdots$ can be obtained using Mathematica-9.

Then the series solution expression by q-HAM can be written in the form

$$u(x,t;n;h) \cong U_M(x,t;n;h) = \sum_{j=0}^{M} u_j(x,t;n;h) \left(\frac{1}{n}\right)^j \tag{4.10}$$

Equation (4.10) is an appropriate solution to the problem (4.8) in terms of convergence parameter h and n.

5. Numerical Results and Discussion

5.1. **Case of $\beta = 1$.** We have obtained the numerical results of the two equations considered in the previous subsection and the comparisons are made with the exact solutions ($\beta = 1$) in the figures below.

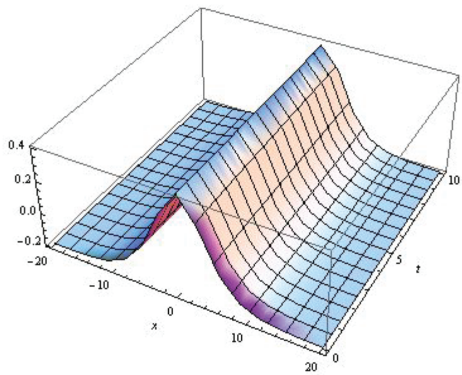

FIGURE 1. q-HAM solution of Ito equation with $h = -0.5$, $n = 1$ and $k = 0.2$.

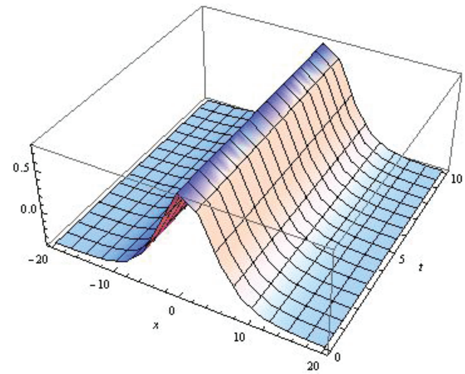

FIGURE 2. Exact solution of Ito equation with $k = 0.2$.

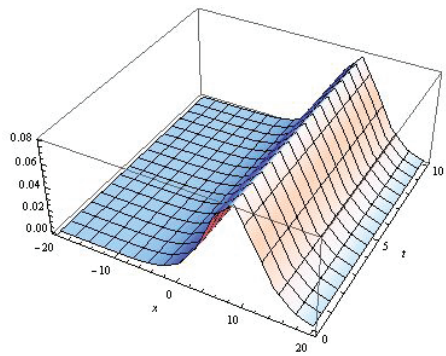

FIGURE 3. q-HAM solution of Sawada-Kotera time-fractional equation with $h = -1$, $n = 1$ and $k = 8$.

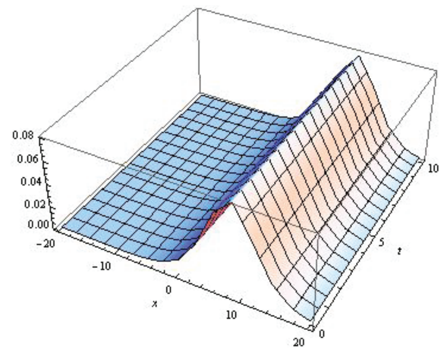

FIGURE 4. Exact solution of Sawada-Kotera time-fractional equation with $k = 8$.

Remark 5.1. *It should be noted that apart from less computational effort required to obtain the series solutions of these equations, we have only used two terms* U_2 ($M = 2$) *to get close as much as possible to the exact solutions.*

5.2. **Case of $\beta = 0.5$.** We have presented below some numerical results for $\beta = 0.5$ to show the effect of the fractional order in time for both Ito equation and Sawada-Kotera equation.

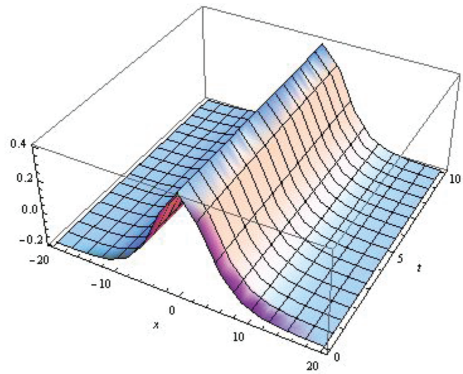

FIGURE 5. q-HAM solution of Ito equation with $h = -0.5$, $\beta = 0.5$, $n = 1$ and $k = 0.2$.

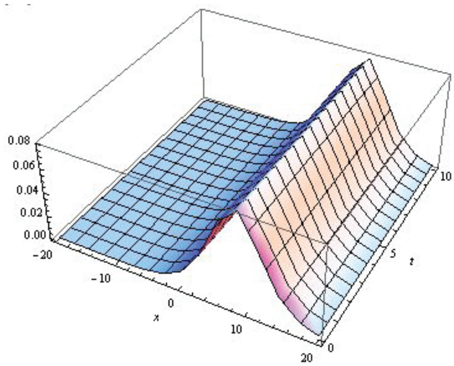

FIGURE 6. q-HAM solution of Sawada-Kotera time-fractional equation with $h = -1$, $\beta = 0.5$, $n = 1$ and $k = 8$.

6. CONCLUSION

In this paper, q-HAM has been successfully developed to solve time-fractional type of both the Ito equation and the Sawada-Kotera equation. The performance of HAM is greatly improved by q-HAM shown using the two well known equations. The results show that the convergence rate of q-HAM is faster than that of HAM due to the presence of fraction factor $\left(\frac{1}{n}\right)^m$.The results of figure(1) to figure(6) are in perfect agreement with those obtained in [2] using MADM, in [3] using HPM and the exact solutions of both equations under suitable initial conditions but with just two terms of the series solutions obtained by q-HAM. The efficiency and accuracy are obvious from the graphs.

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REFERENCES

[1] O. ABDULAZIZ, I. HASHIM, M. S. H. CHOWDHURY, A. K. ZULKIFLE: *Assessment of decomposition method for linear and nonlinear fractional differential equations,* Far East Journal of Applied Mathematics, **28**(1) (2007), 95-112.

[2] H. O. BAKODAH: *Modified Adomian Decomposition Method for the Generalized Fifth Order KdV Equations*, American J. of Comput. Mathematics, **3**(2013), 53-58.

[3] J. Biazar, K. Hosseini, P. Gholamin: *Homotopy Perturbation Method for Solving KdV and Sawada-Kotera Equations*, Journal of Appl. Math., Islamic Azad University, **6**(21), 2009.

[4] S. Dinarvand, S. Khasravi: *The Homotopy Analysis Method for Solving the Sawada-Kotera and Lax's Fifth-Order KdV Equations*, Adv. Theor. Appl. Mech., **1**(7) (2008), 327-335.

[5] M. A. El-Tawil, S. N. Huseen: *The Q-Homotopy Analysis Method (Q-HAM)*, Int. J. of Appl. Math and Mech., **8**(15) (2012), 51-75.

[6] R. Gorenflo, F. Mainardi: *Fractional calculus: integral and differential equations of fractional order,* Fractals and Fractional Calculus in Continuum Mechanics (Udine, 1996), Springer, **378** (1997), 223-276.

[7] J.H. He: *Application of homotopy perturbation method to nonlinear wave equations,* Chaos, Solitons and Fractals, **26**(3) (2005), 695-700.

[8] J.H. He, X.H. Wu: *Construction of solitary solution and compacton-like solution by variationaliteration method*, Chaos, Solitons and Fractals, **29**(1) (2006), 108-113.

[9] S.J. Liao: *An approximate solution technique not depending on small parameters: a special example,* International Journal of Non-linear Mechanics, **30**(3) (1995), 371-380.

[10] S.J. Liao: *Beyond Perturbation: Introduction to Homotopy Analysis Method,* CRC Series: Modern Mechanics and Mathematics, Chapman and Hall/CRC, Boca Raton, Fla, USA, **2**, 2004.

[11] H. Nagashima: *Experiment on Solitary Waves in the Nonlinear Transmission Line Discribed by the Equation* $u_t + uu_\xi - u_{\xi\xi\xi\xi\xi} = 0$, J. of Physical Society of Japan, **47** (1979), 1387-1388.

[12] Z. Odibat: *Exact solitary solutions for variants of the KdV equations with fractional timederivatives,* Chaos, Solitons and Fractals. In press.

[13] Z. Odibat, S. Momani: *Application of variational iteration method to nonlinear differential equations of fractional order*, International Journal of Nonlinear Science and Numerical Simulation, **7**(1) (2006), 27-34.

[14] I. Podlubny: *Fractional Differential Equations,* Vol. 198 of Mathematics in Science and Engineering, Academic Press, San Diego, Calif, USA, 1999.

[15] Q. Wang: *Homotopy perturbation method for fractional KdV-Burgers equation*, Chaos, Solitons and Fractals, **35**(5) (2008), 843-850.

[16] Q. Wang: *Numerical solutions for fractional KdV-Burgers equation by Adomian decomposition method,* Applied Mathematics and Computation, **182**(2) (2006), 1048-1055.

[17] Y. Yamaoto, E. Takizawa: *On a Solution on Non-linear Time-Evolution Equation of Fifth Order*, J. of Phys. Soc. Jpn., **50** (1981), 1421-1422.

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