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PROBLEM OF SHADOW (COMPLEX CASE)

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Abstract. In this review we consider the results on building generalized convex hull for family consisting of convex sets in real and (hyper)complex Euclidean space. The unsolved questions related to those problems are discussed.

1. Introduction

A subject, which is treated here, combines in one bundle some questions of convex, complex, hypercomplex analysis, probability theory and geometry.

Definition 1.1. *We say that the set $E \subset \mathbb{R}^n$ in the n-dimensional real Euclidean space is m-convex with respect to the point $x \in \mathbb{R}^n \backslash E$, if exists a m-dimensional plane L such that $x \in L$ and $L \cap E = \varnothing$.*

Definition 1.2. *We say that the set $E \subset \mathbb{R}^n$ is m-convex if it is m-convex with respect to each point $x \in \mathbb{R}^n \backslash E$.*

Both definitions satisfy the axiom of convexity: the intersection of any non-empty subfamily of these sets satisfies the definition also. For every set $E \subset \mathbb{R}^n$ we can consider the minimal m-convex set containing E and call it m-hull of the set E.

Problem 1 (of shadow). Which minimum number of pairwise disjoint closed balls in the n-dimensional real Euclidean space $\mathbb{R}^n$ with centres on the sphere S^{n-1} and with radii less than the sphere radius is sufficient for any straight line passing through the sphere centre to cross at least one of these balls [1]?

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2. Main results

In [2], the complete answer to this problem for a collection of closed and open balls was obtained.

Theorem 2.1. *[2] In order that the center of a sphere S^{n-1} in the real n-dimensional Euclidean space for $n > 2$ belong to the 1-hull of the family of open (closed) balls with radius at most (less than) the sphere radius and with their centers on the sphere, it is necessary and sufficient to have $n+1$ balls.*

Consider now the previous problem with the change of the family of balls by a family of sets which is obtained from the initial set by the action of some group of transformations or for a family of balls whose centers are not attached to a beforehand given set.

Theorem 2.2. *[3] In order that the chosen point in the n-dimensional real Euclidean space for $n > 2$ belong to the 1- hull of the family of pairwise disjoint closed (open) sets that is obtained from the given convex set with nonempty interior with the help of the group of transformations consisting of movements and homotheties, it is necessary and sufficient to have n elements of the family.*

Generalizing construction above us can consider following classes of sets.

Definition 2.1. *We say that the set of $E \subset \mathbb{C}^n(\mathbb{H}^n)$ is m-complexly (m-hypercomplexly) convex relative to the point $z \in \mathbb{C}^n \backslash E$ $(\mathbb{H}^n)$ if there is a m-dimensional complex (hypercomplex) plane L such that $z \in L$ and $L \cap E = \varnothing$. We say that a set $E \subset \mathbb{C}^n(\mathbb{H}^n)$ is m-complexly (m-hypercomplexly) complex if it is m-complexly (m-hypercomplexly) complex relative to every point $z \in \mathbb{C}^n \backslash E$ $(\mathbb{H}^n \backslash E)$.*

Analogically, for every set $E \subset \mathbb{C}^n$ (or $\mathbb{H}^n$) we can consider the minimal m-complex (m-hypercomplex) convex set containing E and call it m-complex(hypercomplex) hull of the set E.

In the case $m = n - 1$ this class of sets is well known as linearly convex sets [4, 5]. Linearly convex sets are very useful in complex analysis and in the questions of the integral geometry and tomography. On the base of these sets in complex analysis there is built linearly convex complex analysis, similar to real convex analysis. More results of linearly convex analysis can be viewed in the monograph [1]. In the spite of abundance results, there is remained the big amount of the unsolved problems, concerning topological characteristics of these sets, a part from which it is possible to find in [1].

Let us study how the situation with the previous problem is changed if we consider a complex or hypercomplex space instead of a real Euclidean space.

Problem 2 (of shadow, complex case). Which minimum number of pairwise disjoint closed balls with their centers on the sphere $S^{2n-1} \subset \mathbb{C}^n(S^{4n-1} \subset \mathbb{H}^n)$ and with radii less than the sphere radius is sufficient in order that any complex (hypercomplex) straight line passing through the sphere center cross at least one of these balls?

Theorem 2.3. *[3] In order that the chosen point in the 2-dimensional complex (hypercomplex) Euclidean space $\mathbb{C}^2(\mathbb{H}^2)$ belong to the 1-complex (1-hypercomplex) hull of a family of pairwise disjoint open (closed) balls that do not contain the given* *point, it is necessary and sufficient to have two balls with centres on the sphere centred at the chosen point.*

Theorem 2.4. *In order that a point of the n-dimensional (hyper)complex Euclidean space $\mathbb{C}^n, n \geq 3$, belong to the 1-(hyper)complex hull of a family of open or closed balls with centres on the sphere $S^{2n-1} \subset \mathbb{C}^n (S^{4n-1} \subset \mathbb{H}^n)$ that do not contain the chosen point, it is sufficient to have $2n(4n-2)$ balls.*

Proof. Let we consider the sphere $S^{2n-1}(S^{4n-1})$ in n-dimensional complex (hypercomplex) Euclidean space $\mathbb{C}^n(\mathbb{H}^n)$. Now we chose an arbitrary $(2n-1)$(or $(4n-3)$)-dimensional real plane through centre of this sphere. Its intersection with the sphere $S^{2n-1}(S^{4n-1})$ will be the sphere S^{2n-2} (or S^{4n-4}). Let L be an arbitrary complex (hypercomplex) line through centre of the sphere S^{2n-1} or S^{4n-1} respectively. This complex (hypercomplex) line has real dimension two (or four). Intersection L with the sphere $S^{2n-2}(S^{4n-1})$ contains real straight line passing through the sphere center. Now we can use theorem 1. In order that the center of a sphere S^{2n-1} (or S^{4n-1}) (or S^{4n-4}) in the real $(2n-1)$(or $4n-3$)-dimensional Euclidean space L belong to the 1-hull of the family of open or closed balls with radius at most (less than) the sphere radius and with their centers on the respective sphere, it is necessary and sufficient to have $2n$ (or $4n-2$) balls, respectively. So number of $2n$ (or $4n-2$) balls, respectively, is enough for the center of the sphere S^{2n-1} (or S^{4n-1}) belong to the 1-(hyper)complex hull of balls family. This completes the proof of the theorem. □

Following exact result is received in the case of free accommodations of ball's centers.

Theorem 2.5. *[6] In order that a point of the n-dimensional (hyper)complex Euclidean space $\mathbb{C}^n(\mathbb{H}^n)$ belong to the 1-(hyper)complex hull of a family of open (closed) balls that do not contain the chosen point, it is necessary and sufficient to have n balls.*

3. Open questions

It is possible spread the problem on shade for the following situation.

Which minimum number of pairwise disjoint closed balls in the n-dimensional real Euclidean space with centers on the sphere $\mathbb{R}^n$ and with radii less than the sphere radius is sufficient for any straight line passing through the arbitrary point of the ball bounded by sphere S^{n-1} to cross at least one of these balls?

It is easy to verify that if $n = 2$, then this number is equal to three. Let we considers equilateral triangle σ and three balls with radius equal to half of triangle's height and centers into triangle's vertexes. On the one hand it is clear that every straight line passing through the arbitrary point of the convex hull of that family of balls intersects at least one of these balls. On the other hand circumference described around triangle σ lies in this convex hull.

Using this fact is possible to show that under $n = 3$ for solving problem above is sufficient finite number of balls but exact minimal number even its estimate is unknown.

Theorem 3 gives the exact number of balls only for $n = 2$. From Theorem 5, we will obtain the estimate of the number of convex sets or balls whose centers are not given beforehand. Therefore, the following questions remain open.

Question 1. Which minimum number of pairwise disjoint open (closed) balls with their centers on a fixed sphere and with radii not exceeding (less) the sphere radius in the n-dimensional complex (hypercomplex) Euclidean space $\mathbb{C}^n(\mathbb{H}^n)$ for $n > 3$ ensures the membership of the sphere center to their 1-complex (1-hypercomplex) hull?

Question 2. (problem of sphere). Does there exists a linearly convex compact set in the complex Euclidean space $\mathbb{C}^2$ (or 2-convex compact set in $\mathbb{R}^4$) which is homeomorphic to a two-dimensional sphere S^2?

Question 3. Let $K \subset \mathbb{R}^n$ be a compact in the real Euclidean space. Suppose that for each hyperplane $L \cap \mathbb{R}^n$ the intersection $L \cap K$ is $(m-1)$-convex. Find a condition on K which together with the above one would be sufficient for K to be m-convex.

Consider some objects more general relative to the previous definitions.

Definition 3.1. *The set $E \subset \mathbb{R}^n$ in the real Euclidean space is m-semiconvex relative to the point $x \in \mathbb{R}^n \backslash E$ if there exists an m-dimensional half-plane P such that $x \in P$ and $P \cap E = \varnothing$.*

Definition 3.2. *The set $E \subset \mathbb{R}^n$ is m-semiconvex if it is m-semiconvex relative to every point $x \in \mathbb{R}^n \backslash E$.*

Some results about semiconvexity are received in [2,3,4].

Question 4. What is the minimum number of closed (open) balls in n-dimensional real Euclidean space with centres on the sphere and radii, which are smaller (no more) than the radius of the sphere for $n \geq 3$, allow to the origin of the sphere to be in their 1-semiconvex hull? Is this number finite for $n > 3$?

Question 5. Is there an embedding of sphere S^m, where $1 < m < n/2$ in the real Euclidean space $\mathbb{R}^n$, such that the resulting image will be $(n-1)$-semiconvex compact?

Note. The theorem 16.2 [1] implies that for $m \geq n/2$ required embedding is not possible.

Analogously, the numeric estimations of complex and hypercomplex semiconvexities for the families of balls and convex sets which can be introduced similarly to the real semiconvexity remain open.

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