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PROPERTIES OF A GENERAL INTEGRAL OPERATOR

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ABSTRACT. In this paper, we derive sufficient conditions for the univalence and convexity of a new integral operator defined on the space of normalized analytic functions in the open unit disk. Some subordination results for this new integral operator are also given.

1. INTRODUCTION

Let $\mathcal{A}$ be the class of functions which are analytic in the open unit disk $U = \{z : |z| < 1\}$ and have the following form:

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots,\ z \in U.$$

Consider S the subclass of $\mathcal{A}$ consisting of univalent functions. We denote by $S^*(\alpha)$ the class of starlike univalent functions of order α, $0 \le \alpha < 1$,

$$S^*(\alpha) = \left\{ f \in \mathcal{A} : \mathrm{Re}\left[\frac{zf'(z)}{f(z)}\right] > \alpha,\ z \in U \right\}.$$

By $K(\alpha)$ we denote a subclass of $\mathcal{A}$ consisting of convex univalent functions of order α, $0 \le \alpha < 1$ defined as

$$K(\alpha) = \left\{ f \in \mathcal{A} : \mathrm{Re}\left[\frac{zf''(z)}{f'(z)} + 1\right] > \alpha,\ z \in U \right\}.$$

It is well known that $S^*(0) = S^*$ and $K(0) = K$ are the classes of starlike and convex functions in U, respectively.

Recently, Frasin and Jahangiri [6] defined the family $B(\mu, \lambda), \mu \ge 0, 0 \le \lambda < 1$ consisting of all functions $f \in \mathcal{A}$ satisfying the condition

$$\left| f'(z) \left[\frac{z}{f(z)}\right]^{\mu} - 1 \right| < 1 - \lambda, \quad z \in U.$$

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The family $B(\mu, \lambda)$ is a comprehensive class of analytic functions. For instance, we have $B(1, \lambda) = S^*(\lambda)$, $B(2, \lambda) = B(\lambda)$ (see Frasin and Darus [7]), $B(2, 0) = S$ (see Ozaki and Nunokawa [11]).

If f and g are analytic functions in U, we say that f is subordinate to g, written $f \prec g$, if there is a function w analytic in U, with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$, such that $f(z) = g(w(z))$ for all $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$ (see Miller and Mocanu [9]).

Using subordinations, Owa et al. [10] have defined the following subclass $S_b(a)$ of $\mathcal{A}$.

A function $f \in \mathcal{A}$ is said to be in the class $S_b(a)$ if it satisfies

$$\left(f'(z)\right)^b \prec \frac{a(1-z)}{a-z}, \quad z \in U,$$

for some real $a > 1$ and $b > 0$.

Also, Breaz et al. [2] have defined the classes $S_b^*(a)$ and $C_b^*(a)$.
A function $f \in \mathcal{A}$ is said to be in the class $S_b^*(a)$ if it satisfies

$$Re\left[\frac{zf''(z)}{f'(z)}\right] < \frac{a-1}{2b(a+1)}, \quad z \in U, \tag{1.1}$$

for some real $a > 1$ and $b > 0$.

A function $f \in \mathcal{A}$ is said to be in the class $C_b^*(a)$ if it satisfies

$$Re\left[\frac{zf'(z)}{f(z)}\right] > \frac{1-a}{2b(a+1)} + 1, \quad z \in U,$$

for some real $a > 1$ and $b > 0$.

In the present paper, we define a new integral operator given by

$$I_n(z) = \int_0^z \prod_{i=1}^n \left[\frac{tf_i'(t)}{g_i(t)}\right]^{\alpha_i} dt, \tag{1.2}$$

where parameters $\alpha_i \in \mathbb{C}$ and the functions $f_i, g_i \in \mathcal{A}$, $i \in \{1, \ldots, n\}$ are so constrained that the integral (1.2) exists.

The operator I_n extends several integral operators:
(i) For $g_i(t) = t$ and $\alpha_i > 0$ we have $I_n(f)(z) = \int_0^z \prod_{i=1}^n \left[f_i'(t)\right]^{\alpha_i} dt$, that was defined by Breaz et al. in [3], and this operator is a generalization of the integral operator $I(f)(z) = \int_0^z \left[f'(t)\right]^{\alpha} dt$, discussed in [14, 12, 13].
(ii) For $n = 1$ we obtain $I_\alpha(z) = \int_0^z \left[\frac{tf'(t)}{g(t)}\right]^{\alpha} dt$, introduced and studied by Bucur et al. in [4].

In the proof of our main results, we need to recall here the following:

Lemma 1.1. **(Becker** [1]**)** *If the function f is regular in the unit disk U, $f(z) = z + a_2z^2 + \ldots$ and*

$$(1-|z|^2) \cdot \left|\frac{zf''(z)}{f'(z)}\right| \leq 1$$

for all $z \in U$, then the function f is univalent in U.

Lemma 1.2. (General Schwarz Lemma [8]**)** *Let the function f be regular function in the disk $U_R = \{z \in \mathbb{C} : |z| < R\}$, with $|f(z)| < M$ for fixed M. If f has one zero with multiplicity order bigger than m for $z = 0$, then*

$$|f(z)| \leq \frac{M}{R^m}|z|^m, \quad z \in U_R.$$

The equality case holds only if $f(z) = e^{i\theta}\frac{M}{R^m}z^m$, where θ is constant.

Lemma 1.3. (Deniz and Orhan [5]**)** *If $g \in B(\lambda)$, then*

$$\left|\frac{zg^{'}(z)}{g(z)} - 1\right| < \frac{(1-\lambda)(1+|z|)}{1-|z|}, \quad z \in U.$$

For the class $S_b(a)$, Owa et al. [10] proved the following result.

Theorem 1.1. ([10]) *If $f \in \mathcal{A}$ satisfies the inequality (1.1) for some real $a > 1$ and $b > 0$, then $f \in S_b(a)$.*

2. Univalence and Convexity Properties of the Integral Operator I_n

In the following theorems we derive univalence conditions for the operator I_n, defined in (1.2), by using Becker univalence criterion.

Theorem 2.1. *Let the functions $f_i \in \mathcal{A}$, $g_i \in B(\lambda_i)$ and $\alpha_i \in \mathbb{C}$, $M_i \geq 1$, $i \in \{1, \ldots, n\}$. If*

$$\left|\frac{f_i^{''}(z)}{f_i^{'}(z)}\right| \leq M_i, \quad z \in U, \tag{2.1}$$

and

$$\sum_{i=1}^{n} |\alpha_i| \cdot \left[\frac{2\sqrt{3}M_i}{9} + 4(1-\lambda_i)\right] \leq 1, \tag{2.2}$$

for all $i \in \{1, \ldots, n\}$, then the function I_n, given by (1.2) is in the class S.

Proof. It is easily seen that

$$I_n^{'}(z) = \prod_{i=1}^{n}\left[\frac{zf_i^{'}(z)}{g_i(z)}\right]^{\alpha_i}$$

and

$$I_n^{''}(z) = \sum_{i=1}^{n} \alpha_i \left[\frac{zf_i^{'}(z)}{g_i(z)}\right]^{\alpha_i - 1} \cdot \left[\frac{f_i^{'}(z)}{g_i(z)} + \frac{zf_i^{''}(z)}{g_i(z)} - \frac{zf_i^{'}(z)g_i^{'}(z)}{g_i^2(z)}\right] \prod_{\substack{j=1 \\ j\neq i}}^{n}\left[\frac{zf_j^{'}(z)}{g_j(z)}\right]^{\alpha_j}.$$

Thus, we get

$$\frac{zI_n^{''}(z)}{I_n^{'}(z)} = \sum_{i=1}^{n} \alpha_i \left[1 + \frac{zf_i^{''}(z)}{f_i^{'}(z)} - \frac{zg_i^{'}(z)}{g_i(z)}\right]. \tag{2.3}$$

Since $f_i \in B(\lambda_i)$ for all $i \in \{1, \ldots, n\}$, from Lemma 1.3, (2.1) and(2.3), we find that

$$\left|\frac{zI_n''(z)}{I_n'(z)}\right| \le \sum_{i=1}^{n} |\alpha_i| \left\{ |z| \cdot \left|\frac{f_i''(z)}{f_i'(z)}\right| + \left|\frac{zg_i'(z)}{g_i(z)} - 1\right| \right\}$$

$$\le \sum_{i=1}^{n} |\alpha_i| \left\{ |z| \cdot M_i + \frac{(1-\lambda_i)(1+|z|)}{1-|z|} \right\}. \tag{2.4}$$

Multiply the relation (2.4) with $(1-|z|^2)$, we have

$$(1-|z|^2)\left|\frac{zI_n''(z)}{I_n'(z)}\right| \le (1-|z|^2)\cdot|z|\sum_{i=1}^{n}|\alpha_i|\cdot M_i + (1+|z|)^2\sum_{i=1}^{n}|\alpha_i|(1-\lambda_i)$$

$$\le \frac{2\sqrt{3}}{9}\sum_{i=1}^{n}|\alpha_i|\cdot M_i + 4\sum_{i=1}^{n}|\alpha_i|(1-\lambda_i).$$

Now, by using (2.2) and applying Lemma 1.1, we prove that $I_n \in S$. □

If we set $M_i = 1$ and $\lambda_i = 0$, $i \in \{1, \ldots, n\}$ in Theorem 2.1, we obtain:

Corollary 2.1. *Let the functions $f \in \mathcal{A}$, $g \in S$ and $\alpha \in \mathbb{C}$. If*

$$\left|\frac{f''(z)}{f'(z)}\right| \le 1, \quad z \in U,$$

and

$$|\alpha| \le \frac{9}{2\sqrt{3}+36},$$

then the function I_α given by

$$I_\alpha(z) = \int_0^z \left[\frac{tf'(t)}{g(t)}\right]^\alpha dt,$$

is in the class S.

Theorem 2.2. *Let the functions $f_i \in \mathcal{A}$, $g_i \in B(\mu_i, \lambda_i)$, $i \in \{1, \ldots, n\}$. Suppose that $\alpha_i \in \mathbb{C}$, M_i, $N_i \ge 1$ such that*

$$\sum_{i=1}^{n} |\alpha_i| \cdot \left[1 + (2-\lambda_i)N_i^{\mu_i - 1} + \frac{2\sqrt{3}M_i}{9}\right] \le 1, \tag{2.5}$$

for all $i \in \{1, \ldots, n\}$. If

$$\left|\frac{f_i''(z)}{f_i'(z)}\right| \le M_i, \quad \text{and} \quad |g_i(z)| < N_i, \tag{2.6}$$

for all $i \in \{1, \ldots, n\}$, $z \in U$, then the function I_n, given by (1.2) is in the class S.

Proof. Relation (2.3) implies

$$\left|\frac{zI_n''(z)}{I_n'(z)}\right| \le \sum_{i=1}^{n} |\alpha_i| \left\{ 1 + |z| \cdot \left|\frac{f_i''(z)}{f_i'(z)}\right| + \left| g_i'(z) \left[\frac{z}{g_i(z)}\right]^{\mu_i} \right| \cdot \left|\frac{g_i(z)}{z}\right|^{\mu_i - 1} \right\}. \tag{2.7}$$

Now, applying the General Schwarz Lemma to the functions $g_1, \ldots, g_n$, we obtain

$$|g_i(z)| \le N_i |z|, \quad z \in U,\ i \in \{1, \ldots, n\}. \tag{2.8}$$

Using (2.5), (2.6) and (2.8), inequality (2.7) can be rewritten as follows:

$$\begin{aligned}\left|\frac{zI_n''(z)}{I_n'(z)}\right| &\le \sum_{i=1}^{n}|\alpha_i|\left\{1+|z|M_i+\left[\left|g_i'(z)\left(\frac{z}{g_i(z)}\right)^{\mu_i}-1\right|+1\right]N_i^{\mu_i-1}\right\}\\ &\le \sum_{i=1}^{n}|\alpha_i|\left\{1+|z|M_i+(2-\lambda_i)N_i^{\mu_i-1}\right\}.\end{aligned}$$

Therefore,

$$\begin{aligned}(1-|z|^2)\left|\frac{zI_n''(z)}{I_n'(z)}\right| &\le (1-|z|^2)|z|\sum_{i=1}^{n}|\alpha_i|M_i+(1-|z|^2)\sum_{i=1}^{n}|\alpha_i|[1+(2-\lambda_i)N_i^{\mu_i-1}]\\ &\le \frac{2\sqrt{3}}{9}\sum_{i=1}^{n}|\alpha_i|M_i+\sum_{i=1}^{n}|\alpha_i|[1+(2-\lambda_i)N_i^{\mu_i-1}].\end{aligned}$$

If we make use of the inequality (2.5), we obtain

$$(1-|z|^2)\left|\frac{zI_n''(z)}{I_n'(z)}\right| \le 1, \quad z\in U.$$

Finally, applying Lemma 1.1, we yield that I_n is in the class S. □

If we set $\mu_i = M_i = N_i = 1$ and $\lambda_i = \lambda$, $i \in \{1,\dots,n\}$ in Theorem 2.2, we obtain

Corollary 2.2. *Let $\alpha_1,\dots,\alpha_n \in \mathbb{C}$ and $0 \le \lambda < 1$ such that*

$$\sum_{i=1}^{n}|\alpha_i| \le \frac{9}{27+2\sqrt{3}-9\lambda}.$$

If functions $f_i \in \mathcal{A}$, $g_i \in S^*(\lambda)$ satisfies

$$\left|\frac{f_i''(z)}{f_i'(z)}\right| \le 1, \quad \text{and} \quad |g_i(z)| < 1,$$

for all $i \in \{1,\dots,n\}$, $z \in U$, then the function I_n, given by (1.2) is in the class S.

If we set $n = 1$ in Theorem 2.2, we obtain

Corollary 2.3. *Let the functions $f \in \mathcal{A}$, $g \in B(\mu,\lambda)$. Suppose that $\alpha \in \mathbb{C}$, M, $N \ge 1$ such that*

$$|\alpha|\cdot\left[1+(2-\lambda)N^{\mu-1}+\frac{2\sqrt{3}M}{9}\right] \le 1, \quad z\in U.$$

If

$$\left|\frac{f''(z)}{f'(z)}\right| \le M \quad \text{and} \quad |g(z)| < N,$$

for all $z \in U$, then the function I_α is in the class S.

Theorem 2.3. *Let $\alpha_i \in \mathbb{C}$, M_i, $N_i \ge 1$, $i \in \{1,\dots,n\}$. Suppose that $f_i \in \mathcal{A}$ and $g_i \in B(\mu_i,\lambda_i)$ such that*

$$\left|\frac{f_i''(z)}{f_i'(z)}\right| < M_i, \quad |g_i(z)| < N_i,$$

for all $z \in U$ and $i \in \{1, \dots, n\}$. Then the function I_n is in the class $K(\delta)$, where

$$\delta = 1 - \sum_{i=1}^{n} |\alpha_i| \cdot [1 + M_i + (2 - \lambda_i) N_i^{\mu_i - 1}]$$

and

$$0 < \sum_{i=1}^{n} |\alpha_i| \cdot [1 + M_i + (2 - \lambda_i) N_i^{\mu_i - 1}] \leq 1.$$

Proof. Just as in the proof of Theorem 2.2, by using the hypothesis and the General Schwarz Lemma, we obtain

$$\begin{aligned} \left| \frac{z I_n''(z)}{I_n'(z)} \right| &\leq \sum_{i=1}^{n} |\alpha_i| \left\{ 1 + |z| M_i + \left[\left| g_i'(z) \left(\frac{z}{g_i(z)} \right)^{\mu_i} - 1 \right| + 1 \right] N_i^{\mu_i - 1} \right\} \\ &\leq \sum_{i=1}^{n} |\alpha_i| \left\{ 1 + M_i + (2 - \lambda_i) N_i^{\mu_i - 1} \right\} = 1 - \delta. \end{aligned}$$

Therefore, function $I_n \in K(\delta)$. □

If we set $\delta = 0$, $M_i = M$ $i \in \{1, \dots, n\}$ and $g(z) = z$ in Theorem 2.3 we obtain the following result.

Example 2.1. *Let $\alpha_i \in \mathbb{C}$, $M \geq 1$ and $f_i \in \mathcal{A}$, $i \in \{1, \dots, n\}$. If*

$$\left| \frac{f_i''(z)}{f_i'(z)} \right| < M$$

for all $z \in U$ and $i \in \{1, \dots, n\}$, then the function $I_n(f)$ is convex in U, where $\sum_{i=1}^{n} |\alpha_i| = \frac{1}{3+M}$.

Remark 2.1. *Other interesting corollaries of Theorems 2.1- 2.3 can be obtained by suitably specializing the parameters and the functions involved.*

3. Subordination Results

In view of the results due to Breaz et al. [2] and Bucur et al. [4], we obtain sufficient conditions such that integral operator $I_n \in S_{\frac{b}{2\alpha}}(a)$.

Theorem 3.1. *Let $\alpha_i > 0$, $f_i \in S^*_{b_i}(a_i)$ and $g_i \in C^*_{b_i}(a_i)$, $i \in \{1, \dots, n\}$. Then*

$$\operatorname{Re} \left[\frac{z I_n''(z)}{I_n'(z)} \right] < \frac{\alpha (a-1)}{b (a+1)}, \quad z \in U,$$

where

$$\frac{a-1}{b(a+1)} = \max_{1 \leq i \leq n} \frac{a_i - 1}{b_i (a_i + 1)}$$

and $\sum_{i=1}^{n} \alpha_i = \alpha$. This implies that

$$\left\{ \prod_{i=1}^{n} \left[\frac{z f_i'(z)}{g_i(z)} \right]^{\alpha_i} \right\}^{\frac{b}{2\alpha}} \prec \frac{a(1-z)}{a-z}. \tag{3.1}$$

Proof. Taking the real part of both terms in (2.3), we obtain

$$\mathrm{Re}\left[\frac{zI_n''(z)}{I_n'(z)}\right] = \sum_{i=1}^{n}\alpha_i + \sum_{i=1}^{n}\alpha_i \mathrm{Re}\left[\frac{zf_i''(z)}{f_i'(z)}\right] - \sum_{i=1}^{n}\alpha_i \mathrm{Re}\left[\frac{zg_i'(z)}{g_i(z)}\right]$$
$$< \sum_{i=1}^{n}\frac{(a_i-1)\alpha_i}{b_i(a_i+1)} \leq \frac{a-1}{2\frac{b}{2\alpha}(a+1)}.$$

So, I_n is in the class $S^*_{\frac{b}{2\alpha}}(a)$. By using Theorem 1.1 and the definition of the class $S_b(a)$, we deduce that inequality (3.1) take place. □

Corollary 3.1. *If $\alpha_i > 0$, $f_i \in S^*_b(a)$ and $g_i \in C^*_b(a)$, $i \in \{1, \dots, n\}$, then I_n is in the class $S^*_{\frac{b}{2\alpha}}(a)$.*

If we set $n = 1$, we obtain:

Corollary 3.2. [4] *If $\alpha > 0$, $f \in S^*_b(a)$ and $g \in C^*_b(a)$, then*

$$\left[\frac{zf'(z)}{g(z)}\right]^{\frac{b}{2}} \prec \frac{a(1-z)}{a-z}.$$

Example 3.1. *Let us consider the functions f_i and g_i, $i \in \{1, \dots, n\}$ which satisfy*

$$\prod_{i=1}^{n}\left[\frac{zf_i'(z)}{g_i(z)}\right]^{\alpha_i} = (1-z)^{p-1}, \quad z \in U,$$

where $p = \frac{2(1-a)\alpha}{b(a+1)} + 1$. Therefore, we have

$$I_n(z) = \frac{1}{p}\left[1-(1-z)^p\right], \quad z \in U,$$

and this implies that

$$\mathrm{Re}\left[\frac{zI_n''(z)}{I_n'(z)}\right] = \mathrm{Re}\left[\frac{(1-p)z}{1-z}\right] < \frac{\alpha(a-1)}{b(a+1)}, \quad z \in U.$$

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