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GALERKIN METHOD FOR FRACTIONAL DIFFUSION EQUATION

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ABSTRACT. The one-dimensional fractional diffusion equation is studied systematically using the Galerkin method. The Caputo fractional derivative is used for formulation. An example is solved to assess the accuracy of the method. The numerical results are obtained for different values (n) of equation. An effective and easy-to-use method for solving such equations is needed.

1. INTRODUCTION

Fractional diffusion equations have attracted during the last few decades for modelling many physical and chemical processes and in engineering. Many authors have presented the existence and approximations of the solutions to one-dimensional fractional diffusion equation. In [1] two-step Adomian decomposition method is used analytical solution for the space fractional diffusion equation. Mingrong Cui [2] proposed high-order compact finite difference scheme and analysis the condition for stability. Finite difference method is presented for this problem and some examples are given in [3]. Also in [4] a class of initial-boundary value fractional diffusion equations with variable coefficients on a finite domain are examined using numerical method and analysis of stability, consistency and convergence. The analytical solutions of the space fractional diffusion equations are presented by modified decomposition method [5]. Ray examined the analytical solutions of the space fractional diffusion equations by two-step Adomian decomposition method [6]. In this paper, we consider one-dimensional fractional diffusion equation:

$$\frac{\partial u(x,t)}{\partial t} = d(x)\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} + q(x,t) \tag{1.1}$$

with initial condition $u(x,0) = f(x)$, $0 \le x \le 1$ and boundary conditions $u(0,t) = g_0(t)$, $u(1,t) = g_1(t)$, $t \ge 0$ where $d(x)$ represents the diffusion coefficent and $q(x,t)$ the source/sink function. Sources provide energy or material to the system where sinks

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absorb energy or material. Eq. (1.1) becomes the classical diffusion equation for $\alpha = 2$. It models a superdiffuse flow for $1 < \alpha < 2$ and a classical advective flow for $\alpha = 1$ [4].

In this paper, finite element Galerkin method is considered for numerical solution of one-dimensional fractional diffusion equation.

The paper has been organized as follows: In section 3, Caputo fractional derivative is given briefly. In section 4, finite element Galerkin method is investigated and analysis of the method is given. A numerical example is given in section 5 and conclusion is given in section 6.

2. Caputo Fractional Derivative

There are various kind of fractional derivatives that widely used ones are the Grunwald-Letnikov, the Riemann-Liouville and the Caputo fractional derivatives. Caputo fractional derivative is a regularization in the time origin for the Riemann-Liouville derivative [7, 8]. A nice comparison of these definitions from the view point of their applications in physics and engineering can be found in [9]. In this study, we use the Caputo fractional derivative that is defined as follow [10]:

$$D^{\alpha}_{*x} f(x) = J^{m-\alpha} D^{m} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t) dt$$

for $m-1 < \alpha \leq m$ and $m \in N$.

3. Galerkin Method

A usual scalar product for two real valued functions $u(x)$ and $v(x)$ is defined by $< u, v >= \int_0^T u(x)v(x)dx$, $u(x)$ and $v(x)$ are orthogonal if $< u, v >= 0$. And a norm associated with this scalar product is defined by

$$\|u\| = \sqrt{< u, u >} = (\int_0^T |u(x)|^2 dx)^{\frac{1}{2}}.$$

Let

(i) $T_h : 0 = x_0 < x_1 < ... < x_M < x_{M+1} = 1$ be a partition of $(0, 1)$, $h_j = x_j - x_{j-1}$.
(ii) $V_h^0 = v : v$, continuous and piecewise linear function on T_h. with $v(0) = v(1) = 0$
(iii) $\{\varphi_j\}$, $j = 1, ..., M$ be a basis function for V_h where

$$\varphi_j(x) = \begin{cases} \frac{x-x_{j-1}}{h_j}, & x_{j-1} \leq x \leq x_j \\ \frac{x_{j+1}-x}{h_{j+1}}, & x_j \leq x \leq x_{j+1} \\ 0 \quad , & \text{otherwise} \end{cases} .$$

To illustrate the application of the Galerkin method, firstly, we should modify the equation (1.1). At the grid point (x_i, u_i), the proposed equation may be discretized by using Caputo fractional derivative

$$\frac{u_i - f_i}{k} = d_i \left[\frac{1}{\Gamma(0.2)} \int_0^{x_i} (x_i - \mu)^{-0.8} u_i''(\mu) d\mu \right] + q_i, \tag{3.1}$$

where $d_i := d(x_i)$, $f_i := f(x_i) = u_{i-1}$, and $q_i := q(x_i, t_j)$. For $\alpha = 1.8$ the previous discretization will be as follow:

$$\frac{u_i - f_i}{k} = \frac{d_i}{\Gamma(0.2)} I + q_i,$$

where $I = \int_0^{x_i} (x_i - \mu)^{-0.8} u_i^{''}(\mu) d\mu$. Applying integration by parts two times, we obtain

$$I = \frac{36}{25} \int_0^{x_i} (x_i - \mu)^{-2.8} u_i(\mu) d\mu.$$

Expanding $u_i(\mu)$ in Taylor series about a point $\mu = x_i$ and then substituting I in (3.1), we obtain

$$\begin{aligned}\frac{u_i - f_i}{k} =& \frac{d_i}{\Gamma(0.2)} \frac{36}{25} \left[(u_i(x_i) \int_0^{x_i} (x_i - \mu)^{-2.8} d\mu + u_i^{'}(x_i) \int_0^{x_i} (x_i - \mu)^{-1.8} d\mu \right. \\ &+ \left. \frac{u_i^{''}(x_i)}{2} \int_0^{x_i} (x_i - \mu)^{-0.8} d\mu) \right] + q_i.\end{aligned}$$

After algebraic manipulations we get

$$2akx^3 u^{''} - akx^2 u^{'} + (1 - kxb)u - f(x) - k(1 + x)e^{-t} x^3 = 0 \tag{3.2}$$

where $a = \frac{-3\Gamma(2.2)}{10\Gamma(0.2)}$, $b = \frac{-2\Gamma(2.2)}{15\Gamma(0.2)}$.

Let apply Galerkin method to the equation (3.2): Find the approximate solution $U(x) \in V_h^0$ $\forall W(x) \in V_h^0$ such that

$$\int_0^1 W \left(2akx^3 U^{''} - akx^2 U^{'} + (1 - kxb)U - f(x) - k(1 + x)e^{-t} x^3 \right) dx = 0.$$

So we get

$$\begin{aligned}&\int_0^1 \left(2akx^3 W U^{''} - akx^2 W U^{'} + (1 - kxb) W U \right) dx \\ &= \int_0^1 (W f(x) + W k(1 + x)e^{-t} x^3) dx.\end{aligned} \tag{3.3}$$

By applying integration by parts to $\int_0^1 2akx^3 W U^{''} dx$, we get

$$\int_0^1 2akx^3 W U^{''} dx = [2akx^3 W U^{'}]_0^1 - \int_0^1 (6akx^2 W U^{'} + 2akx^3 W^{'} U^{'}) dx \tag{3.4}$$

$[2akx^3 W U^{'}]_0^1 = 0$ since $W(0) = W(1) = 0$. By substituting (3.4) in (3.3), we get

$$\begin{aligned}&\int_0^1 \left((1 - kxb) W U - 7akx^2 W U^{'} - 2akx^3 W^{'} U^{'} \right) dx \\ &= \int_0^1 (W f(x) + W k(1 + x)e^{-t} x^3) dx.\end{aligned} \tag{3.5}$$

We may find the approximate solution $U(x) \in V_h^0$ by using basis functions $\varphi_j(x)$ as

$$U(x) = \sum_{j=1}^n c_j \varphi_j(x), \quad U'(x) = \sum_{j=1}^n c_j \varphi_j'(x), \quad W(x) = \sum_{i=1}^n s_i \varphi_i(x), \quad W'(x) = \sum_{i=1}^n s_i \varphi_i'(x).$$

If we use these identities in (3.5), then we get

$$\int_0^1 [\sum_{i=1}^n (1-kxb)s_i\varphi_i(x)\sum_{j=1}^n c_j\varphi_j(x) - \sum_{i=1}^n 7akx^2 s_i\varphi_i(x)\sum_{j=1}^n c_j\varphi_j'(x)$$
$$-\sum_{i=1}^n 2akx^3 s_i\varphi_i'(x)\sum_{j=1}^n c_j\varphi_j'(x)]dx$$
$$=\int_0^1 \left[\sum_{i=1}^n s_i\varphi_i(x)f(x) + \sum_{i=1}^n s_i\varphi_i(x)k(1+x)e^{-t}x^3\right] dx.$$

For $|i-j|>1$ we have $\int_0^1 \varphi_j'\varphi_i dx = 0$ and $\int_0^1 \varphi_j\varphi_i dx = 0$, since if so then we have that φ_j and φ_i have non-overlapping supports. So we get

$$\sum_{i=1}^n s_i \int_0^1 \sum_{j=1}^n c_j \left((1-kxb)\varphi_j(x)\varphi_i(x) - 7akx^2\varphi_j'(x)\varphi_i(x) - 2akx^3\varphi_j'(x)\varphi_i'(x)\right) dx$$
$$=\sum_{i=1}^n s_i \int_0^1 \left(\varphi_i(x)f(x) + \varphi_i(x)k(1+x)e^{-t}x^3\right) dx.$$

The method is described in matrix form in the following way: For $i=2$, $j=1,..,n$

$$\alpha_{12} = \int_h^{2h} (a_1(x)\varphi_1\varphi_2 + a_2(x)\varphi_1'\varphi_2 + a_3(x)\varphi_1'\varphi_2')dx,$$
$$\alpha_{22} = \int_h^{3h} (a_1(x)\varphi_2\varphi_2 + a_2(x)\varphi_2'\varphi_2 + a_3(x)\varphi_2'\varphi_2')dx,$$
$$\alpha_{32} = \int_{2h}^{3h} (a_1(x)\varphi_3\varphi_2 + a_2(x)\varphi_3'\varphi_2 + a_3(x)\varphi_3'\varphi_2')dx,$$

for $i=m$, $j=1,..,n$

$$\alpha_{(m-1)m} = \int_{(m-1)h}^{mh} (a_1(x)\varphi_{m-1}\varphi_m + a_2(x)\varphi_{m-1}'\varphi_m + a_3(x)\varphi_{m-1}'\varphi_m')dx,$$
$$\alpha_{mm} = \int_{(m-1)h}^{(m+1)h} (a_1(x)\varphi_m\varphi_m + a_2(x)\varphi_m'\varphi_m + a_3(x)\varphi_m'\varphi_m')dx,$$
$$\alpha_{(m+1)m} = \int_{mh}^{(m+1)h} (a_1(x)\varphi_{m+1}\varphi_m + a_2(x)\varphi_{m+1}'\varphi_m + a_3(x)\varphi_{m+1}'\varphi_m')dx,$$

and for $i=n-1$, $j=1,..,n$

$$\alpha_{(n-2)(n-1)} = \int_{(n-1)h}^{(n-2)h} (a_1(x)\varphi_{n-2}\varphi_{n-1} + a_2(x)\varphi_{n-2}'\varphi_{n-1} + a_3(x)\varphi_{n-2}'\varphi_{n-1}')dx,$$
$$\alpha_{(n-1)(n-1)} = \int_{(n-2)h}^{(n)h} (a_1(x)\varphi_{n-1}\varphi_{n-1} + a_2(x)\varphi_{n-1}'\varphi_{n-1} + a_3(x)\varphi_{n-1}'\varphi_{n-1}')dx,$$
$$\alpha_{(n)(n-1)} = \int_{nh}^{(n-1)h} (a_1(x)\varphi_n\varphi_{n-1} + a_2(x)\varphi_n'\varphi_{n-1} + a_3(x)\varphi_n'\varphi_{n-1}')dx,$$

where $a_1(x) := 1 - kx_i b$, $a_2(x) := -7akx_i^2$ and $a_3(x) := -2akx_i^3$. So we get the matrices:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \alpha_{12} & \alpha_{22} & \alpha_{32} & 0 & \dots & 0 & 0 \\ 0 & \alpha_{23} & \alpha_{33} & \alpha_{43} & 0 & \dots & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & \dots & 0 & 0 & \alpha_{(n-2)(n-1)} & \alpha_{(n-1)(n-1)} & \alpha_{n(n-1)} \\ . & . & . & . & 0 & 0 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} u(0,t) = 0 \\ \int_h^{3h} \varphi_2(f(x) + k(1+x)e^{-t}x^3)dx \\ \int_{2h}^{4h} \varphi_3(f(x) + k(1+x)e^{-t}x^3)dx \\ . \\ . \\ . \\ \int_{(n-1)h}^{(n+1)h} \varphi_n(f(x) + k(1+x)e^{-t}x^3)dx \\ u(1,t) = e^{-t} \end{bmatrix}$$

$$C = [c_1, c_2, ..., c_n]'$$

and

$$AC = B.$$

Finally the approximate solution U is obtained by solving C by using Matlab 9.1. The maximum absolute errors are listed in Table 1.

Table 1: Maximum absolute errors, $k = 0.01$

n	Galerkin metodu	[11]	[4]
11	9.242e-04	0.10446	1.822e-03
21	5.762e-04	0.10518	1.168e-03
61	1.723e-04		8.644e-04
121	6.175e-05		6.848e-04

4. Conclusion

In this paper, finite element method with Galerkin formula is applied for the numerical solution of the fractional diffusion equation and the maximum absolute errors have shown in Table 1, which shows that this method approximate the exact solution very well. The implementation of the present method is more computational than the existing methods.

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