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QUASI CONFORMAL HARMONIC MAPPINGS RELATED TO CONVEX FUNCTIONS

ARZU YEMISCI

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Abstract. Let $f(z) = h(z) + \overline{g(z)}$ be an univalent sense-preserving harmonic mapping in the open unit disc $\mathbb{D} = \{z| \ |z| < 1\}$. If f satisfies the condition $|\omega(z)| = \left|\frac{g'(z)}{h'(z)}\right| < k$ $(0 \leq k < 1)$, then f is called $k-$quasiconformal harmonic mapping in $\mathbb{D}$ [4]. The class of such mappings is denoted by $S_{H(k)}$.

The aim of this paper is to give some properties of the solution of non-linear partial differential equation $\overline{f_{\overline{z}}} = \omega(z) f_z$ under the condition $|\omega| < k$ $(0 \leq k < 1)$, $\omega(z) \prec \frac{k^2(b_1 - z)}{k^2 - \overline{b_1} z}$, $h(z) \in \mathcal{C}$ and $h(z) \in \mathcal{K}$. The proofs of this paper are based on the idea of Robinson [5].

1. Introduction

Let Ω be the family of functions $\phi(z)$ regular in the open unit disc $\mathbb{D}$ that satisfy the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in \mathbb{D}$.

Next, let $\mathcal{A}$ denote the class of analytic functions of the form $s(z) = z + \sum_{n=2}^{\infty} c_n z^n$ in the open unit disc $\mathbb{D}$. Let $\mathcal{P}$ designate the class of functions $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ which are analytic, have positive real part in $\mathbb{D}$ and such that $p(z)$ is in $\mathcal{P}$ if and only if

$$p(z) = \frac{1 + \phi(z)}{1 - \phi(z)}$$

for some function $\phi(z) \in \Omega$ and every $z \in \mathbb{D}$.

Moreover, let $\mathcal{C}$ denote the family of functions $s(z) \in \mathcal{A}$, such that $s(z)$ is in $\mathcal{C}$ if and only if

$$1 + z\frac{s''(z)}{s'(z)} = p(z)$$

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for some $p(z) \in \mathcal{P}$ and all $z \in \mathbb{D}$. A function $s_1(z)$ from $\mathcal{A}$ that satisfies the condition

$$Re\frac{s_1'(z)}{s_2'(z)} > 0,$$

where $s_2(z) \in \mathcal{C}$, is called a close-to-convex function [2]. The class of all such functions is denoted by $\mathcal{K}$. Let $F_1(z)$ and $F_2(z)$ be elements of $\mathcal{A}$. If there exists a function $\phi(z) \in \Omega$ such that $F_1(z) = F_2(\phi(z))$ for every $z \in \mathbb{D}$, then we say that $F_1(z)$ is subordinate to $F_2(z)$ and we write $F_1(z) \prec F_2(z)$. Specially, if $F_2(z)$ is univalent in $\mathbb{D}$, then $F_1(z) \prec F_2(z)$ if and only if $F_1(\mathbb{D}) \subset F_2(\mathbb{D})$ and $F_1(0) = F_2(0)$ implies $F_1(\mathbb{D}_r) \subset F_2(\mathbb{D}_r)$, where $\mathbb{D}_r = \{z| \; |z| < r, \quad 0 < r < 1\}$ (subordination and Lindelöf principle [2]).

Finally, a planar harmonic mapping in the open unit disc $\mathbb{D}$ is a complex-valued harmonic function f, which maps $\mathbb{D}$ onto the some planar domain $f(\mathbb{D})$. Since $\mathbb{D}$ is a simply connected domain the mapping f has a canonical decomposition $f(z) = h(z) + \overline{g(z)}$, where $h(z)$ and $g(z)$ are analytic in $\mathbb{D}$ and have the following power series expansions

$$h(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=0}^{\infty} b_n z^n,$$

where $a_n, b_n \in C$, $n = 0, 1, 2, \ldots$. As usual, we call $h(z)$ the analytic part of f and $g(z)$ the co-analytic part of f. An elegant and complete account of the theory of the harmonic mappings is given Duren's monograph [1]. Lewi proved in 1936 that the harmonic function f is locally univalent if and only if its Jacobian

$$J_f = |h'(z)|^2 - |g'(z)|^2$$

is different from zero in $\mathbb{D}$. In view of this result, locally univalent harmonic mappings in the open unit disc $\mathbb{D}$ are either sense-preserving if $|g'(z)| < |h'(z)|$ or sense-reversing if $|g'(z)| > |h'(z)|$ in $\mathbb{D}$.

In this paper we will restrict ourselves to the study of sense-preserving harmonic mappings. We will also note that $f(z) = h(z) + \overline{g(z)}$ is sense-preserving in $\mathbb{D}$ if and only if $h'(z)$ does not vanish in $\mathbb{D}$ and the second dilation $\omega(z) = \frac{g'(z)}{h'(z)}$ has the property $|\omega(z)| < 1$ for all $z \in \mathbb{D}$. Therefore the class of all sense-preserving harmonic mappings in the open unit disc $\mathbb{D}$ with $a_0 = b_0 = 0$ and $a_1 = 1$ will be denoted by S_H. Thus, S_H contains standard class S of univalent functions. The family of all mappings S_H with additional property $g'(0) = 0$, i.e., $b_1 = 0$, is denoted by S_H^0. Hence, it is clear that $S \subset S_H^0 \subset S_H$. For the aim of this paper we will need the following lemma and theorems.

Lemma 1.1. *([3]) Let $\phi(z)$ be a non-constant analytic function in the open unit disc $\mathbb{D}$ with $\phi(0) = 0$. If $|\phi(z)|$ attains its maximum value on the circle $|z| = r$ at z_0, then $z_0 \cdot \phi'(z_0) = m\phi(z_0)$, $m \geq 1$.*

Theorem 1.1. *([2]) Let $s(z)$ be an element of $\mathcal{C}$. Then* $Re\left(z\frac{s'(z)}{s(z)}\right) > \frac{1}{2}$.

Theorem 1.2. *Let $s(z)$ be an element of $\mathcal{C}$. Then,*

$$\frac{r}{1+r} \leq |s(z)| \leq \frac{r}{1-r},$$

$$\frac{1}{(1+r)^2} \leq |s'(z)| \leq \frac{1}{(1-r)^2}.$$

2. Main Results

Theorem 2.1. *Let $f(z) = h(z) + \overline{g(z)}$ be an element of S_H and $h(z) \in C$. Then the solution of the non-linear elliptic partial differential equation $\overline{f}_{\overline{z}} = \omega(z) f_z$ under the condition $|\omega(z)| < k$ $(0 \le k < 1)$, $\omega(z) \prec \dfrac{k^2(b_1 - z)}{k^2 - \overline{b_1} z}$ is*

$$\frac{g(z)}{h(z)} = \frac{k^2(b_1 - \phi(z))}{k^2 - \overline{b_1}\phi(z)}, \quad \phi(z) \in \Omega$$

Proof. We consider the linear transformation $w = \dfrac{k^2(b_1 - z)}{k^2 - b_1 z}$. This transformation maps $|z| < k$ onto itself and

$$\omega(z) = \frac{(b_1 z + b_2 z^2 + \dots)'}{(z + a_2 z^2 + \dots)'} = \frac{b_1 + 2b_2 z + 3b_3 z^2 + \dots}{1 + 2a_2 z + 3a_3 z^2 + \dots} \quad \Longrightarrow \quad \omega(0) = b_1.$$

Therefore, the function

$$\phi(z) = \frac{k^2(b_1 - \omega(z))}{k^2 - \overline{b_1}\omega(z)}$$

satisfies the condition of Schwarz lemma, and we have

$$\omega(z) = \frac{g'(z)}{h'(z)} \prec \frac{k^2(b_1 - z)}{k^2 - \overline{b_1} z}.$$

On the other hand, the transformation $w = \dfrac{k^2(b_1 - z)}{k^2 - \overline{b_1} z}$ maps $|z| = r$ onto the disc with centre

$$C(r) = \left(\frac{k^2(1 - r^2) Re b_1}{k^2 - |b_1|^2 r^2}, \frac{k^2(1 - r^2) Im b_1}{k^2 - |b_1|^2 r^2} \right)$$

and radius

$$\rho(r) = \frac{k(k^2 - |b_1|^2) r}{k^2 - |b_1|^2 r^2}.$$

Using the subordination principle, we have

$$\omega(\mathbb{D}_r) = \left\{ \frac{g'(z)}{h'(z)} \mid \left| \omega(z) - \frac{k^2(1 - r^2) b_1}{k^2 - |b_1|^2 r^2} \right| < \frac{k(k^2 - |b_1|^2) r}{k^2 - |b_1|^2 r^2}, \quad 0 < r < 1 \right\}. \tag{2.1}$$

Now, we define the function $\phi(z)$ by

$$\frac{g(z)}{h(z)} = \frac{k^2(b_1 - \phi(z))}{k^2 - \overline{b_1}\phi(z)}. \tag{2.2}$$

Then $\phi(z)$ is analytic and $\phi(0) = 0$. If we take the derivative from (2.2), after brief calculations we get

$$\omega(z) = \frac{g'(z)}{h'(z)} = \frac{k^2(b_1 - \phi(z))}{k^2 - \overline{b_1}\phi(z)} + \frac{k^2(|b_1|^2 + k^2 - 2b_1\phi(z)) z\phi'(z)}{(k^2 - \overline{b_1}\phi(z))^2} \cdot \frac{h(z)}{zh'(z)}. \tag{2.3}$$

On the other hand, since $h(z) \in C$ then $h(z)$ satisfies the condition $Re\left(z\frac{h'(z)}{h(z)}\right) > \dfrac{1}{2}$ (Theorem 1.1), then $z\frac{h'(z)}{h(z)} < \dfrac{1}{1 - z} \Longrightarrow \dfrac{h(z)}{zh'(z)} = 1 - \phi(z)$, thus the equality (2.3) can be written in the following form

$$\omega(z) = \frac{g'(z)}{h'(z)} = \frac{k^2(b_1 - \phi(z))}{k^2 - \overline{b_1}\phi(z)} + \frac{k^2(|b_1|^2 + k^2 - 2b_1\phi(z))z\phi'(z)}{(k^2 - \overline{b_1}\phi(z))^2} \cdot (1 - \phi(z)).$$

Now it is easy to realize that the subordination

$$\frac{g(z)}{h(z)} = \frac{k^2(b_1 - \phi(z))}{k^2 - \overline{b_1}\phi(z)} \Longleftrightarrow \frac{g(z)}{h(z)} \prec \frac{k^2(b_1 - z)}{k^2 - \overline{b_1}z}$$

is equivalent to $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. Indeed, if we assume the contrary, then there exists $z_0 \in \partial\mathbb{D}_r$ such that $|\phi(z_0)| = 1$. So, by Jack lemma (Lemma 1.1), $z_0\phi'(z_0) = m\phi(z_0)$, $m \geq 1$. For such z_0 we have

$$\begin{aligned} \omega(z_0) &= \frac{g'(z_0)}{h'(z_0)} \\ &= \frac{k^2(b_1 - \phi(z_0))}{k^2 - \overline{b_1}\phi(z_0)} + \frac{k^2(|b_1|^2 + k^2 - 2b_1\phi(z_0))m\phi(z_0)}{(k^2 - \overline{b_1}\phi(z_0))^2} \cdot (1 - \phi(z_0)) \notin \omega(\mathbb{D}_r), \end{aligned}$$

but this contradicts with (2.1). So, our assumption is wrong, i.e., $|\phi(z)| < 1$ for every $z \in \mathbb{D}$. □

Remark 2.1. *The solution set of the non-linear elliptic partial differential equation* $\overline{f_{\overline{z}}} = \omega(z)f_z$ *under the conditions* $|\omega(z)| < k$, $(0 \leq k < 1)$, $\omega(z) \prec \frac{k^2(b_1 - z)}{k^2 - \overline{b_1}z}$ *can be denoted by*

$$\mathcal{B} = \left\{ f(z) = h(z) + \overline{g(z)} \mid \frac{g(z)}{h(z)} = \frac{k^2(b_1 - \phi(z))}{k^2 - \overline{b_1}\phi(z)} \text{ for some } \phi(z) \in \Omega \text{ and every } z \in \mathbb{D} \right\}.$$

Corollary 2.1. *Let* $f(z) = h(z) + \overline{g(z)}$ *be an element of* $\mathcal{B}$. *Then*

$$\left.\begin{aligned} rF(k, |b_1|, -r) \leq |g(z)| \leq rF(k, |b_1|, r) \\ G(k, |b_1|, -r) \leq |g'(z)| \leq G(k, |b_1|, r) \end{aligned}\right\}, \tag{2.4}$$

where $F(k, |b_1|, r) = \frac{k(|b_1| + kr)}{(1 - r)(k + |b_1|r)}$ *and* $G(k, |b_1|, r) = \frac{k(|b_1| + kr)}{(1 - r)^2(k + |b_1|r)}$.

Proof. Since $f(z) = h(z) + \overline{g(z)} \in \mathcal{B}$, we have

$$\left.\begin{aligned} \frac{k(|b_1| - kr)}{(k + |b_1|r)} \leq |\omega(z)| = \left|\frac{g'(z)}{h'(z)}\right| \leq \frac{k(|b_1| + kr)}{(k + |b_1|r)} \\ \frac{k(|b_1| - kr)}{(k + |b_1|r)} \leq \left|\frac{g(z)}{h(z)}\right| \leq \frac{k(|b_1| + kr)}{(k + |b_1|r)} \end{aligned}\right\}. \tag{2.5}$$

Using Theorem 1.2 in the inequalities (2.4) we get (2.5). □

Corollary 2.2. *Let* $f(z) = h(z) + \overline{g(z)}$ *be an element of* $\mathcal{B}$. *Then*

$$\frac{1}{(1 + r)^4}F_2(k, |b_1|, r) \leq |J_f| \leq \frac{1}{(1 - r)^4}F_1(k, |b_1|, r) \tag{2.6}$$

where

$$F_1(k,|b_1|,r)=\frac{\left[(k+k|b_1|)-(|b_1|+k^2)r\right]\left[(k-k|b_1|)-(|b_1|-k^2)r\right]}{(k-|b_1|r)^2}$$

and

$$F_2(k,|b_1|,r)=\frac{\left[(k+k|b_1|)+(|b_1|+k^2)r\right]\left[(k-k|b_1|)+(|b_1|-k^2)r\right]}{(k+|b_1|r)^2}.$$

Proof. Using Theorem 2.1, we can write

$$F_2(k,|b_1|,r)\le(1-|\omega(z)|^2)\le F_1(k,|b_1|,r). \tag{2.7}$$

On the other hand

$$J_f=|h'(z)|^2-|g'(z)|^2=|h'(z)|^2(1-|\omega(z)|^2). \tag{2.8}$$

Considering (2.7), (2.8) and Theorem 1.2, after calculations we get (2.6). □

Corollary 2.3. *Let* $f(z)=h(z)+\overline{g(z)}\in\mathcal{B}$. *Then*

$$\int_0^r\frac{k(1-|b_1|)+(|b_1|-k^2)\rho}{(1+\rho)^2(k+|b_1|\rho)}d\rho\le|f|\le\int_0^r\frac{k(1+|b_1|)+(|b_1|+k^2)\rho}{(1-\rho)^2(k+|b_1|\rho)}d\rho. \tag{2.9}$$

Proof. Using Theorem 2.1 we obtain

$$\frac{k(1+|b_1|)-(|b_1|+k^2)r}{k-|b_1|r}\le(1+|\omega(z)|)\le\frac{k(1+|b_1|)+(|b_1|+k^2)r}{k+|b_1|r} \tag{2.10}$$

and

$$\frac{k(1-|b_1|)+(|b_1|-k^2)r}{k+|b_1|r}\le(1-|\omega(z)|)\le\frac{k(1-|b_1|)-(|b_1|-k^2)r}{k-|b_1|r}. \tag{2.11}$$

On the other hand we have

$$(|h'(z)|-|g'(z)|)|dz|\le d|f|\le(|h'(z)|+|g'(z)|)|dz|\Longrightarrow$$

$$|h'(z)|(1-|\omega(z)|)|dz|\le d|f|\le|h'(z)|(1+|\omega(z)|)|dz|. \tag{2.12}$$

Considering (2.10), (2.11) and (2.12), after integration, we get (2.9). □

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Department of Mathematics and Computer Sciences
Istanbul Kultur University
Istanbul, Turkey
E-mail address: a.sen@iku.edu.tr