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PRESERVING PROPERTIES OF SUBCLASSES OF p-VALENT FUNCTIONS BY USING INTEGRAL OPERATORS

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ABSTRACT. In this paper we study the preserving properties of a subclass of p-valent functions by using integral operators.

1. INTRODUCTION

Let A_p denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic and p-valent in the unit disk $U = \{z : |z| < 1\}$. Also denote by T_p the class of functions of the form

$$f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k, (a_k \geq 0, z \in U), \tag{1.2}$$

which are analytic and p-valent in U.

Definition 1.1. ([2]) *Let $I_{A,p}$ be a Alexander type integral operator defined as:*

$$I_{A,p} : A_p \to A_p,\ I_{A,p}(F) = f,\ p \in \mathbb{N},$$

where

$$f(z) = p \int_0^z \frac{F(t)}{t} dt. \tag{1.3}$$

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Definition 1.2. ([2]) *Let $I_{a,p}$ be a Bernardi type integral operator defined as:*

$$I_{a,p} : A_p \to A_p,\ I_{a,p}(F) = f,\ a = 1, 2, 3, ..., p \in \mathbb{N},$$

where

$$f(z) = \frac{p+a}{z^a} \int_0^z F(t) \cdot t^{a-1} dt. \tag{1.4}$$

Definition 1.3. ([2]) *Let $L_{a,p}$ be a generalization of the previously integral operator defined as:*

$$L_{a,p} : A_p \to A_p, L_{a,p}(F) = f,\ a \in \mathbb{R},\ a \geq 0,\ p \in \mathbb{N},$$

where

$$f(z) = \frac{p+a}{z^a} \int_0^z F(t) \cdot t^{a-1} dt. \tag{1.5}$$

Definition 1.4. ([1]) *Let $I_{c+\delta,p}$, be the integral operator defined as:*

$$I_{c+\delta,p} : A_p \to A_p,\ 0 < u \leq 1,\ 1 \leq \delta < \infty,\ 0 < c < \infty,$$

$$f(z) = I_{c+\delta,p}(F)(z) = (c + \delta + p - 1) \int_0^1 u^{c+\delta-2} F(uz) du. \tag{1.6}$$

Remark 1.1. ([2]) *For $\delta = 1$ and $c = 1, 2, ...$, from the integral operator $I_{c+\delta,p}$ we obtain the Bernardi integral operator defined by (1.4).*

2. Preliminary results

The Sălăgean differential operator [3] can be generalized for a function $f(z) \in A_p$ as follows

$$S^0_{\delta,p} f(z) = f(z),$$
$$S^1_{\delta,p} f(z) = (1 - \delta) f(z) + \delta \tfrac{z f'(z)}{p} = S_{\delta,p} f(z),$$
$$\vdots$$
$$S^n_{\delta,p} f(z) = S_{\delta,p}(S^{n-1}_{\delta,p} f(z)),\ (n \in \mathbb{N}, \delta \geq 0, z \in U).$$

The n-th Ruscheweyh derivative for a function $f(z) \in A_p$, is defined by

$$R^n_p f(z) = \frac{z^p}{n!} \frac{d^n}{dz^n}(z^{n-p} f(z)), (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, z \in U).$$

It can be easily seen that the operators S^n_p and R^n_p on the function $f(z) \in A_p$ are given by

$$S^n_{\delta,p} f(z) = z^p + \sum_{k=p+1}^{\infty} \left(1 + (\frac{k}{p} - 1)\delta\right)^n a_k z^k,$$

and

$$R^n_p f(z) = z^p + \sum_{k=p+1}^{\infty} C^n_{n+k-p} a_k z^k,$$

where $C^n_{n+k-p} = \frac{(n+k-p)!}{n!(k-p)!}$.

Definition 2.1. ([3]) *Let $n \in \mathbb{N}_0$ and $\lambda \geq 0$. Let $D^n_{\lambda,\delta,p}f$ denote the operator defined by*

$$D^n_{\lambda,\delta,p} : A_p \to A_p,$$

$$D^n_{\lambda,\delta,p}f(z) = (1-\lambda)S^n_{\delta,p}f(z) + \lambda R^n_p f(z),\ z \in U.$$

Notice that $D^n_{\lambda,\delta,p}$ is a linear operator and for $f(z) \in A_p$ we have

$$D^n_{\lambda,\delta,p}f(z) = z^p + \sum_{k=p+1}^{\infty} \phi_k(n,\lambda,\delta,p)a_k z^k,$$

where

$$\phi_k(n,\lambda,\delta,p) = \left[(1-\lambda)\left(1+(\frac{k}{p}-1)\delta\right)^n + \lambda C^n_{n+k-p}\right]. \tag{2.1}$$

It is clear that $D^0_{\lambda,\delta,p}f(z) = f(z)$ and $D^1_{\lambda,1,p}f(z) = \frac{z}{p}f'(z)$.

Definition 2.2. ([3]) *For $-p \leq \alpha < p$, $\beta \geq 0$, we let $S^n_p(\alpha,\beta,\lambda,\delta)$, be the subclass of A_p consisting of functions $f(z)$ of the form (1.1) and satisfying the following condition*

$$Re\left\{\frac{z(D^n_{\lambda,\delta,p}f(z))'}{D^n_{\lambda,\delta,p}f(z)} - \alpha\right\} \geq \beta\left|\frac{z(D^n_{\lambda,\delta,p}f(z))'}{D^n_{\lambda,\delta,p}f(z)} - p\right|,$$

also let $T^n_p(\alpha,\beta,\lambda,\delta) = S^n_p(\alpha,\beta,\lambda,\delta) \cap T_p$.

Theorem 2.1. ([3]) *A function $f(z)$ defined by (1.2) is in the class $T^n_p(\alpha,\beta,\lambda,\delta)$, $-p \leq \alpha < p$, $\beta \geq 0$, if and only if*

$$\sum_{k=p+1}^{\infty} \{k(1+\beta) - (\alpha+p\beta)\}\phi_k(n,\lambda,\delta,p)a_k \leq (p-\alpha), \tag{2.2}$$

where $\phi_k(n,\lambda,\delta,p)$ is given by (2.1) and the result is sharp.

Corollary 2.1. ([3]) *Let the function $f(z)$ defined by (1.2) be in the class $T^n_p(\alpha,\beta,\lambda,\delta)$, $-p \leq \alpha < p$, $\beta \geq 0$, then*

$$a_k \leq \frac{(p-\alpha)}{\{k(1+\beta) - (\alpha+p\beta)\}\phi_k(n,\lambda,\delta,p)},\ k \geq p+1.$$

3. Main results

Theorem 3.1. *The Alexander type integral operator defined by (1.3) preserves the class $T^n_p(\alpha,\beta,\lambda,\delta)$, that is: If $F \in T^n_p(\alpha,\beta,\lambda,\delta)$, then $f(z) = I_{A,p}F(z) \in T^n_p(\alpha,\beta,\lambda,\delta)$, for $F(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k$, $(a_k \geq 0, p \in \mathbb{N} = \{1,2,3,...\})$.*

Proof. Let $\subset T_p$, $F(z) = z^p - \sum\limits_{k=p+1}^{\infty} a_k z^k$, $a_k \geq 0$. Then

$$f(z) = I_{A,p}F(z) = p\int_0^z \frac{F(t)}{t}dt = p\int_0^z \frac{1}{t}\left(t^p - \sum_{k=p+1}^{\infty} a_k t^k\right)dt$$

$$= p\left(\frac{z^p}{p} - \sum_{k=p+1}^{\infty} \frac{a^k}{k} z^k\right) = z^p - \sum_{k=p+1}^{\infty} b_k z^k,$$

with $b_k = p\frac{a_k}{k} \geq 0$, $k \geq p+1$. It follows that $f \in T_p$. We have now to prove that $f \in T_p^n(\alpha, \beta, \lambda, \delta)$. Using Theorem 2.1 we need to prove that:

$$\sum_{k=p+1}^{\infty} \{k(1+\beta) - (\alpha + p\beta)\}\phi_k(n, \lambda, \delta, p)b_k \leq (p - \alpha), \tag{3.1}$$

for $-p \leq \alpha < p$, $\beta \geq 0$. This means:

$$\sum_{k=p+1}^{\infty} \{k(1+\beta) - (\alpha + p\beta)\}\phi_k(n, \lambda, \delta, p)p\frac{a_k}{k} \leq (p - \alpha),$$

But we have $p\frac{a_k}{k} \leq a_k$, for $k \geq p+1$, and by using (2.2), we observe that inequality (3.1) is fulfilled. This means that $f \in T_p^n(\alpha, \beta, \lambda, \delta)$. . □

Theorem 3.2. *The integral operator $I_{c+\delta,p}$, defined by (1.6) preserves the class $T_p^n(\alpha, \beta, \lambda, \delta)$, that is: If $F \in T_p^n(\alpha, \beta, \lambda, \delta)$, then $f(z) = I_{c+\delta,p}F(z) \in T_p^n(\alpha, \beta, \lambda, \delta)$, for $F(z) = z^p - \sum\limits_{k=p+1}^{\infty} a_k z^k$, $(a_k \geq 0, p \in \mathbb{N} = \{1, 2, 3, ...\})$.*

Proof. Let $F \in T_p^n(\alpha, \beta, \lambda, \delta)$, $F(z) = z^p - \sum\limits_{k=p+1}^{\infty} a_k z^k$, $a_k \geq 0$. We have, from Theorem 2.1:

$$\sum_{k=p+1}^{\infty} \{k(1+\beta) - (\alpha + p\beta)\}\phi_k(n, \lambda, \delta, p)a_k \leq (p - \alpha). \tag{3.2}$$

From (1.6) we obtain $f(z) = I_{c+\delta,p}, F(z) = z^p - \sum\limits_{k=p+1}^{\infty} \frac{c+\delta+p-1}{c+k+\delta-1} a_k z^k$, where $0 < c < \infty$, $1 \leq \delta < \infty$.

We also remark that for $0 < c < \infty$, $k \geq p+1$ and $1 \leq \delta < \infty$, we have

$$0 < \frac{c + \delta + p - 1}{c + k + \delta - 1} < 1. \tag{3.3}$$

Thus $f \in T_p$ and by using Theorem 2.1 we have only to prove that:

$$\sum_{k=p+1}^{\infty} \{k(1+\beta) - (\alpha + p\beta)\}\phi_k(n, \lambda, \delta, p)\frac{c + \delta + p - 1}{c + k + \delta - 1}a_k \leq (p - \alpha), \tag{3.4}$$

where $-p \leq \alpha < p$, $\beta \geq 0$, $0 < c < \infty$ and $1 \leq \delta < \infty$. By using the relation (3.3) we have

$$\frac{c + \delta + p - 1}{c + k + \delta - 1} \cdot a_k < a_k,$$

for $0 < c < \infty$, $k \geq p+1$, $1 \leq \delta < \infty$, and thus from (3.2) we conclude that the condition (3.4) take place and thus the proof it is complete. □

The following theorem is proved similarly (see Remark 1.1):

Theorem 3.3. *The Bernardi type integral operator defined by (1.4) preserves the class* $T_p^n(\alpha,\beta,\lambda,\delta)$*, that is: If* $F \in T_p^n(\alpha,\beta,\lambda,\delta)$*, then* $f(z) = I_{a,p}F(z) \in T_p^n(\alpha,\beta,\lambda,\delta)$*, for* $F(z) = z^p - \sum\limits_{k=p+1}^{\infty} a_k z^k$*,* $(a_k \geq 0, p \in \mathbb{N} = \{1,2,3,...\})$.

Theorem 3.4. *Let* $T_p^n(\alpha,\beta,\lambda,\delta)$ *with* $-p \leq \alpha < p$*,* $\beta \geq 0$*,* $F(z) = z^p - \sum\limits_{k=p+1}^{\infty} b_k z^k$*,* $b_k \geq 0$*. For* $f(z) = L_{a,p}(F)(z)$*,* $f(z) = z^p - \sum\limits_{k=p+1}^{\infty} a_k z^k$*,* $a_k \geq 0$*,* $z \in \mathbb{U}$*, where the integral operator* $L_{a,p}$ *it is defined by (1.5), we have:*

$$a_k \leq \frac{(p-\alpha)}{\{k(1+\beta)-(\alpha+p\beta)\}\phi_k(n,\lambda,\delta,p)} \cdot \frac{a+p}{a+k},\ k \geq p+1.$$

Proof. For $f = L_{a,p}(F)(z)$ with $F(z) = z^p - \sum\limits_{k=p+1}^{\infty} b_k z^k$ and $f(z) = z^p - \sum\limits_{k=p+1}^{\infty} a_k z^k$ we have

$$a_k = b_k \cdot \frac{a+p}{a+k},$$

where $a \in \mathbb{R}$, $a \geq 0$, $k \geq p+1$. The coefficient bounds for the functions belonging to the class $T_p^n(\alpha,\beta,\lambda,\delta)$ are

$$b_k \leq \frac{(p-\alpha)}{\{k(1+\beta)-(\alpha+p\beta)\}\phi_k(n,\lambda,\delta,p)}.$$

For $k \geq p+1$ we obtain

$$a_k = b_k \cdot \frac{a+p}{a+k} \leq$$
$$\leq \frac{(p-\alpha)}{\{k(1+\beta)-(\alpha+p\beta)\}\phi_k(n,\lambda,\delta,p)} \cdot \frac{a+p}{a+k}.$$

Hence the theorem is proved. □

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