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CERTAIN NEW CLASSES OF ANALYTIC FUNCTIONS DEFINED BY USING SIGMOID FUNCTION

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Abstract. Using an operator involving modified sigmoid function introduced by Fadipe-Joseph et al [2], new family of analytic functions $f(z)$ which is analytic and univalent in the open unit disk $U = \{z : |z| < 1\}$ are defined. Coefficient inequalities, growth and distortion theorems for this class are provided.

1. Introduction

In twentieth century, the theory of special functions was overshadowed by many other field like real and functional analysis, topology, algebra and differential equations. Can we also explore a sigmoid function in Geometric function theory?

Murugusundaramoorthy and Janani [3] obtained initial coefficients of λ- pseudo starlike functions related to sigmoid functions for certain normalized analytic functions defined on the open unit disk. Here, coefficient inequality, growth and distortion theorems for the class of analytic functions defined by using sigmoid function are established.

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

and let $\mathcal{T}$ denote the class of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \tag{1.2}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ (c.f [5]).

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1.1. **The Sigmoid Function.** A sigmoid function is a mathematical function having an "S" shape (sigmoid curve). The sigmoid function, also called the sigmoidal curve or logistic function is the function of the form $\gamma(s) = \frac{1}{1+e^{-s}};\ s \in \Re$. A sigmoid function is a bounded differentiable real function that is defined for all real input values and has a positive derivative at each point. It is useful in compressing, or squashing outputs. It is a monotone function The sigmoid function is the most popular of the three activation function in the hardware implementation of artificial neural network. The sigmoid function is defined as

$$G(s) = \frac{1}{1+e^{-s}} = \frac{1}{2} + \frac{1}{4}s - \frac{1}{48}s^3 + \frac{1}{480}s^5 - \frac{17}{80640}s^7 + \cdots .$$

Let $\gamma(s)$ be a modified sigmoid function.That is,

$$\gamma(s) = \frac{2}{1+e^{-s}} = 1 + \frac{1}{2}s - \frac{1}{24}s^3 + \frac{1}{240}s^5 - \frac{17}{40320}s^7 + \cdots .$$

For details see [1], [2] and [3]

$$f_\gamma(z) = z + \sum_{k=2}^{\infty} \gamma(s) a_k z^k \quad (a_k \geq 0),$$

which is analytic and univalent in the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Then $f_\gamma(z)$ belongs to the class A_γ of the form (1.1) for $\lim_{s\to\infty} \gamma(s) = 1$.

Similarly,

$$f_\gamma(z) = z - \sum_{k=2}^{\infty} \gamma(s) a_k z^k;\ a_k \geq 0 \tag{1.3}$$

is analytic and univalent in the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Then $f_\gamma(z)$ belongs to the class T_γ which is a subclass of the normalised univalent functions of the form (1.2) for $\lim_{s\to\infty} \gamma(s) = 1$.

Definition 1.1. *A function f_γ defined by (1.3) belongs to the class $T_\gamma(\lambda, \beta, \alpha, \mu)$ if*

$$\left| \frac{\frac{D^{n+1} f_\gamma(z)}{D^n f_\gamma(z)} - \mu}{\left(\frac{D^{n+1} f_\gamma(z)}{D^n f_\gamma(z)} + \lambda\right) - 2\alpha\left(\frac{D^{n+1} f_\gamma(z)}{D^n f_\gamma(z)} - \mu\right)} \right| > \beta \quad (z \in U),$$

for $|z| < 1,\ 0 < \lambda \leq 1, \gamma(s) = \frac{2}{1+e^{-s}}\ , 0 \leq \beta \leq 1, \frac{1}{2} \leq \alpha \leq 1,\ \mu \geq 1,\ n \in N_0 = N \cup \{0\}$.

1.2. **Salagean Differential Operator Involving Modified Sigmoid Function.** Let $D^n f_\gamma(z)$ denote the Salagean differential operator involving modified sigmoid function, then

$$\begin{aligned} &D^0 f_\gamma(z) = f_\gamma(z) \\ &D^1 f_\gamma(z) = \gamma(s) z f'_\gamma(z) \\ &\vdots \\ &D^n f_\gamma(z) = D[D^{n-1} f_\gamma(z)] = \gamma(s) z [D^{n-1} f_\gamma(z)]'. \end{aligned}$$

Therefore,

$$D^n f_\gamma(z) = \gamma^n(s)z + \sum_{k=2}^{\infty} \gamma^{n+1}(s)k^n a_k z^k.$$

Taking $\lim_{s\to\infty} \gamma(s) = 1$, we have the Salagean differential operator ([4], [6]).

2. Main Results

2.1. Coefficient estimates for class $T_\gamma(\lambda, \beta, \alpha, \mu)$.

Theorem 2.1. *Let $f_\gamma \in T_\gamma$ defined as $f(z) = z + \sum_{k=2}^{\infty} \gamma(s) a_k z^k$ ($a_k \geq 0$). If $f_\gamma \in T_\gamma(\lambda, \beta, \alpha, \mu)$, then*

$$\begin{aligned}&\sum_{k=2}^{\infty} \gamma(s)k^n \left[\gamma(s)k\left[1-\beta(1-2\alpha)\right] - \beta\left(\lambda + 2\alpha\mu + \frac{1}{\beta}\right)\right]|a_k| \\ &< \gamma(s)\left(1-\beta\right) + \beta\left[2\alpha(\gamma(s)-\mu) - \lambda\right] - \mu.\end{aligned}$$

Proof. First note that:

$$\begin{aligned}&\frac{D^{n+1}f_\gamma(z)}{D^n f_\gamma(z)} - \mu \\ &= \frac{\gamma^{n+1}(s)z - \sum_{k=2}^{\infty}\gamma^{n+2}(s)k^{n+1}a_k z^k - \mu\gamma^n(s)z + \sum_{k=2}^{\infty}\mu\gamma^{n+1}(s)k^n a_k z^k}{\gamma^n(s)z - \sum_{k=2}^{\infty}\gamma^{n+1}(s)k^n a_k z^k} \\ &= \frac{\gamma^n(s)[\gamma(s)-\mu]z - \sum_{k=2}^{\infty}\gamma^{n+1}(s)[k\gamma(s)-\mu]k^n a_k z^k}{\gamma^n(s)z - \sum_{k=2}^{\infty}\gamma^{n+1}(s)k^n a_k z^k},\end{aligned}$$

$$\begin{aligned}&\frac{D^{n+1}f_\gamma(z)}{D^n f_\gamma(z)} + \lambda \\ &= \frac{\gamma^{n+1}(s)z - \sum_{k=2}^{\infty}\gamma^{n+2}(s)k^{n+1}a_k z^k + \gamma^n(s)\lambda z - \sum_{k=2}^{\infty}\lambda\gamma^{n+1}(s)k^n a_k z^k}{\gamma^n(s)z - \sum_{k=2}^{\infty}\gamma^{n+1}(s)k^n a_k z^k} \\ &= \frac{\gamma^n(s)[\gamma(s)+\lambda]z - \sum_{k=2}^{\infty}\gamma^{n+1}(s)[k\gamma(s)+\lambda]k^n a_k z^k}{\gamma^n(s)z - \sum_{k=2}^{\infty}\gamma^{n+1}(s)k^n a_k z^k}\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}&2\alpha\left(\frac{D^{n+1}f_\gamma(z)}{D^n f_\gamma(z)} - \mu\right) \\ &= \frac{2\alpha\gamma^{n+1}(s)z - \sum_{k=2}^{\infty}2\alpha\gamma^{n+2}(s)k^{n+1}a_k z^k - 2\alpha\mu\gamma^n(s)z + \sum_{k=2}^{\infty}2\alpha\mu\gamma^{n+1}k^n a_k z^k}{\gamma^n(s)z - \sum_{k=2}^{\infty}\gamma^{n+1}(s)k^n a_k z^k} \\ &= \frac{2\alpha\gamma^n(s)[\gamma(s)-\mu]z - \sum_{k=2}^{\infty}2\alpha\gamma^{n+1}(s)[k\gamma(s)-\mu]k^n a_k z^k}{\gamma^n(s)z - \sum_{k=2}^{\infty}\gamma^{n+1}(s)k^n a_k z^k}.\end{aligned} \tag{2.2}$$

Subtracting (2.2) from (2.1) we obtain,

$$\frac{[\gamma^{n+1}(s)+\lambda\gamma^{n}(s)-2\alpha\gamma^{n+1}(s)+2\alpha\mu\gamma^{n}(s)]z}{\gamma^{n}(s)z-\sum_{k=2}^{\infty}\gamma^{n+1}(s)k^{n}a_kz^k}$$
$$-\frac{\sum_{k=2}^{\infty}[k\gamma^{n+2}(s)+\lambda\gamma^{n+1}(s)-2\alpha k\gamma^{n+2}(s)+2\alpha\mu\gamma^{n+1}(s)]k^{n}a_kz^k}{\gamma^{n}(s)z-\sum_{k=2}^{\infty}\gamma^{n+1}(s)k^{n}a_kz^k}$$
$$=\frac{\gamma^{n}(s)[\gamma(s)+\lambda-2\alpha\gamma(s)+2\alpha\mu]z-\sum_{k=2}^{\infty}\gamma^{n+1}(s)[k\gamma(s)+\lambda-2k\alpha\gamma(s)+2\alpha\mu]k^{n}a_kz^k}{\gamma^{n}(s)z-\sum_{k=2}^{\infty}\gamma^{n+1}(s)k^{n}a_kz^k}.$$

Dividing (2.1) by (2.2), we have

$$\left|\frac{\gamma^{n+1}(s)z-\sum_{k=2}^{\infty}\gamma^{n+2}(s)k^{n+1}a_kz^k-\mu\gamma^{n}(s)z+\sum_{k=2}^{\infty}\gamma^{n+1}(s)k^{n}a_kz^k}{\gamma z\left[\gamma(s)-2\alpha\gamma(s)+2\alpha\mu+\lambda\right]-\sum_{k=2}^{\infty}k^{n}\gamma^{n+1}(s)\left[\gamma(s)k+\lambda-2\alpha\gamma(s)k+2\alpha\mu\right]a_kz^k}\right|>\beta$$

which implies that

$$\left|\frac{[\gamma(s)-\mu]-\sum_{k=2}^{\infty}\gamma(s)k^{n}\left[\gamma k-1\right]a_kz^{k-1}}{[\gamma(s)-2\alpha\gamma(s)+2\alpha\mu+\lambda]-\sum_{k=2}^{\infty}k^{n}\gamma(s)\left[\gamma(s)k+\lambda-2\alpha\gamma(s)k+2\alpha\mu\right]a_kz^{k-1}}\right|>\beta.$$

Since $|z|=r<1$, then as $|z|\to 1$, we have

$$\beta\left[\gamma(s)-2\alpha\gamma(s)+2\alpha\mu+\lambda\right]-\beta\sum_{k=2}^{\infty}k^{n}\gamma(s)\left[\gamma(s)+\lambda-2\alpha\gamma(s)k+2\alpha\mu\right]|a_k|$$
$$<(\gamma(s)-\mu)-\sum_{k=2}^{\infty}\gamma(s)k^{n}\left[\gamma(s)k-1\right]|a_k|,$$

i.e.,

$$\sum_{k=2}^{\infty}\gamma(s)k^{n}\left[\gamma(s)k-1\right]|a_k|-\beta\sum_{k=2}^{\infty}k^{n}\gamma(s)\left[\gamma(s)+\lambda-2\alpha\gamma(s)k+2\alpha\mu\right]|a_k|<$$
$$(\gamma(s)-\mu)-\beta\left[\gamma(s)-2\alpha\gamma(s)+2\alpha\mu+\lambda\right]$$

which implies

$$\sum_{k=2}^{\infty}\gamma(s)k^{n}\left[\gamma(s)k\left[1-\beta(1-2\alpha)\right]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]|a_k|$$
$$<\gamma(s)\left(1-\beta\right)+\beta\left[2\alpha(\gamma(s)-\mu)-\lambda\right]-\mu$$

and completes the proof. □

Corollary 2.1. *Let the function $f(z)$ defined by (1.3) belongs to the class $T_\gamma(\lambda,\beta,\alpha,\mu)$ then*

$$a_k\le\frac{\gamma(s)\left(1-\beta\right)+\beta\left[2\alpha(\gamma(s)-\mu)-\lambda\right]-\mu}{\gamma(s)k^{n}\left[\gamma(s)k\left[1-\beta(1-2\alpha)\right]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]},$$

$0<\lambda\le 1,\gamma(s)=\frac{2}{1+e^{-s}}$ $,0\le\beta\le 1,\frac{1}{2}\le\alpha\le 1,$ $\mu\ge 1,$ $n\in N_0=N\cup\{0\}$. *Equality holds for the function*

$$f_\gamma(z)=z+\frac{\gamma(s)\left(1-\beta\right)+\beta\left[2\alpha(\gamma(s)-\mu)-\lambda\right]-\mu}{\gamma(s)k^{n}\left[\gamma(s)k\left[1-\beta(1-2\alpha)\right]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}a_kz^k.$$

Corollary 2.2. *If* $f_\gamma \in T_\gamma(1,\beta,\alpha,\mu)$, *then*

$$\sum_{k=2}^{\infty}\gamma(s)k^n\left[\gamma(s)k\left[1-\beta(1-2\alpha)\right]-\beta\left(1+2\alpha\mu+\frac{1}{\beta}\right)\right]|a_k|$$
$$<\gamma(s)\left(1-\beta\right)+\beta\left[2\alpha(\gamma(s)-\mu)-1\right]-\mu,$$

which implies

$$a_k\le\frac{\gamma(s)\left(1-\beta\right)+\beta\left[2\alpha(\gamma(s)-\mu)-1\right]-\mu}{\gamma(s)k^n\left[\gamma(s)k\left[1-\beta(1-2\alpha)\right]-\beta\left(1+2\alpha\mu+\frac{1}{\beta}\right)\right]},$$

$0<\lambda\le 1, \gamma(s)=\frac{2}{1+e^{-s}}$ $, 0\le\beta\le 1, \frac{1}{2}\le\alpha\le 1,\ \mu\ge 1.$

Corollary 2.3. *If* $f_\gamma \in T_\gamma(\lambda,0,\alpha,\mu)$, *then*

$$\sum_{k=2}^{\infty}\gamma(s)k^n\left(\gamma(s)k\right)|a_k|<\gamma(s)-\mu,$$

which implies

$$a_k\le\frac{\gamma(s)-\mu}{\gamma^2k^{n+1}},$$

$0<\lambda\le 1, \gamma(s)=\frac{2}{1+e^{-s}}$ $, 0\le\beta\le 1, \frac{1}{2}\le\alpha\le 1,\ \mu\ge 1.$

Corollary 2.4. *If* $f_\gamma \in T_\gamma(\lambda,\beta,\alpha,1)$, *then*

$$\sum_{k=2}^{\infty}\gamma(s)k^n\left[\gamma(s)k\left[1-\beta(1-2\alpha)\right]-\beta\left(\lambda+2\alpha+\frac{1}{\beta}\right)\right]|a_k|$$
$$<\gamma(s)\left(1-\beta\right)+\beta\left[2\alpha(\gamma(s)-1)-\lambda\right]-1,$$

which implies

$$a_k\le\frac{\gamma(s)\left(1-\beta\right)+\beta\left[2\alpha(\gamma(s)-1-\lambda\right]-1}{\gamma(s)k^n\left[\gamma(s)k\left[1-\beta(1-2\alpha)\right]-\beta\left(\lambda+2\alpha+\frac{1}{\beta}\right)\right]},$$

$0<\lambda\le 1, \gamma(s)=\frac{2}{1+e^{-s}}$ $, 0\le\beta\le 1, \frac{1}{2}\le\alpha\le 1,\ \mu\ge 1.$

Corollary 2.5. *If* $f_\gamma \in T_\gamma(1,0,\alpha,1)$, *then*

$$\sum_{k=2}^{\infty}\gamma(s)k^n\left(\gamma(s)k\right)|a_k|<\gamma(s)-1$$

which implies

$$a_k\le\frac{\gamma(s)-1}{\gamma^2k^{n+1}},$$

$0<\lambda\le 1, \gamma(s)=\frac{2}{1+e^{-s}}$ $, 0\le\beta\le 1, \frac{1}{2}\le\alpha\le 1,\ \mu\ge 1.$

2.2. Growth theorem for class $T_\gamma(\lambda,\beta,\alpha,\mu)$.

Theorem 2.2. *Let $f_\gamma \in T_\gamma(\lambda,\beta,\alpha,\mu)$, then*

$$r-\gamma(s)\frac{\gamma(s)\,(1-\beta)+\beta\,[2\alpha(\gamma(s)-\mu)-\lambda]-\mu}{\gamma(s)2^n\left[2\gamma(s)\,[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}r^2$$

$$\leq r+\gamma(s)\frac{\gamma(s)\,(1-\beta)+\beta\,[2\alpha(\gamma(s)-\mu)-\lambda]-\mu}{\gamma(s)2^n\left[2\gamma(s)\,[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}r^2.$$

2.3. Distortion theorem for class $T_\gamma(\lambda,\beta,\alpha,\mu)$.

Theorem 2.3. *Let $f_\gamma(z)$ belongs to the class $T_\gamma(\lambda,\beta,\alpha,\mu)$, then*

$$1-\gamma\frac{\gamma(s)\,(1-\beta)+\beta\,[2\alpha(\gamma(s)-\mu)-\lambda]-\mu}{\gamma(s)\left[\gamma(s)2\,[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}2^{1-n}r \leq |f'_\gamma(z)|$$

$$\leq 1+\gamma\frac{\gamma(s)\,(1-\beta)+\beta\,[2\alpha(\gamma(s)-\mu)-\lambda]-\mu}{\gamma(s)\left[\gamma(s)2\,[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}2^{1-n}r.$$

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