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GROWTH AND DISTORTION THEOREMS FOR GENERALIZED q-STARLIKE FUNCTIONS

YAŞAR POLATOĞLU

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Abstract. For every $q \in (0,1)$ we define a class of analytic functions, so called $q-$ starlike functions on the open unit disc $\mathbb{D} = \{z| \quad |z| < 1\}$. This class was introduced by M.E.H. Ismail, E. Merkes and D.Styer [4] . In the present paper we will give basic characterization, growth theorem and distortion theorem for this class.

1. Introduction

Let Ω be the family of functions which are regular in $\mathbb{D}$ and satisfying the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for every $z \in \mathbb{D}$, and let $\mathcal{A}$ be the class of functions $s(z) = z + c_2z^2 + c_3z^3 + \cdots$ which are regular in $\mathbb{D}$. Let $s_1(z)$ and $s_2(z)$ be an element of $\mathcal{A}$, if there exists a function $\phi(z) \in \Omega$ such that $s_1(z) = s_2(\phi(z))$ for every $z \in \mathbb{D}$, then we say that $s_1(z)$ subordinate to $s_2(z)$ and we write $s_1(z) \prec s_2(z)$. We also note that if $s_2(z)$ is univalent in $\mathbb{D}$, then $s_1(z) \prec s_2(z)$ if and only if $s_1(0) = s_2(0)$, $s_1(\mathbb{D}) \subset s_2(\mathbb{D})$ implies $s_1(\mathbb{D}_r) \subset s_2(\mathbb{D}_r)$ where $\mathbb{D}_r = \{z \,|\, |z| < r,\ 0 < r < 1\}$. Let $s_1(z) = z + \sum_{n=2}^{\infty} a_nz^n$ and $s_2(z) = z + \sum_{n=2}^{\infty} b_nz^n$ be elements of $\mathcal{A}$, then the convolution of these functions is defined by

$$s_1(z) * s_2(z) = z + \sum_{n=2}^{\infty} a_nb_nz^n.$$

Denote by $\mathcal{P}$ the family of functions $p(z) = 1 + p_1z + p_2z^2 + \cdots$ in $\mathbb{D}$ and such that $p(z)$ is in $\mathcal{P}$ if and only if

$$p(z) \prec \frac{1+z}{1-z} \Leftrightarrow p(z) = \frac{1+\phi(z)}{1+\phi(z)}$$

for some function $\phi(z) \in \Omega$ and every $z \in \mathbb{D}$ [3].

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For analytic function $s(z)$ in $\mathbb{D}$, we recall here q-fractional calculus. If $\mu \in \mathbb{R}$ is fixed a subset B of $\mathbb{C}$ is called μ-geometric if $\mu z \in B$ whenever $z \in B$. If a subset B of $\mathbb{C}$ is a μ-geometric then it contains all geometric sequences $\{z\mu^n\}_{n=0}^{\infty}$, $z \in B$. Let s be a function, real or complex valued, defined on a q-geometric set B , $|q| \neq 1$. The q-difference operator, which was reintroduced by Jackson, F. H., and may go back to E. Heine or Euler [1] defined by

$$D_q s(z) = \frac{s(z) - s(qz)}{z - qz} \quad \text{for} \quad z \in B/\{0\}. \tag{1.1}$$

The q-difference operator (1.1) sometimes called Jackson q-difference operator. If $0 \in \mathcal{A}$, the q-derivative at zero is defined for $|q| < 1$ by

$$D_q s(0) = \lim_{n \to \infty} \frac{s(zq^n) - s(0)}{zq^n} \quad \text{for} \quad z \in B/\{0\}.$$

Provided the limit exists and does not depend on z. In addition, the q-derivative at zero defined by $|q| > 1$ by

$$D_q s(0) = D_{q^{-1}} s(0).$$

Under the hypotheses of the definition of q-difference operator then we have the following rules [1].

(1) $D_q z^k = \frac{1-q^k}{1-q} . z^{k-1}$
(2) $D_q(s_1(z) + s_2(z)) = D_q s_1(z) + D_q s_2(z)$
(3) $D_q(s_1(z) - s_2(z)) = D_q s_1(z) - D_q s_2(z)$
(4) $D_q(s_1(z).s_2(z)) = s_2(qz) D_q s_1(z) + s_1(z) D_q s_2(z)$
$D_q(s_1(z).s_2(z)) = s_1(qz) D_q s_2(z) + s_2(z) D_q s_1(z)$
(5) $D_q(\frac{s_1(z)}{s_2(z)}) = \frac{s_2(z) D_q s_1(z) - s_1(z) D_q s_2(z)}{s_2(qz).s_2(z)}$

Let $s(z)$ be an element of $\mathcal{A}$, if $s(z)$ satisfies the condition

$$\left| z \frac{D_q s(z)}{s(z)} - \frac{1}{1-q} \right| \leq \frac{1}{1-q},$$

for all $z \in \mathbb{D}$, then $s(z)$ is called q-starlike function in $\mathbb{D}$, the class of such functions will be denoted by $S_q^{\star}$. Clearly, $D_q \to \frac{d}{\phi(z)}$ as $q \to 1^-$.

Theorem 1.1. *([4]) For a function $s(z)$ to belongs $S_q^{\star}$ it is necessary and sufficient that*

$$\left| \frac{s(qz)}{s(z)} \right| \leq 1$$

for all $z \in \mathbb{D}$.

2. Main Results

Theorem 2.1. *Let $s(z)$ be an element of $\mathcal{A}$, then we have $s(z) \in S_q^{\star}$ if and only if*

$$z \frac{D_q s(z)}{s(z)} \prec \frac{1+z}{1-qz}.$$

Proof. Let $s(z)$ be an element of $S_q^\star$. Then we have

$$\left|z\frac{D_q s(z)}{s(z)} - \frac{1}{1-q}\right| \leq \frac{1}{1-q} \Leftrightarrow \left|z\frac{D_q s(z)}{s(z)} - M\right| \leq M, M = \frac{1}{1-q}, M > 1.$$

Therefore, the function

$$\Psi(z) = \frac{1}{M} z \frac{D_q s(z)}{s(z)} - 1$$

has modulus at most 1 in the unit disc $\mathbb{D}$ and so

$$\phi(z) = \frac{\Psi(z) - \Psi(0)}{1 - \overline{\Psi(0)}\Psi(z)} = \frac{\frac{1}{M} z \frac{D_q s(z)}{s(z)} - (\frac{1}{M} - 1)}{1 - (\frac{1}{M} - 1)(\frac{1}{M} z \frac{D_q s(z)}{s(z)} - 1)}. \tag{2.1}$$

Then $\phi(0) = 0$, $|\phi(z)| < 1$ and by the Schwarz lemma,

$$|\phi(z)| \leq |z|. \tag{2.2}$$

From (2.1) and (2.2) we obtain

$$z\frac{D_q s(z)}{s(z)} = \frac{1 + \phi(z)}{1 - (1 - \frac{1}{M})\phi(z)} = \frac{1 + \phi(z)}{1 - q\phi(z)}. \tag{2.3}$$

The equality (2.3) shows that

$$z\frac{D_q s(z)}{s(z)} \prec \frac{1+z}{1-qz}.$$

Conversely $z\frac{D_q s(z)}{s(z)} \prec \frac{1+z}{1-qz}$. Then we have

$$z\frac{D_q s(z)}{s(z)} = \frac{1 + \phi(z)}{1 - (1 - \frac{1}{M})\phi(z)} \Rightarrow z\frac{D_q s(z)}{s(z)} - M = M\frac{\frac{1-M}{M} + \phi(z)}{1 + \frac{1-M}{M}\phi(z)}.$$

On the other hand, the function $\left(\frac{\frac{1-M}{M} + \phi(z)}{1 + \frac{1-M}{M}\phi(z)}\right)$ maps the unit circle onto itself. So,

$$\left|z\frac{D_q s(z)}{s(z)} - M\right| = |M\frac{\frac{1-M}{M} + \phi(z)}{1 + \frac{1-M}{M}\phi(z)}| < M \Rightarrow$$

$$\left|z\frac{D_q s(z)}{s(z)} - \frac{1}{1-q}\right| < \frac{1}{1-q}.$$

□

Corollary 2.2. *Let $s(z)$ be an element of $S_q^\star$. Then*

$$\frac{1-r}{1+qr} \leq \left|z\frac{D_q s(z)}{s(z)}\right| \leq \frac{1+r}{1-qr}. \tag{2.4}$$

Proof. The linear transformation $w(z) = \frac{1+z}{1-qz}$ maps $|z| = r$ onto the circle with the centre $C(r) = \left(\frac{1+q^2r^2}{1-q^2r^2}, 0\right)$ and the radius $\rho(r) = \frac{2qr}{1-q^2r^2}$. Using the subordination principle, we have

$$\left|z\frac{D_q s(z)}{s(z)} - \frac{1+q^2r^2}{1-q^2r^2}\right| \leq \frac{2qr}{1-q^2r^2},$$

which gives (2.4). □

Corollary 2.3. *The radius of q-starlike of the class $S_q^\star$ is $R_{ST} = \frac{1}{q}$. This radius is sharp due to the extremal function is $s(z) = \frac{1+z}{1-qz}$.*

Proof. Since

$$\left| z\frac{D_q s(z)}{s(z)} - \frac{1+q^2r^2}{1-q^2r^2} \right| \leq \frac{2qr}{1-q^2r^2},$$

we have

$$\mathrm{Re}\left(z\frac{D_q s(z)}{s(z)} \right) \geq \frac{1-qr}{1+qr}.$$

Hence, for $r < R_{ST}$ the right hand side of the preceding inequality is positive, implying that

$$R_{ST} = \frac{1}{q}.$$

□

Theorem 2.4. *Let $s(z)$ be an element of $S_q^{\star}$. Then*

$$\frac{r-q}{1-rq} \leq \left| \frac{s(qz)}{s(z)} \right| \leq \frac{r+q}{1+rq}. \tag{2.5}$$

Proof. Since

$$w(z) = \frac{s(qz)}{s(z)} = \frac{qz + a_2q^2z^2 + a_3q^3z^3 + \cdots}{z + a_2z^2 + a_3z^3 + \cdots} = \frac{q + a_2q^2z + a_3q^3z^2 + \cdots}{1 + a_2z + a_3z^2 + \cdots} \Rightarrow w(0) = q$$

and using Theorem 1.1, the function

$$\phi(z) = \frac{w(z) - w(0)}{1 - \overline{w(0)}.w(z)} = \frac{w(z) - q}{1 - qw(z)} \tag{2.6}$$

satisfies the conditions: $\phi(0) = 0$ and $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. Therefore, by Schwarz lemma,

$$|\phi(z)| \leq |z|. \tag{2.7}$$

From (2.6) and (2.7) we obtain

$$w(z) = \frac{q + \phi(z)}{1 + q\phi(z)} \Leftrightarrow w(z) = \frac{s(qz)}{s(z)} \prec \frac{q+z}{1+qz}.$$

On the other hand, the linear transformation $w(z) = \frac{q+z}{1+qz}$ maps $|z| = r$ onto the circle with the centre $C(r) = \left(\frac{q(1-r^2)}{1-r^2q^2}, 0 \right)$ and the radius $\rho(r) = \frac{r(1-q^2)}{1-r^2q^2}$. Using subordination principle, then we can write

$$\left| \frac{s(qz)}{s(z)} - \frac{q(1-r^2)}{1-r^2q^2} \right| \leq \frac{r(1-q^2)}{1-r^2q^2},$$

which gives (2.5). □

PREPARATION: Let $s(z) = z + a_2z^2 + a_3z^3 + \cdots$ be defined on a q-geometric set B, $|q| \neq 1$. Using the rule (1), then we can write

$$\begin{aligned}
s(z) &= z + a_2z^2 + a_3z^3 + ... + a_nz^n + \cdots \\
D_qs(z) &= D_qz + a_2D_qz^2 + a_3D_qz^3 + ... + a_nD_qz^n + \cdots \\
&= \frac{1-q}{1-q}z^{1-1} + a_2\frac{1-q^2}{1-q}z^{2-1} + a_3\frac{1-q^3}{1-q}z^{3-1} + ... + a_n\frac{1-q^n}{1-q}z^{n-1} + ... \\
&= 1 + a_2\frac{1-q^2}{1-q}z + a_3\frac{1-q^3}{1-q}z^2 + ... + a_n\frac{1-q^n}{1-q}z^{n-1} + ... \Rightarrow \\
D_qs(z) &= 1 + a_2\frac{1-q^2}{1-q}z + a_3\frac{1-q^3}{1-q}z^2 + ... + a_n\frac{1-q^n}{1-q}z^{n-1} + ... \\
zD_qs(z) &= z + a_2\frac{1-q^2}{1-q}z^2 + a_3\frac{1-q^3}{1-q}z^3 + ... + a_n\frac{1-q^n}{1-q}z^n + ... \Rightarrow \\
D^qs(z) &= zD_qs(z) = z + \sum_{n=2}^{\infty} a_n\frac{1-q^n}{1-q}z^n
\end{aligned}$$

On the other hand, if $s_1(z) = \frac{z}{1-z}$, then we have

$$D^qs(z) = z + \sum_{n=2}^{\infty} \frac{1-q^n}{1-q}z^n.$$

Therefore, using the definition of convolution we can write

$$D^qs(z) = zD_qs(z) = z + \sum_{n=2}^{\infty} a_n\frac{1-q^n}{1-q}z^n = s(z) * s_1(z).$$

Theorem 2.5. *Let $s(z)$ be an element of $S_q^\star$. Then*

$$\left(\frac{q-q^k}{1-q}\right)^2 |a_k|^2 \leq (1+q)^2 + \sum_{n=2}^{k-1}\left(\frac{1+q}{1-q}\right)(1-q^{2n})|a_n|^2. \tag{2.8}$$

Proof. Using Theorem 2.1 and preparation, then we can write

$$\frac{D^qs(z)}{s(z)} = \frac{1+\phi(z)}{1-q\phi(z)} \Leftrightarrow D^qs(z) - q\phi(z)D^qs(z) = s(z) + \phi(z)s(z) \Rightarrow$$

$$\sum_{n=2}^{k} a_k\left(\frac{q-q^n}{1-q}\right)z^n + \sum_{n=k+1}^{\infty} c_nz^n = \phi(z)\left(\sum_{n=1}^{k-1} a_n(\frac{q-q^{n+1}}{1-q})z^n\right)$$

where the sum $(\sum_{k=n+1}^{\infty} c_kz^k)$ is convergent in $\mathbb{D}$. Let $z = re^{i\theta}$. Then, since $|\phi(z)| < 1$,

$$\sum_{n=2}^{k}\left(\frac{q-q^n}{1-q}\right)^2 |a_n|^2r^{2k} \leq \sum_{n=1}^{k-1}\left(\frac{q-q^{n+1}}{1-q}\right)^2 |a_n|^2r^{2k}. \tag{2.9}$$

Passing to the limit in (2.9) as $r \to 1$ we conclude that

$$\sum_{n=2}^{k}\left(\frac{q-q^n}{1-q}\right)^2 |a_n|^2 \leq \sum_{n=1}^{k-1}\left(\frac{q-q^{n+1}}{1-q}\right)^2 |a_n|^2$$

and we have (2.6). The proof of this theorem is based on a method introduced by Clunie [2]. □

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Department of Mathematics and Computer Sciences,
İstanbul Kültür University,
Istanbul, Turkey
E-mail address: y.polatoglu@iku.edu.tr