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SOME PROPERTIES OF THE SOLUTION SET OF THE NON-LINEAR ELLIPTIC DIFFERENTIAL EQUATION $\overline{f_{\overline{z}}} = w(z)f_z$

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ABSTRACT. Let $f = f + \overline{g}$ be the sense-preserving harmonic mapping, then it satisfies that non-linear elliptic differential equation $\overline{f_{\overline{z}}} = w(z)f_z$. We will obtain the solution of this differential equation by using subordination method, under the condition $w(z) = \frac{g'(z)}{h'(z)} \prec b_1\left(\frac{1+Az}{1+Bz}\right)^{\alpha}$, $\alpha \geq 0$, $-1 \leq B < A \leq 1$, and we will investigate the properties of the solution of this differential equation.

1. INTRODUCTION

Let Ω be the family of functions $\phi(z)$ regular in the disc $\mathbb{D}$ and satisfying the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. Next, $\mathcal{A}$ is denote the class of analytic functions of the form $s(z) = z + a_2z^2 + a_3z^3 + \cdots$ in the open unit disc $\mathbb{D}$. Let $\mathcal{P}(A, B)(1- \leq B < A \leq 1)$ designate the class of functions $p(z) = 1 + p_1z + p_2z^2 + \cdots$ which are analytic, have positive real part in $\mathbb{D}$, and such that $p(z)$ is in $\mathcal{P}(A, B)$ if and only if

$$p(z) = \frac{1 + A\phi(z)}{1 + B\phi(z)}$$

for some $\phi(z) \in \Omega$ and every $z \in \mathbb{D}$.

Moreover, let $\mathcal{C}$ denote the family of functions $s(z) \in \mathcal{A}$ and such that $s(z)$ is in $\mathcal{C}$ if and only if

$$1 + z\frac{s''(z)}{s'(z)} = p(z)$$

for some $p(z)$ is in $\mathcal{P}$ and all $z \in \mathbb{D}$, and let $s_1(z)$ be an element of $\mathcal{A}$ and satisfies the condition

$$Re\left(\frac{s_1'(z)}{s_1(z)}\right) > 0,$$

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where $s_2(z) \in \mathcal{C}$, then $s_1(z)$ is called close-to-convex function. The class of such functions is denoted by $\mathcal{K}$. Let $F_1(z)$ and $F_2(z)$ be an elements of $\mathcal{A}$, if there exists a function $\phi(z) \in \Omega$ such that $F_1(z) = F_2(\phi(z))$ for all $z \in \mathbb{D}$, then we say that $F_1(z)$ is subordinate to $F_2(z)$ and we write $F_1(z) \prec F_2(z)$ if and only if $F_1(\mathbb{D}) \subset F_2(\mathbb{D})$ and $F_1(0) = F_2(0)$ implies $F_1(\mathbb{D}_r) \subset F_2(\mathbb{D}_r)$, where $\mathbb{D}_r = \{z||z| < r, 0 < r < 1\}$. (Subordination and Lindelof principle[2]).

Finally, a planar harmonic mapping in the open unit disc $\mathbb{D}$ is a complex-valued harmonic function f, which maps $\mathbb{D}$ onto the some planar domain $f(\mathbb{D})$. Since $\mathbb{D}$ is a simply-connected domain the mapping f has a canonical decomposition $f(z) = h(z) + \overline{g(z)}$, where $h(z)$ and $g(z)$ are analytic in $\mathbb{D}$ and have the following power series expansions

$$h(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=0}^{\infty} b_n z^n,$$

where $a_n, b_n \in C$, $n = 0, 1, 2, 3, ...$ as usual we call $h(z)$ the analytic part of $f(z)$ and $g(z)$ is co-analytic part of $f(z)$. An elegant and complete account of the theory of harmonic mappings is given in Duren's monograph [1] proved in 1936 that the harmonic function $f(z)$ is locally univalent in $\mathbb{D}$ if and only if its Jacobian

$$J_f = |h'(z)|^2 - |g'(z)|^2$$

is different from zero in $\mathbb{D}$. In view of this result, locally univalent harmonic mappings in the open unit disc $\mathbb{D}$ are either sense-reserving if $|g'(z)| > |h'(z)|$ in $\mathbb{D}$ or sense-preserving if $|g'(z)| < |h'(z)|$ in $\mathbb{D}$.

Throughout this paper we will restrict ourselves to the study of sense-preserving harmonic mappings. We will also note that $f(z) = h(z) + \overline{g(z)}$ is sense-preserving in $\mathbb{D}$ if and only if $h'(z)$ doesn't vanish in $\mathbb{D}$ and the second dilatation $w(z) = \dfrac{g'(z)}{h'(z)}$ has the property $|w(z)| < 1$ for all $z \in \mathbb{D}$. Therefore, the class of all sense-preserving harmonic mappings in the open unit disc with $a_0 = b_0 = 0$ and $a_1 = 1$ will be denoted by $\mathcal{S}_H$. Thus $\mathcal{S}_H$ contains standard class $\mathcal{S}$ of univalent functions. The family of all mappings $f \in \mathcal{S}_H$ with the additional property $g'(0) = 0$, i.e, $b_1 = 0$ is denoted by $\mathcal{S}_H^0$. Hence it is clear that $\mathcal{S} \subset \mathcal{S}_H^0 \subset \mathcal{S}_H$.

We consider the following class of harmonic mappings

$$\mathcal{S}_{HK(A,B)} = \left\{ f = h + \overline{g} | w(z) = \frac{g'(z)}{h'(z)} \prec b_1 \left(\frac{1 + Az}{1 + Bz} \right)^{\alpha}, \alpha \geq 0, -1 \leq B < A \leq 1, h(z) \in \mathcal{C} \right\}.$$

The class of $\mathcal{S}_{HK(A,B)}$ is the solution set of the non-linear elliptic partial differential equation $\overline{f_{\overline{z}}} = w(z) f_z$ under the conditions $\frac{g'(z)}{h'(z)} \prec b_1 \left(\frac{1+Az}{1+Bz} \right)^{\alpha}$, $\alpha \geq 0$, $-1 \leq B < A \leq 1$, $h(z) \in \mathcal{C}$. The aim of this paper is to investigate the class of $\mathcal{S}_{H(K(B))}$, for this aim we need the following lemma and theorems.

Lemma 1.1. *([3]) Let $\phi(z)$ be a non-constant and analytic function in the unit disc $\mathbb{D}$ with $\phi(0) = 0$. If $|\phi(z)|$ attains its maximum value on the circle $|z| = r$ at the point z_0, then $z_0 \phi'(z_0) = m\phi(z_0)$, $m \geq 1$.*

Theorem 1.2 ([2]). *Let $h(z)$ be an element of $\mathcal{C}$, then*

$$\frac{r}{1+r} \le |h(z)| \le \frac{r}{1-r},$$

$$\frac{r}{(1+r)^2} \le |h'(z)| \le \frac{r}{(1-r)^2}$$

and

$$Re\left(z\frac{h'(z)}{h(z)}\right) > \frac{1}{2}.$$

2. Main Results

Theorem 2.1. *Let $f = h(z) + \overline{g(z)}$ be an element of $\mathcal{S}_{HK(A,B)}$, then*

$$\frac{g(z)}{h(z)} \prec b_1 \left(\frac{1+Az}{1+Bz}\right)^{\alpha},$$

where $b_1 \in \mathbb{R}$, $\alpha \ge 0$, $-1 \le B < A \le 1$.

Proof. Since $f = h(z) + \overline{g(z)} \in \mathcal{S}_{HK(A,B)}$, then we have

$$\frac{g'(z)}{h'(z)} \prec b_1 \left(\frac{1+Az}{1+Bz}\right)^{\alpha},$$

and

$$Re\left(z\frac{h'(z)}{h(z)}\right) > \frac{1}{2} \Rightarrow z\frac{h'(z)}{h(z)} \prec \frac{1}{1-z} \Rightarrow \left|z\frac{h'(z)}{h(z)} - \frac{1}{1-r^2}\right| \le \frac{r}{1-r^2},$$

therefore the boundary value of $\left(z\frac{h'(z)}{h(z)}\right)$ on the circle is $\left(\frac{1+re^{i\theta}}{1-r^2}\right)$. On the other hand, if we investigate the properties of the linear transformation $\left(\frac{1+Az}{1+Bz}\right)^{\alpha}$, and using subordination principle or Lindelöf principle with $0 < r < 1$, $0 < \frac{1-Ar}{1-Br} < 1$, $\frac{1+Ar}{1+Br} > 1$ we obtain

$$\left(\frac{1-Ar}{1-Br}\right)^{\alpha} \le \frac{1-Ar}{1-Br} < |p(z)| \le \frac{1+Ar}{1+Br} \le \left(\frac{1+Az}{1+Bz}\right)^{\alpha}.$$

Therefore

$$w(\mathbb{D}_r) = \left\{\frac{g'(z)}{h'(z)}| \left(\frac{1-Ar}{1-Br}\right)^{\alpha} \le \frac{1-Ar}{1-Br} < \left|\frac{g'(z)}{h'(z)}\right| \le \frac{1+Ar}{1+Br} \le \left(\frac{1+Az}{1+Bz}\right)^{\alpha}\right\}, \tag{2.1}$$

where $\alpha \ge 0$, $-1 \le B < A \le 1$, $0 < r < 1$. Now we define the function $\phi(z)$ by

$$\frac{g(z)}{h(z)} = b_1 \left(\frac{1+A\phi(z)}{1+B\phi(z)}\right)^{\alpha},$$

then $\phi(z)$ analytic and $\phi(0) = 0$. Now, it is easy to realize that the subordination

$$\frac{g(z)}{h(z)} = b_1 \left(\frac{1+A\phi(z)}{1+B\phi(z)}\right)^{\alpha}$$

is equivalent to $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. Indeed assume the contrary that there exists a $z_0 \in \mathbb{D}$ such that $|\phi(z_0)| = 1$. then by using I. S. Jack Lemma we have

$$w(z_0) = \frac{g'(z_0)}{h'(z_0)} = b_1 \left(\frac{1+A\phi(z_0)}{1+B\phi(z_0)}\right)^{\alpha} \left(1 + \frac{k\alpha(A-B)\phi(z_0)}{(1+A\phi(z_0))(1+B\phi(z_0))}.\frac{1-r^2}{1+re^{i\theta}}\right) \notin w(\mathbb{D}_r)$$

because $|\phi(z_0)| = 1$, $k \geq 1$ and the relation (2.1). But this is contradiction to the condition of the definition of $\mathcal{S}_{HK(A,B)}$ and so assumption is wrong, i.e., $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. □

Corollary 2.2. *Let $f = h(z) + \overline{g(z)}$ be an element of $\mathcal{S}_{HK(A,B)}$ then*

$$rF(A, B, -r) \leq |g(z)| \leq rF(A, B, r), \tag{2.2}$$

$$rG(A, B, -r) \leq |g'(z)| \leq rG(A, B, r), \tag{2.3}$$

where

$$F(A, B, r) = \frac{1}{1-r}\left(\frac{1+Ar}{1+Br}\right)^{\alpha},$$
$$G(A, B, r) = \frac{1}{1-r}^{2}\left(\frac{1+Ar}{1+Br}\right)^{\alpha}.$$

Proof. Since $\frac{g(z)}{h(z)} \prec b_1\left(\frac{1+Ar}{1+Br}\right)^{\alpha}$, and $\frac{g'(z)}{h'(z)} \prec b_1\left(\frac{1+Ar}{1+Br}\right)^{\alpha}$ then we have (using Theorem 2.1)

$$|h(z)|\left(\frac{1-Ar}{1-Br}\right)^{\alpha} \leq |g(z)| \leq |h(z)|\left(\frac{1+Ar}{1+Br}\right)^{\alpha}, \tag{2.4}$$

$$|h'(z)|\left(\frac{1-Ar}{1-Br}\right)^{\alpha} \leq |g'(z)| \leq |h'(z)|\left(\frac{1+Ar}{1+Br}\right)^{\alpha}. \tag{2.5}$$

If we use Theorem 2.1 in the inequalities (2.4) and (2.5), then we obtain (2.2) and (2.3). □

Corollary 2.3. *Let $f = h(z) + \overline{g(z)}$ be an element of $\mathcal{S}_{HK(A,B)}$ then*

$$r^2F(A, B, -r) \leq J_f \leq r^2F(A, B, r), \tag{2.6}$$

where

$$F(A, B, r) = \frac{1}{(1+r)^4}\left[1 - \left(\frac{1+Ar}{1+Br}\right)^{2\alpha}\right].$$

Proof. Using Theorem 2.1 we can write

$$\left(\frac{1-Ar}{1-Br}\right)^{\alpha} = |w(z)| = \left|\frac{g'(z)}{h'(z)}\right| \leq \left(\frac{1+Ar}{1+Br}\right)^{\alpha}.$$

Therefore we have

$$\left[1 - \left(\frac{1+Ar}{1+Br}\right)^{2\alpha}\right] \leq (1 - |w(z)|^2) \leq \left[1 - \left(\frac{1-Ar}{1-Br}\right)^{2\alpha}\right]. \tag{2.7}$$

On the other hand

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2 = |h'(z)|^2(1 - |w(z)|^2). \tag{2.8}$$

considering (2.7), (2.8) and Theorem 2.1 we get (2.6). □

Corollary 2.4. *If $f = h(z) + \overline{g(z)} \in \mathcal{S}_{HK(A,B)}$, then*

$$\int_0^r \frac{\rho}{(1+\rho)^2}\left(1 - \left(\frac{1+A\rho}{1+B\rho}\right)^{\alpha}\right)d\rho \leq |f| \leq \int_0^r \frac{\rho}{(1-\rho)^2}\left(1 + \left(\frac{1+A\rho}{1+B\rho}\right)^{\alpha}\right)d\rho. \tag{2.9}$$

Proof. Since

$$1+\left(\frac{1-A\rho}{1-B\rho}\right)^{\alpha} \leq 1+|w(z)| \leq 1+\left(\frac{1+A\rho}{1+B\rho}\right)^{\alpha}, \tag{2.10}$$

$$1-\left(\frac{1+A\rho}{1+B\rho}\right)^{\alpha} \leq 1-|w(z)| \leq 1-\left(\frac{1-A\rho}{1-B\rho}\right)^{\alpha}, \tag{2.11}$$

and $(|h'(z)|-|g'(z)|)|dz| \leq |df| \leq (|h'(z)|+|g'(z)|)|dz| \Rightarrow$

$$|h'(z)|(1-|w(z)|)|dz| \leq |df| \leq |h'(z)|(1+|w(z)|)|dz|. \tag{2.12}$$

Considering (2.10), (2.11) and (2.12) we obtain

$$\frac{r}{(1+r)^2}\left(1-\left(\frac{1+Ar}{1+Br}\right)^{\alpha}\right)|dz| \leq |df| \leq \frac{r}{(1-r)^2}\left(1+\left(\frac{1+Ar}{1+Br}\right)^{\alpha}\right)|dz|. \tag{2.13}$$

Integrating (2.13) we get (2.9). □

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