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SOME PROPERTIES OF INTEGRAL OPERATORS OF p-VALENT FUNCTIONS

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ABSTRACT. In this paper, we study some properties of a new general integral operator of p-valent functions on some classes of p-valently starlike, k-uniformly p-valent starlike and p-valently convex functions.

1. INTRODUCTION

Let $\mathcal{U} = \{z : |z| < 1\}$ be the open unit disk and $\mathcal{A}$ be the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,\ z \in \mathcal{U}$$

which are analytic in $\mathcal{U}$ and satisfy the conditions $f(0) = f^{'}(0) - 1 = 0$.

Let $\mathcal{A}_p$ the class of functions of the form:

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad (p \in \mathbb{N}^*),$$

which are analytic in $\mathcal{U}$.

A function $f \in \mathcal{A}_p$ is said to be p-valently starlike of order β $(0 \le \beta < p)$ if and only if

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta,\ z \in \mathcal{U}.$$

We denote by $\mathcal{S}_p^{\star}(\beta)$ the class of all such functions.

A function $f \in \mathcal{A}_p$ is said to be p-valently convex of order β, $(0 \le \beta < p)$ if and only if

$$\mathrm{Re}\left(1 + \frac{zf^{''}(z)}{f^{'}(z)}\right) > \beta, z \in \mathcal{U}.$$

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Let $\mathcal{C}_p(\beta)$ denote the class of all those functions which are p-valently convex of order β in $\mathcal{U}$.

We note that $\mathcal{S}_p^\star(0) = \mathcal{S}_p^\star$ and $\mathcal{C}_p(0) = \mathcal{C}_p$ are respectively, the classes of p-valently starlike and p-valently convex functions in $\mathcal{U}$. Also, we note that $\mathcal{S}_1^\star = \mathcal{S}^\star$ si $\mathcal{C}_1 = \mathcal{C}$ are, respectively the usual classes of starlike and convex functions in $\mathcal{U}$.

A function $f \in \mathcal{A}_p$ is in the class $\mathcal{US}_p(\beta, k)$ of k-uniformly p-valent starlike of order β, $(-1 \le \beta < p)$ in $\mathcal{U}$, if the condition

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)} - \beta\right) \ge k\left|\frac{zf'(z)}{f(z)} - p\right| \quad (k \ge 0,\, z \in \mathcal{U}).$$

is satisfied.

For uniformly starlike functions, we refer to the papers [11] - [13], [1].

Note that $\mathcal{US}_1(\beta, k) = \mathcal{UST}(\beta, k)$, where the classes $\mathcal{UST}(\beta, k)$ are the classes k-uniformly starlike of order β, $0 \le \beta < 1$ studied in [2].

2. Main results

For $\alpha_i > 0$, $f_i \in \mathcal{A}_p$ and $g_i, h_i \in \mathcal{A}$, for all $i = 1, 2, \ldots .n$, we define the following general integral operator

$$F_{p,n}(z) = \int_0^z pt^{(1-n)(p-1)} g_1^{p-1}(t)\left(\frac{f_1(t)}{h_1^p(t)}\right)^{\alpha_1} \ldots g_n^{p-1}(t)\left(\frac{f_n(t)}{h_n^p(t)}\right)^{\alpha_n} dt. \tag{2.1}$$

If we consider $p = 1$ and $g_i(z) = z$, $h_i(z) = z$, $i = 1, 2, \ldots, n$, in relation (2.1), we obtain of the general integral operator $F_{1,n}(z) = F_n(z)$, introduced and studied by D. Breaz and N. Breaz in [4] and D. Breaz et al. in [7] (see also [[3]-[10]]).

Also, considering $p = n = 1$, $\alpha_1 = \alpha \in [0, 1]$ in (2.1) we obtain the integral operator $\int_0^z \left(\frac{f(t)}{h(t)}\right)^\alpha dt$ studied in [15] and for $h(z) = z$, we obtain the integral operator $\int_0^z \left(\frac{f(t)}{t}\right)^\alpha dt$ studied in [14].

In this paper, we obtain the order of convexity of the operator $F_{p,n}$ on the classes $S_p^*(\beta)$, $\mathcal{US}_p(\beta, k)$. Also, as particular cases, the order of convexity of the operators $\int_0^z \left(\frac{f(t)}{t}\right)^\alpha dt$, $\int_0^z \left(\frac{f(t)}{h(t)}\right)^\alpha dt$, are given.

2.1. P-valent convexity of the operator $F_{p,n}$.

Theorem 2.1. *Let $\alpha_i > 0$, $-1 \le \beta_i < p$, $0 \le \gamma_i < 1$ and $k_i > 0$ for all $i = 1, 2, \ldots .n$ and $f_i \in \mathcal{US}_p(\beta_i, k_i)$, $g_i, h_i \in \mathcal{S}^*(\gamma_i)$ for all $i = 1, 2, \ldots, n$. If*

$$0 \le p + n - pn + \sum_{i=1}^n (p - 1 - p\alpha_i)\gamma_i + \sum_{i=1}^n \alpha_i\beta_i < p$$

then the integral operator $F_{p,n}$ is p-valently convex of order

$$p + n - pn + \sum_{i=1}^n (p - 1 - p\alpha_i)\gamma_i + \sum_{i=1}^n \alpha_i\beta_i.$$

Proof. From relation (2.1), we observe that $F_{p,n}(z) \in \mathcal{A}_p$ and easily calculate the first derivative of $F_{p,n}$. We have:

$$F^{'}_{p,n}(z) = pz^{(1-n)(p-1)} \prod_{i=1}^{n} g_i^{p-1}(z) \left(\frac{f_i(z)}{h_i^p(z)} \right)^{\alpha_i}. \tag{2.2}$$

We differentiate (2.2) logarithmically and multiply by z and we get:

$$\frac{zF^{''}_{p,n}(z)}{F^{'}_{p,n}(z)} = (1-n)(p-1) + \sum_{i=1}^{n} \left[(p-1)\frac{zg_i'(z)}{g_i(z)} + \alpha_i \left(\frac{zf_i'(z)}{f_i(z)} - p\frac{zh_i'(z)}{h_i(z)} \right) \right].$$

Futher, we obtain:

$$1 + \frac{zF^{''}_{p,n}(z)}{F^{'}_{p,n}(z)} = p + n - pn + \sum_{i=1}^{n} \left[(p-1)\frac{zg_i'(z)}{g_i(z)} + \alpha_i \left(\frac{zf_i^{'}(z)}{f_i(z)} - \beta_i \right) - p\alpha_i \frac{zh_i'(z)}{h_i(z)} \right]$$

$$+ \sum_{i=1}^{n} \alpha_i \beta_i. \tag{2.3}$$

Calculating the real part of both sides of (2.3), we have:

$$\mathrm{Re}\left(1 + \frac{zF^{''}_{p,n}(z)}{F^{'}_{p,n}(z)} \right) = p + n - pn + \sum_{i=1}^{n} \left[(p-1)\mathrm{Re}\left(\frac{zg_i'(z)}{g_i(z)} \right) + \alpha_i \mathrm{Re}\left(\frac{zf_i^{'}(z)}{f_i(z)} - \beta_i \right) \right]$$

$$- \sum_{i=1}^{n} p\alpha_i \mathrm{Re}\left(\frac{zh_i'(z)}{h_i(z)} \right) + \sum_{i=1}^{n} \alpha_i \beta_i. \tag{2.4}$$

Since $f_i \in \mathcal{US}_p(\beta_i, k_i)$ and $g_i, h_i \in \mathcal{S}^*(\gamma_i)$ for all $i = 1, 2, \dots n$, using (2.4), we have:

$$\mathrm{Re}\left(1 + \frac{zF^{''}_{p,n}(z)}{F^{'}_{p,n}(z)} \right) > p + n - pn + \sum_{i=1}^{n} (p-1)\gamma_i$$

$$+ \sum_{i=1}^{n} \alpha_i k_i \left| \frac{zf_i^{'}(z)}{f_i(z)} - p \right| + \sum_{i=1}^{n} p\alpha_i \gamma_i + \sum_{i=1}^{n} \alpha_i \beta_i. \tag{2.5}$$

Because $\sum_{i=1}^{n} \alpha_i k_i \left| \frac{zf_i^{'}(z)}{f_i(z)} - p \right| > 0$, for all $i = 1, 2, \dots n$, from (2.5), we obtain:

$$\mathrm{Re}\left(1 + \frac{zF^{''}_{p,n}(z)}{F^{'}_{p,n}(z)} \right) > p + n - pn + \sum_{i=1}^{n} (p - p\alpha_i - 1)\gamma_i + \sum_{i=1}^{n} \alpha_i \beta_i.$$

Therefore $F_{p,n}$ is p-valently convex of order

$$p + n - pn + \sum_{i=1}^{n} (p - p\alpha_i - 1)\gamma_i + \sum_{i=1}^{n} \alpha_i \beta_i.$$

□

Considering $p = n = 1$, $\alpha_1 = \alpha$, $\beta_1 = \beta$,$\gamma_1 = \gamma$ $k_1 = k$, $f_1 = f$ and $h_1(z) = z, g1(z) = z$ in Theorem 2.1, we have:

Corollary 2.1. *Let $\alpha > 0$, $-1 \leq \beta < 1$, $0 \leq \gamma < 1$, $k > 0$ and $f \in \mathcal{UST}(\beta, k)$, $g, h \in \mathcal{S}^*(\gamma)$. If $0 \leq 1+\alpha(\beta-\gamma) < 1$ then the integral operator $\int_0^z \left(\frac{f(t)}{t}\right)^\alpha dt$ is convex of order $1+\alpha(\beta-\gamma)$ in $\mathcal{U}$.*

Theorem 2.2. *Let $\alpha_i > 0$, $-1 \leq \beta_i < p$, $0 \leq \gamma_i < 1$ and $k_i > 0$ for all $i = 1, 2, \ldots n$, and*

$$\left|\frac{zf_i^{'}(z)}{f_i(z)} - p\right| > -\frac{p+n-pn+\sum_{i=1}^n (p-p\alpha_i-1)\gamma_i + \sum_{i=1}^n \alpha_i\beta_i}{\sum_{i=1}^n \alpha_i k_i} \tag{2.6}$$

for $i = 1, 2, \ldots, n$, then the integral operator $F_{p,n}$ is p-valently convex in U.

Proof. From (2.5) and (2.6) we easily get $F_{p,n} \in \mathcal{C}_p$. □

From Theorem 2.2, using $\mathrm{Re} z \leq |z|$ we get

Corollary 2.2. *Let $\alpha_i > 0$, $-1 \leq \beta_i < p$, $0 \leq \gamma_i < 1$ and $k_i > 0$ for all $i = 1, 2, \ldots n$, and*

$$\mathrm{Re}\left(\frac{zf_i^{'}(z)}{f_i(z)}\right) > p - \frac{p+n-pn+\sum_{i=1}^n (p-p\alpha_i-1)\gamma_i + \sum_{i=1}^n \alpha_i\beta_i}{\sum_{i=1}^n \alpha_i k_i}$$

that is $f_i \in \mathcal{S}_p^*(\sigma)$, with $\sigma = p - \frac{p+n-pn+\sum_{i=1}^n (p-p\alpha_i-1)\gamma_i + \sum_{i=1}^n \alpha_i\beta_i}{\sum_{i=1}^n \alpha_i k_i}$ and $0 \leq \sigma < p$, for all $i = 1, 2, \ldots, n$, then the integral operator $F_{p,n}$ is p-valently convex in $\mathcal{U}$.

Considering $n = p = 1$, $\alpha_1 = \alpha$, $\beta_1 = \beta$, $\gamma_1 = \gamma$, $k_1 = k$ and $f_1 = f$, $g1(z) = z$, $h1(z) = z$ in Corollary 2.2, we have:

Corollary 2.3. *Let $\alpha > 0$, $-1 \leq \beta < 1$,$0 \leq \gamma < 1$, $k > 0$ and $f \in \mathcal{S}^\star(\delta)$, where $\delta = 1 - \frac{1}{\alpha k}$;$0 \leq \delta < 1$, then the integral operator $\int_0^z \left(\frac{f(t)}{t}\right)^\alpha dt$ is convex in $\mathcal{U}$.*

Remark 2.1. *If we consider $g_i(t) = t$, $h_i(t) = t$, for all $i = 1, 2, \ldots .n$ in relation (2.1) we get the integral operator $F_p(z) = \int_0^z pt^{p-1}\left(\frac{f_1(t)}{t^p}\right)^{\alpha_1} \ldots. \left(\frac{f_n(t)}{t^p}\right)^{\alpha_n} dt$ defined and studied by B. A. Frasin in* [8].

Remark 2.2. *If we consider $p = n = 1$, $\alpha_1 = \alpha$, $\beta_1 = \beta$, $k_1 = k$ and $f_1 = f$, $g_1 = h_1 = h$, $f \in \mathcal{UST}(\beta, k)$ in Theorem 2.1, then we get the convexity of $I_1(f, h)(z) = \int_0^z \left(\frac{f(t)}{h(t)}\right)^\alpha dt$, which is a particular case of the integral operator*

$$I(f, g)(z) = \int_0^z \prod_{i=1}^n \left(\frac{f_i(t)}{g_i(t)}\right)^{\alpha_i} dt$$

defined and studied by N. Ularu and D. Breaz in [15].

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