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# REPRODUCING KERNELS FOR HOLOMORPHIC VECTOR BUNDLES

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Abstract. The maps defined by reproducing kernels on total and base spaces of holomorphic vector bundles into some Hilbert and Grassmann spaces are considered and the main results concerning basic properties of this maps are proved.

## 1. Introduction

This work deals with mappings defined by reproducing kernels of the Bergman function type for holomorphic sections of complex vector bundles. Such mappings seems to be very interesting from the geometric as well as physical point of view (see [4], [5] or [9]). In Section 3 we show that the mappings mentioned above are holomorphic (Theorem 3.2) and describe how to use them in the proof of Kodaira embedding theorem (Theorem 3.5). In Section 4 we use this mappings to define Kählerian on the base manifold and the new Hermitian structure on the considered bundle. Section 2 contains the description of main results of [7]. Without any other explanation we use the following symbols: $\mathbf{N}$-the set of natural numbers; $\mathbf{R}$-the set of reals; $\mathbf{C}$-the complex plain.

## 2. Preliminaries

All proofs of theorems and propositions presented in this section are given in [7].

Assume that there are given:

- $\mathbf{E} = (E, \pi, M)$ - a holomorphic vector bundle over a complex manifold $M$;
- $\mu \in \Gamma^\infty(\bigwedge^{2n} T^*M)$ - a volume form on $M$, where $n := dim_{\mathbf{C}} M$;
- $h$ - a Hermitian structure on $\mathbf{E}$.

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We denote by $L^2(\mathbf{E}) = L^2(\mathbf{E}, h, \mu)$ the Hilbert space being a completion of the space $\Gamma_0^\infty(\mathbf{E})$ of all smooth sections of $\mathbf{E}$ with a compact support with respect to the norm $\|\cdot\|$ defined by the scalar product

$$< s \mid t >:= \int\limits_M h(s,t)\mu, \qquad s, t \in \Gamma_0^\infty(\mathbf{E}).$$

It is known that $L^2(\mathbf{E})$ can be identify with the space (of classes) of all Lebesgue measurable sections $s$ of $\mathbf{E}$ for which the integral

$$\|s\|^2 = \int\limits_M h(s,s)\mu(s)$$

is finite. When $M$ has a countable basis of topology one can prove that $L^2(\mathbf{E})$ is a separable Hilbert space.

Let $L^2H(\mathbf{E}) = L^2H(\mathbf{E}, h, \mu)$ denotes the space of all elements of $L^2(\mathbf{E})$ which can be identify with holomorphic sections of $\mathbf{E}$, i.e.,

$$L^2H(\mathbf{E}) = L^2(\mathbf{E}) \cap H(M, \mathbf{E}),$$

where $H(M, \mathbf{E}) = H^o(M, \mathbf{E})$ is the space of all holomorphic sections of $\mathbf{E}$. We call $L^2H(\mathbf{E})$ *the* $(h, \mu)$*-Bergman space of sections of* $\mathbf{E}$.

For any element $v^*$ of the bundle $\mathbf{E}^* = (E^*, \pi', M)$ dual to $\mathbf{E}$ we define the evaluation functional $\mathcal{E}_{v^*}$ on $L^2H(\mathbf{E})$ by the following formula

$$\mathcal{E}_{v^*}s := v^*(s(\pi'(v^*))), \qquad s \in L^2H(\mathbf{E}),$$

where $\pi' : E^* \to M$ is the vector bundle projection. It is clear that $\mathcal{E}_{v^*}$ is a linear functional. Using similar methods as in the case of an ordinary Bergman space (see [2] or [6]) one can prove the following

**Proposition 2.1.** *For any $v^* \in E^*$ there exists a neighbourhood $Y$ of $v^*$ in $E^*$ and a constant $C > 0$ such that for any $w^* \in Y$ and any $s \in L^2H(\mathbf{E})$*

$$|\mathcal{E}_{w^*}s| \le C\|s\|.$$

As a simple corollary of this proposition we obtain the following

**Theorem 2.1.**

(*i*) *For any $v^* \in E^*$ the evaluation functional $\mathcal{E}_{v^*}$ is continuous;*
(*ii*) *$L^2H(\mathbf{E})$ is a closed subspace of $L^2(\mathbf{E})$.*

Since $L^2H(\mathbf{E})$ is a Hilbert space we can use the Riesz theorem on the representation of linear functionals on this space. Hence for any $f \in L^2H(\mathbf{E})^*$ there exists an unique element $f^\#$ of $L^2H(\mathbf{E})$ such that for each $s \in L^2H(\mathbf{E})$

$$f(s) =< f^\# | s > .$$

Moreover the map $L^2H(\mathbf{E})^* \ni f \mapsto f^\# \in L^2H(\mathbf{E})$ is an antilinear isometry.

Let for a given $v^* \in E^*$

$$k_{v^*} := \mathcal{E}_{v^*}^\# \in L^2H(\mathbf{E}).$$

Since the map $\mathbf{E}_x^* \ni v^* \mapsto \mathcal{E}_{v^*} \in L^2H(\mathbf{E})^*$ is linear we obtain that $\mathbf{E}_x^* \ni v^* \mapsto k_{v^*} \in L^2H(\mathbf{E})$ is an antilinear mapping for any $x \in M$. Hence the mapping

$$\mathbf{E}_x^* \ni v^* \mapsto \overline{k_{v^*}(y)} \in \overline{\mathbf{E}}_y \tag{2.1}$$

is linear for any $x, y \in M$, where $\overline{\mathbf{E}}$ denotes the complex vector bundle over $M$ which is complex conjugated to $\mathbf{E}$ (we recall that the total space of $\overline{\mathbf{E}}$ coincides with the total space of $\mathbf{E}$ as a $C^\infty$-manifold with the same $\pi$ as a vector bundle projection, but the multiplication of elements of $\overline{\mathbf{E}}$ by complex numbers is given by the formula: $\lambda\overline{v} = \overline{\overline{\lambda}v}$ or $\overline{\lambda v} = \overline{\lambda}\overline{v}$, where $\lambda \in \mathbf{C}$ and $v \in E$; see [3]). Consequently the map (2.1) can be identyfied with a tensor $K(x,y) \in \mathbf{E}_x \otimes \overline{\mathbf{E}}_y$. Since $\mathbf{E}_x \otimes \overline{\mathbf{E}}_y$ is in a natural way a fibre of the vector bundle $\mathbf{E}\tilde{\otimes}\overline{\mathbf{E}}; = pr_1^*\mathbf{E} \otimes pr_2^*\overline{\mathbf{E}}$ over $M \times M$, where $pr_j; M \times M \to M, j = 1,2$ are ordinary projections ($pr_j(x_1, x_2) = x_j$ for $j = 1, 2$) we can identify the map $M \times M \ni (x, y) \mapsto K(x,y) \in \mathbf{E}_x \otimes \overline{\mathbf{E}}_y$ with a section of this bundle.

**Definition 2.1.** *Section $K$ will be called the $(h,\mu)$-Bergman section of the bundle $\mathbf{E}\tilde{\otimes}\overline{\mathbf{E}}$.*

Let us define the transposition $^t : \mathbf{E}\tilde{\otimes}\overline{\mathbf{E}} \to \overline{\mathbf{E}}\tilde{\otimes}\mathbf{E}$ as a vector bundle isomorphism given on homogeneous tensors by the formula

$$(v_x \otimes \overline{v_y})^t := \overline{v_y} \otimes v_x, \qquad v_x \in \mathbf{E}_x, \quad v_y \in \mathbf{E}_y, \quad x, y \in M.$$

The main properties of $K$ are described in the following

**Theorem 2.2.** *The $(h,\mu)$-Bergman section $K$ has the following properties:*

(i) *$K(y,x) = \overline{K(x,y)^t}$, $x, y \in M$;*
(ii) *$K(x,y)$ is holomorphic in $x$ and antiholomorphic in $y$;*
(iii) *$K$ is $\mathbf{R}$-analytic on $M \times M$.*

## 3. Maps given by the Bergman section and the Kodaira embeding theorem

For the proofs of results presented in this section see [8].

Let us denote

$$\overline{L^2H(\mathbf{E})} := \{\overline{f} : f \in L^2H(\mathbf{E})\}.$$

We will consider this vector space as a subspace of

$$\overline{L^2(\mathbf{E})} := \{\overline{f} : f \in L^2(\mathbf{E})\},$$

where the last space is the Hilbert space with the scalar product

$$\begin{aligned} < \overline{s}|\overline{t} > &:= \int_M \overline{h}(\overline{s}, \overline{t})\mu \\ &= \int_M h(t,s)\mu = < t|s >, \qquad s, t \in L^2(\mathbf{E}). \end{aligned}$$

It is easy to show that

$$\overline{L^2(\mathbf{E})} = L^2(\overline{\mathbf{E}}) = L^2(\overline{\mathbf{E}}, \overline{h}, \mu)$$

and that the complex conjugation

$$L^2(\mathbf{E}) \ni s \mapsto \overline{s} \in L^2(\overline{\mathbf{E}})$$

is an antilinear isometry of Hilbert spaces.

Let ua consider the map

$$E^* \ni v^* \mapsto \mathcal{J}(v^*) := \overline{k_{v^*}} \in \overline{L^2H(\mathbf{E})}.$$

We have

$$[\mathcal{J}(v^*)](y) = K(x,y)v^*, \quad v^* \in \mathbf{E}^* \quad x := \pi'(v^*), \quad y \in M.$$

**Theorem 3.1.** *The map $\mathcal{J}$ is continuous.*

Let $(U, \varphi)$ be *a vector bundle chart* on $\mathbf{E}$ i.e.: (i) $U$ is an open nonempty subset of $M$; (ii) $\varphi : \pi^{-1}(U) \to V \times \mathbf{C}^r$ is a biholomorphism, where $V$ is an open subset of $\mathbf{C}^n$; (iii) if $\varphi = (\varphi_1, \varphi_2, ..., \varphi_{n+r})$ then for any $x \in U$ the map $(\varphi_{n+1}, ..., \varphi_{n+r})_{|\mathbf{E}_x}$ is an isomorphism of $\mathbf{E}_x$ onto $\mathbf{C}^r$. Then the map $\tilde{\varphi} = (\varphi_1, ..., \varphi_n) : U \to V$ is a holomorphic chart on $M$. Let $\mathbf{e} := (e_1, .., e_r)$ be a holomorphic frame of $\mathbf{E}$ defined on $U$ as follows

$$e_k(x) := \varphi^{-1}(\tilde{\varphi}(x), 0, ..., \underbrace{1}_{(k-th)-place}, ..., 0), \quad x \in U, \quad k = 1, 2, ..., r.$$

For any $x \in U$ the sequence $(e_1(x), ..., e_r(x))$ is the ordered basis in the vector space $\mathbf{E}_x$.

The vector bundle chart $(U, \varphi)$ canonicaly defines a vector bundle chart $(U, \varphi')$ on the bundle $\mathbf{E}^*$ . Namely, if $(e_1, ..., e_r)$ is a frame on $U$ defined by $\varphi$ then for any $x \in U$ and any $v^* \in \mathbf{E}^*_x$ $\varphi'_{j+n}(v^*)$ is the j-th coordinate of $v^*$ with respect to the base $(e^{*1}(x), e^{*2}(x), ..., e^{*r}(x))$ of $\mathbf{E}^*_x$ dual to $(e_1(x), ..., e_r(x))$ for $j = 1, 2, .., r$. Moreover $\varphi'_j(v^*) := \tilde{\varphi}'_j(\pi'(v^*)) := \tilde{\varphi}_j(x)$ for $j = 1, 2, ..., r$. We have $\tilde{\varphi'} = \tilde{\varphi} : U \to V$ and

$$\varphi'\left(\sum_{j=1}^{r} \alpha_j e^{*j}(x)\right) = (\tilde{\varphi}(x), \alpha_1, \alpha_2, ..., \alpha_r), \qquad v^* \in \mathbf{E}^*_x, \quad x \in U.$$

If $z \in V$ then for any $(\alpha_1, .., \alpha_r), (\beta^1, ..., \beta^r) \in \mathbf{C}^r$

$$[\varphi'^{-1}(z, \alpha_1, ..., \alpha_r)](\varphi^{-1}(z, \beta^1, ..., \beta^r)) = \sum_{j=1}^{r} \alpha_j \beta^j.$$

If $D$ is an open subset in $\mathbf{C}^m$, $H$ is a complex Hilbert space and $F : D \to H$ then we say that $F$ is *holomorphic* if for any $z_0 = (z_{01} \ldots, z_{0m}) \in D$ there exists a polydisc $P \subset D$ with center at $z_0$ such that $F_{|P}$ can be expressed as the power series of the form

$$F(z) = F(z_1 \ldots, z_m) = \sum_{k_1,\ldots,k_m=1}^{\infty} \alpha_{k_1,\ldots,k_m}(z_1 - z_{01})^{k_1} \ldots (z_m - z_{0m})^{k_m}.$$

A map $F : M \to H$, where $M$ is a complex manifold, is holomorphic if for any holomorphic chart $(U, \varphi)$ the superposition $F \circ \varphi^{-1}$ is holomorphic on $\varphi(U)$. Using this definition and the previous considerations one can prove the following:

**Theorem 3.2.** *The map $\mathcal{J}$ is holomorphic.*

In the proof one can use the Cauchy integral formula

$$\mathcal{J}(\tilde{e}_1^{*m}(z)) = \frac{1}{(2\pi i)^n} \int\limits_{\Gamma(a,\rho)} \frac{1}{(w^1 - z^1)\cdots(w^n - z^n)} \mathcal{J}(\tilde{e}_1^{*m}(w))dw,$$

the expansion of the function

$$f(z) = \frac{1}{(w^1 - z^1)\cdots(w^n - z^n)}, \quad z \in P,$$

into the power series (with respect to the powers of $(z - z_0)$) and the standard arguments concerning the integration of power series to obtain that $\mathcal{J} \circ \tilde{e}_1^{*m}$ is the $\mathbf{C}$-analytic mapping on $V$ for $m = 1, 2, ..., r$. To complete the proof of the theorem it is enough to note that for any $(z, \alpha) = (z, \alpha_1, ..., \alpha_r) \in V \times \mathbf{C}^r$

$$\mathcal{J}(\varphi_1'^{-1}(z, \alpha) = \mathcal{J}\Big(\sum_{m=1}^{r} \alpha_m \tilde{e}_1^{*m}(z)\Big) = \sum_{m=1}^{r} \alpha_m \mathcal{J}(\tilde{e}_1^{*m}(z)). \qquad \square$$

Let us consider the following condition:
$(\mathbf{A})$ *for any* $v^* \in E^*$ $\quad v^* \neq 0$ *there exists* $s \in L^2H(\mathbf{E})$ *such that*

$$\mathcal{E}_{v^*} s = v^*\Big(s(\pi'(v^*))\Big) \neq 0.$$

**Proposition 3.1.** *If the condition* $(\mathbf{A})$ *is satisfied then for any* $x \in M$

$$dim \mathcal{J}(\mathbf{E}_x^*) = dim \mathbf{E}_x^* = r.$$

*Proof.* The condition $(\mathbf{A})$ implies that for any $v^* \neq 0$ we have $k_{v^*} \neq 0$. Consequently the linear mapping

$$\mathbf{E}_x^* \ni v^* \mapsto \mathcal{J}(v^*) = \overline{k_{v^*}} \in \overline{L^2H(\mathbf{E})}$$

is an isomorphism onto its image. $\square$

**Assumption.** In the remaining part of this section we suppose that the condition $(\mathbf{A})$ is satisfied.

For any vector bundle chart $(U, \varphi)$ on $\mathbf{E}$ we define a map $B_\varphi : U \to \Big(\overline{L^2H(\mathbf{E})}\Big)^r$ as follows

$$B_\varphi := (\underbrace{\mathcal{J}, J, ..., J}_{r-times}) \circ (e^{*1}, e^{*2}, ..., e^{*r}),$$

i.e.

$$B_\varphi(x) := (\overline{k_{e^{*1}(x)}}, ..., \overline{k_{e^{*r}(x)}}), \qquad x \in U.$$

By Theorem 3.2 $B_\varphi$ is a holomorphic mapping. By Proposition 3.1 the value $B_\varphi(x)$is a sequence of linearly independent vectors in $\overline{L^2H(\mathbf{E})}$ for any $x \in U$. Consequently we can consider $B_\varphi$ as a holomorphic map on $U_\varphi$ into the space $\mathcal{B}_r(\overline{L^2H(\mathbf{E})})$ of all $r$-element sequences of linearly independent vectors in $\overline{L^2H(\mathbf{E})}$ ($\mathcal{B}_r(\overline{L^2H(\mathbf{E})})$ is an open subset in the Hilbert space $(\overline{L^2H(\mathbf{E})})^r$). Let $G_r(\overline{L^2H(\mathbf{E})})$ denotes the Grassmann space of all $r$-dimensional subspaces in $\overline{L^2H(\mathbf{E})}$ and let

$$\alpha_r : \mathcal{B}_r(\overline{L^2H(\mathbf{E})}) \to G_r(\overline{L^2H(\mathbf{E})})$$

be the natural projection, which assignees to any sequence $(s_1, ..., s_r) \in \mathcal{B}_r(\overline{L^2H(\mathbf{E})})$ the vector subspace $\alpha_r(s_1, ..., s_r) \subset \overline{L^2H(\mathbf{E})}$ spanned by vectors $s_1, ..., s_r$. It is well known that $\alpha_r$ is holomorphic with respect to natural complex structures on $\mathcal{B}_r(\overline{L^2H(\mathbf{E})})$ and $G_r(\overline{L^2H(\mathbf{E})})$.

Now we are ready to define the map

$$\mathcal{Z} : M \to G_r(\overline{L^2H(\mathbf{E})})$$

as follows: *for any vector bundle chart $(U, \varphi)$ on $\mathbf{E}$*

$$\mathcal{Z}_{|U} := \alpha_r \circ B_\varphi. \tag{3.1}$$

Since for any $x \in M$

$$\mathcal{Z}(x) = \alpha_r(B_\varphi(x)) = \mathcal{J}(\mathbf{E}_x^*)$$

we see that $\mathcal{Z}(x)$ does not depend on $\varphi$. By the previous considerations it is clear that the map $\mathcal{Z}$ is holomorphic.

Let us recall the famous Kodaira embedding theorem

**Theorem 3.3.** *(Kodaira) If on a complex compact manifold $M$ there exists a positive line bundle $\mathbf{L}$ then for some $N \in \mathbf{N}$ there exists an embedding $\mathcal{Z}$ of $M$ into the complex projective space $\mathbf{P}^N$. (see [1])*

The most important and difficult step in the proof of this theorem is to show that

**Theorem 3.4.** *If $\mathbf{L}$ is a positive holomorphic line bundle over a complex compact manifold $M$ then there exists $k_o \in \mathbf{N}$ such that for any $k \geq k_o$ the bundle $\mathbf{L}^k = \underbrace{\mathbf{L} \otimes \cdots \otimes \mathbf{L}}_{k-times}$ has the following properties:*

- (K1) *for any $x, y \in M$ there exists a holomorphic section $s$ of $\mathbf{L}^k$ such that $s(x) = 0$ and $s(y) \neq 0$;*
- (K2) *for any $x \in M$ and any covector $v^* \in T_x^*M$ there exists a vector bundle chart $(U, \varphi)$ and a holomorphic section $s = s_1e_1$ of $\mathbf{L}^k$ such that $s(x) = 0$ and $ds_1(x) = v^*$, where $e_1$ is a frame of $\mathbf{L}^k$ defined on $U$ by $\varphi$.*

For the proof see [1] Chapter 1, Section 4.

We will write now conditions which are equivalent to (K1) and (K2) in the case when $M$ is compact but are more appropriate in our approach. Namely, let $\mathbf{L}$ be a holomorphic line vector bundles over $M$ with the hermitian structure $h$ and let $\mu$ be a volume form on $M$. We say that the triple $(\mathbf{L}, h, \mu)$ satisfies the conditions (K1') and (K2') if: *there exists $k_o \in \mathbf{N}$ such that for any $k \geq k_o$*

- (K1') *and for any $x, y \in M$ there exists a section $s \in L^2H(\mathbf{L}^k)$ such that $s(x) = 0$ and $s(y) \neq 0$;*
- (K2') *for any $x \in M$ and any covector $v^* \in T_x^*M$ there exists a vector bundle chart $(U, \varphi)$ on $\mathbf{L}^k$ and a section $s \in L^2H(\mathbf{L}^k)$ such that $x \in U$, $s(x) = 0$ and $ds_1(x) = v^*$, where $s_{|U} = s_1e_1$ and $e_1$ is a frame of $\mathbf{L}^k$ defined on $U$ by $\varphi$.*

In the above definition we suppose that $\mathbf{L}^k$ is a hermitian bundle with the hermitian metric $h^k := \underbrace{h \otimes \cdots \otimes h}_{k-times}$.

If Theorem 3.4 is proved then the Kodaira theorem is a consequence of the following result

**Theorem 3.5.** *Let the triple* $(\mathbf{L}, h, \mu)$ *satisfies the conditions (K1') and (K2'). Then for any* $k \geq k_o$ *the map* $\mathcal{Z}$ *defined for* $\mathbf{L}^k$ *by (3.1) is an embeding of* $M$ *into the projective space* $\mathbf{P}(\overline{L^2H(\mathbf{L}^k)})$.

*Proof.* Let $x, y \in M$ and $x \neq y$. Let $s \in L^2H(\mathbf{L}^k)$ be such that $s(x) = 0$ and $s(y) \neq 0$. Then for any $v^* \in \mathbf{L}_x^{-k}$

$$< \overline{\mathcal{J}(v^*)}|s >=< k_{v^*}|s >= v^*(s(x)) = 0$$

This means that $\overline{\mathcal{J}(\mathbf{L}_x^{-k})}$ is a subspace of $L^2H(\mathbf{L}^k)$ orthogonal to $s$. On the other hand there exists $w^* \in \mathbf{L}_y^{-k}$ such that

$$< \overline{\mathcal{J}(w^*)}|s >=< k_{w^*}|s >= w^*(s(y)) \neq 0$$

Then $\overline{\mathcal{J}(\mathbf{L}_y^{-k})}$ is not orthogonal to $s$, which implies that

$$\mathcal{Z}(x) = \mathcal{J}(\mathbf{L}_x^{-k}) \neq \mathcal{J}(\mathbf{L}_y^{-k}) = \mathcal{Z}(y).$$

Hence $\mathcal{Z}$ is one-to-one map.

Suppose now that there exists a tangent vector $v \in T_xM$, where $x = \pi(v)$, such that $v \neq 0$ and $\mathcal{Z}_* v = 0$. Let $(U, \varphi)$ be a vector bundle chart on $\mathbf{L}^k$ such that $\tilde{\varphi}(x) = 0$. Let $V := \tilde{\varphi}(U)$ and $v' := \tilde{\varphi}_* v \in T_0 V$. Then

$$[D(\mathcal{J} \circ \tilde{e}^{*1})(0)]v' = 0$$

which implies that

$$\begin{aligned} \overline{[D(\mathcal{J} \circ \tilde{e}^{*1})(0)]v'} &= [D(\overline{\mathcal{J}} \circ \tilde{e}^{*1})(0)]\overline{v'} \\ &= [D_z k_{e^{*1}(\cdot)}(0)]\overline{v'} = 0. \end{aligned}$$

Hence for any $s \in L^2H(\mathbf{L}^k)$ we have

$$\overline{v}(s_1) = \overline{v}[e^{*1}(x)(s(x))] = \overline{v}(< k_{e^{*1}(x)}|s >=< [D_z k_{e^{*1}(\cdot)}(0)]\overline{v'}|s >= 0,$$

where $s_1 \in H(V)$ is such that $s = s_1 e_1$. This, however contradics the condition (K2'). $\square$

## 4. Riemannian and Hermitian structures defined by the embeding $\mathcal{Z}$

It is well known that if $H$ is a complex Hilbert space then on the Grassmann space $G_r(H)$ there exists canonical Kählerian structure $\kappa$ defined by the scalar product $< \cdot|\cdot >$ on $H$. Similarly on the tautological bundle $\tau(G_r(H))$ there exists canonical Hermitian structure $h_0$ also defined by the scalar product on $H$. For example, if $Y_0$ is a given $r$-dimensional subspace in $H$ spanned by orthonormal vectors $y_1, \dots, y_r \in H$ then one of coordinate neighbourhoods of $Y_0$ consists of all $r$-dimensional subspaces $Y_{w_1 \dots, w_r}$ spanned by vectors of the form: $y_1 + w_1, \dots, y_r + w_r$, where $w_1 \dots, w_r$ are arbitrary elements of $Y_0^\perp$. Then the matrix $\chi$ of the canonical Hermitian structure $h_0$ with respect to the basis $y_1 + w_1, \dots, y_r + w_r$ in $Y_{w_1 \dots, w_r}$ treated as the fiber of $\tau(G_r(H))$ is of the form

$$\chi = I_r + [< w_i|w_j >]_{i,j=1}^r.$$

Taking pull-backs $\mathcal{Z}^*\tau(G_r(H))$, $\mathcal{Z}^*\kappa$ and $\mathcal{Z}^*h_0$ we obtain that $\mathcal{Z}^*\tau(G_r(H))$ is isomorphic as a vector bundle to $\mathbf{E}$ but is endowed with the new Kählerian (and then Riemannian) structure on the base manifold $M$ and the new Hermitian structure. This structures seems to be very interesting from the geometric as well as physical point of view.

## References


[1] P. Griffiths, J. Harris: *Principles of algebraic geometry*, A Wiley-Interscience Publication, John Wiley & Sons, New York, 1978.

[2] S.G. Krantz: *Function theory of several complex variables*, Intersciences-Willey, New York, 1982.

[3] J.W. Milnor, J.D. Stasheff: *Characteristic classes*, Princeton University Press and University of Tokyo Press, Princeton, New Jersey 1974.

[4] A. Odzijewicz: *On reproducing kernels and quantizaton of states*, Commun. Math. Phys., **114** (1988), 577–597.

[5] A. Odzijewicz: *Coherent states and geometric quantization*, Commun. Math. Phys., **150** (1992), 385–413.

[6] Z. Pasternak-Winiarski: *On weights which admit the reproducing kernel of Bergman type*, Internat. J. Math. & Math. Sci., **15**(1) (1992), 1–14.

[7] Z. Pasternak-Winiarski, J. Wojcieszyński: *Bergman spaces and kernels for holomorphic vector bundles*, Demonstratio Math. **30**(1) (1997), 199–214.

[8] M. Monastyrski, Z. Pasternak-Winiarski: *Maps on complex manifolds into Grassmann spaces defined by reproducing kernels of Bergman type*, Demonstratio Math. **30** (2) (1997), 465–474.

[9] J. H. Rawnsley: *Coherent states and Kahler manifolds*, Quart. J. Math., Oxford **28** (1977), 403–415.



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