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A CERTAIN CLASS OF HARMONIC MAPPINGS RELATED TO THE CLOSE-TO-CONVEX FUNCTIONS OF ORDER BETA

YASEMIN KAHRAMANER

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Abstract. In the present paper we will investigate a certain class of harmonic mappings for which the second dilatation is close -to convex functions of order $\beta(\beta \geq 0)$ The aim of this paper is to give some properties of the class of functions

$$\begin{aligned} \mathcal{S}_{H(\mathcal{K}(\beta))} &= \{f = h(z) + \overline{g(z)} \in \mathcal{S}_H | w(z) \\ &= \frac{g'(z)}{h'(z)} = b_1(p(z))^\beta, \beta \geq 0, p(z) \in \tilde{P}, h(z) \in \mathcal{C}\}. \end{aligned}$$

1. Introduction

Let Ω be the family of functions $\phi(z)$ which are analytic in the open unit disc $\mathbb{D} = \{z| \quad |z| < 1\}$ and satisfying the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in \mathbb{D}$.

Next,$\mathcal{A}$ denote the class of analytic functions of the form $s(z) = z + a_2z^2 + a_3z^3 + \cdots$ in the open unit disc $\mathbb{D}$. Let $\mathcal{P}$ designate the class of functions $p(z) = 1 + p_1z + p_2z^2 + \cdots$ which are analytic, have positive real part in $\mathbb{D}$, and such that $p(z)$ is in $\mathcal{P}$ if and only if

$$p(z) = \frac{1 + \phi(z)}{1 - \phi(z)},$$

for some function $\phi(z) \in \Omega$ and every $z \in \mathbb{D}$.

Moreover, let $\mathcal{C}$ denote the family of functions $s(z) \in \mathcal{A}$ and such that $s(z)$ is in $\mathcal{C}$ if and only if

$$1 + z\frac{s''(z)}{s'(z)} = p(z),$$

for some $p(z) \in \mathcal{P}$ and all $z \in \mathbb{D}$, and let $s_1(z)$ be an element of $\mathcal{A}$ and satisfies the condition

$$\operatorname{Re}\frac{s_1'(z)}{s_2(z)} > 0,$$

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where $s_2(z) \in \mathcal{C}$, then $s_1(z)$ is called close -to-convex function. The class of such functions is denoted by $\mathcal{K}$. Let $F_1(z)$ and $F_2(z)$ be an elements of $\mathcal{A}$, if there exists a function $\phi(z) \in \Omega$ such that $F_1(z) = F_2(\phi(z))$ for every $z \in \mathbb{D}$, then we say that $F_1(z)$ is subordinate to $F_2(z)$ and we write $F_1(z) \prec F_2(z)$. Specially $F_2(z)$ is univalent in $\mathbb{D}$, then $F_1(z) \prec F_2(z)$ if and only if $F_1(\mathbb{D}) \subset F_2(\mathbb{D})$ and $F_1(0) = F_2(0)$ implies $F_1(\mathbb{D}_r) \subset F_2(\mathbb{D}_r)$ where $\mathbb{D}_r = \{z \mid |z| < r,\ 0 < r < 1\}$.(Subordination and Lindelöf principle [1])

Finally, a planar harmonic mapping in the open unit disc $\mathbb{D}$ is a complex-valued harmonic function f, which maps $\mathbb{D}$ onto the some planar domain $f(\mathbb{D})$. Since $\mathbb{D}$ is a simply connected domain the mapping f has a canonical decomposition $f(z) = h(z) + \overline{g(z)}$, where $h(z)$ and $g(z)$ are analytic in $\mathbb{D}$ and have the following power seies expansions

$$h(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=0}^{\infty} b_n z^n,$$

where $a_n, b_n \in \mathbb{C}$, $n = 0, 1, 2, ...$, as usual we call $h(z)$ analytic part of f and $g(z)$ is co-analytic part of f, an elegant and complete account of the theory of harmonic mappings is given Duren' s monograph [2]. Lewy proved in 1936 [2] that the harmonic function f is locally univalent if and only if its Jacobian

$$J_f = |h'(z)|^2 - |g'(z)|^2$$

is different from zero in $\mathbb{D}$. In wiew of this result, locally univalent harmonic mappings in the open disc $\mathbb{D}$ are either sense-preserving if $|h'(z)| > |g'(z)|$ or sense-reserving if $|h'(z)| < |g'(z)|$ in $\mathbb{D}$. In this paper we will restrict ourselves to the study of sense-preserving harmonic mappings. We will also note that $f(z) = h(z) + \overline{g(z)}$ is sense-preserving in $\mathbb{D}$ if and only if $h'(z)$ does not vanish in $\mathbb{D}$, and the second dilatation $w(z) = (\frac{g'(z)}{h'(z)})$ has the property $|w(z)| < 1$ for all $z \in \mathbb{D}$. Therefore the class of all sense-preserving harmonic mappings in the open unit disc with $a_0 = b_0 = 0$, $a_1 = 1$ will be denoted by $\mathcal{S}_H$, thus $\mathcal{S}_H$ contains standart class $\mathcal{S}$ of univalent functions. The family of all mappings $\mathcal{S}_H$ with the additional property $g'(0) = 0$, i.e, $b_1 = 0$ is denoted by $\mathcal{S}_H^0$. Hence it is clear that $\mathcal{S} \subset \mathcal{S}_H^0 \subset \mathcal{S}_H$.

Now we consider the following class of harmonic mappings

$$\mathcal{S}_{H(\mathcal{K}(\beta))} = \{f = h(z) + \overline{g(z)} \in \mathcal{S}_H | w(z) = \frac{g'(z)}{h'(z)} = b_1(p(z))^{\beta}, \beta \geq 0, p(z) \in \tilde{P}, h(z) \in \mathcal{C}\}$$

where $\tilde{P}$ denote the family of functions $p(z)$ which are normalized by $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ and there exists a complex number a such that the rotated function $(ap(z))$ has a positive real part ,i.e,$(ap(z)) \in \mathcal{P}$.

The aim of this paper is to give some properties of the class $\mathcal{S}_{H(\mathcal{K}(\beta))}$. For this aim we will need following lemma and theorems.

Lemma 1.1. *([3]) Let $\phi(z)$ be regular in the open unit disc $\mathbb{D}$. Then if $|\phi(z)|$ attains its maximum value on the circle at the point z_0 , one has $z_0.\phi'(z_0) = k\phi(z_0)$, $k \geq 1$.*

Theorem 1.1. *([1]) Let $s(z)$ be an element $\mathcal{C}$, then*

$$\frac{r}{1+r} \leq |s(z)| \leq \frac{r}{1-r}$$

and

$$\frac{r}{(1+r)^2} \leq |s'(z)| \leq \frac{r}{(1-r)^2}.$$

Theorem 1.2. *([1]) If $s(z) \in \mathcal{C}$, then* $\operatorname{Re}\left(z\frac{s'(z)}{s(z)}\right) > \frac{1}{2}$ *and* $\operatorname{Re}\left(\frac{s(z)}{z}\right) > \frac{1}{2}$.

2. Main Results

Theorem 2.1. *Let $f(z) = h(z) + \overline{g(z)}$ be an element of $\mathcal{S}_{H(\mathcal{K}(\beta))}$, then*

$$\frac{g(z)}{h(z)} = (p(z))^{\beta}$$

where $\beta \geq 0$ and $p(z) \in \tilde{P}$.

Proof. Since $f(z) = h(z) + \overline{g(z)} \in \mathcal{S}_{H(\mathcal{K}(\beta))}$, then

$$\frac{g'(z)}{h'(z)} = (p(z))^{\beta}, \quad p(z) \in \tilde{P}, \quad h(z) \in \mathcal{C}.$$

So, we have $ap(z) \in \mathcal{P}$, $|a|^{\beta} = 1$, and using Theorem 1.2 we obtain $\frac{h(z)}{zh'(z)} = 1 - \phi(z)$.
On the other hand, using Subordination and Lindelöf principle with

$$0 < r < 1, \frac{1+r}{1-r} > 1, 0 < \frac{1-r}{1+r} < 1 \Rightarrow (ap(z))^{\beta} < \left(a\frac{1+z}{1-z}\right)^{\beta},$$

we get

$$\left(\frac{1-r}{1+r}\right)^{\beta} < |ap(z)|^{\beta} < \left(\frac{1+r}{1-r}\right)^{\beta}.$$

Therefore

$$w(\mathbb{D}_r) = \left\{\frac{g'(z)}{h'(z)} \mid \left(\frac{1-r}{1+r}\right)^{\beta} < |ap(z)|^{\beta} < \left(\frac{1+r}{1-r}\right)^{\beta}, |a|^{\beta} = 1\right\}. \tag{2.1}$$

Now, we define the function $\phi(z)$ by

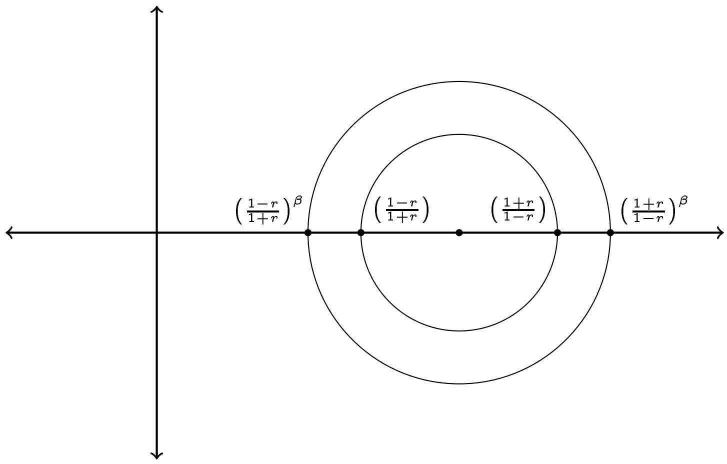
$$\frac{g(z)}{h(z)} = b_1\left(a\frac{1+\phi(z)}{1-\phi(z)}\right)^{\beta}, \quad \beta > 0, \quad a^{\beta} = 1.$$

Therefore, we have $\phi(0) = 0$, $\phi(z)$ analytic and

$$\frac{g'(z)}{h'(z)} = \frac{2\beta(1+\phi(z))^{\beta-1}.z\phi'(z)}{(1-\phi(z))^{\beta}} + \frac{g(z)}{h(z)}$$

or

$$\frac{g'(z)}{h'(z)} = b_1\left(a\frac{1+\phi(z)}{1-\phi(z)}\right)^{\beta} + \frac{2\beta(1+\phi(z))^{\beta-1}.z\phi'(z)}{(1-\phi(z))^{\beta}}.$$



Now, it is easy to realize that the subordination

$$\frac{g'(z)}{h'(z)} \prec b_1(a\frac{1+\phi(z)}{1-\phi(z)})^{\beta}$$

(from the definition of $\mathcal{S}_{H(\mathcal{K}(\beta))}$) is equivalent to $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. Indeed, assume the contrary, i.e., assume that there exists a $z_0 \in \mathbb{D}_r$ such that $|\phi(z_0)| = 1$. Then by the Jack lemma (Lemma 1.1) $z_0\phi'(z_0) = k\phi(z_0)$, $k \geq 1$, such that for z_0 we have

$$w(z_0) = b_1(a\frac{1+\phi(z_0)}{1-\phi(z_0)})^{\beta} + \frac{2\beta(1+\phi(z_0))^{\beta-1}k\phi(z_0)}{(1-\phi(z_0))^{\beta}} \notin w(\mathbb{D}_r)$$

because $|\phi(z_0)| = 1$, $k \geq 1$ and the relations (2.1). But, this is a contradiction to the definition of $\mathcal{S}_{H(\mathcal{K}(\beta))}$ and so the assumption is wrong, i.e, $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. ☐

Corollary 2.1. *Let $f = h(z) + \overline{g(z)}$ be an element of $\mathcal{S}_{H(\mathcal{K}(\beta))}$, then*

$$\frac{r(1-r)^{\beta}}{(1+r)^{\beta+1}} \leq |g(z)| \leq \frac{r(1+r)^{\beta}}{(1-r)^{\beta+1}} \tag{2.2}$$

and

$$\frac{r(1-r)^{\beta}}{(1+r)^{\beta+2}} \leq |g'(z)| \leq \frac{r(1+r)^{\beta}}{(1-r)^{\beta+2}}. \tag{2.3}$$

Proof. Using Theorem 2.1 we can write

$$|h(z)|\frac{(1-r)^{\beta}}{(1+r)^{\beta}} \leq |g(z)| \leq \frac{(1+r)^{\beta}}{(1-r)^{\beta}}.|h(z)| \tag{2.4}$$

and

$$|h'(z)|\frac{(1-r)^{\beta}}{(1+r)^{\beta}} \leq |g'(z)| \leq \frac{(1+r)^{\beta}}{(1-r)^{\beta}}|h'(z)|. \tag{2.5}$$

If we use Theorem 1.2 in the equalities (2.4) and (2.5) , then we obtain (2.2) and (2.3). ☐

Corollary 2.2. *Let $f(z) = h(z) + \overline{g(z)}$ be an element of $\mathcal{S}_{H(\mathcal{K}(\beta))}$. Then,*

$$\frac{r^2}{(1+r)^4}.(1-(\frac{1+r}{1-r})^{2\beta}) \leq J_f \leq \frac{r^2}{(1-r)^4}.(1-(\frac{1-r}{1+r})^{2\beta}) \tag{2.6}$$

and

$$\int_0^r \frac{\rho}{(1+\rho)^2}.(1-(\frac{1+\rho}{1-\rho})^{\beta})d\rho \le |f| \le \int_0^r \frac{\rho}{(1-\rho)^2}.(1+(\frac{1+\rho}{1-\rho})^{\beta})d\rho. \tag{2.7}$$

Proof. Using Theorem 2.1 we get

$$1-(\frac{1+r}{1-r})^{2\beta} \le (1-|w(z)|^2) \le 1-(\frac{1-r}{1+r})^{2\beta}, \tag{2.8}$$

$$(1+(\frac{1-r}{1+r})^{\beta}) \le (1+|w(z)|) \le (1+(\frac{1+r}{1-r})^{\beta}) \tag{2.9}$$

and

$$1-(\frac{1+r}{1-r})^{\beta} \le (1-|w(z)|) \le 1-(\frac{1-r}{1+r})^{\beta}. \tag{2.10}$$

On the other hand, we have

$$J_f = |h'(z)|^2(1-|w(z)|^2) \tag{2.11}$$

and

$$|h'(z)|(1-|w(z)|)|dz| \le |df| \le |h'(z)|(1+|w(z)|)|dz|. \tag{2.12}$$

Using (2.8), (2.11) and Theorem 1.2 and using (2.9), (2.10), (2.12) and Theorem 1.2, we get (2.6) and (2.7) respectively. □

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Department of Mathematics
İstanbul Ticaret University
İstanbul, Turkey
E-mail address: ykahramaner@iticu.edu.tr