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APPLICATION OF SALAGEAN OPERATOR ON UNIVALENT FUNCTIONS WITH RESPECT TO k-SYMMETRIC POINTS

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ABSTRACT. In this paper, the authors introduce two new subclasses $\Omega_k(\alpha)$ and $\Omega'_k(\alpha)$ of holomorphic functions by using salagean operator and k-symmetric points. Coefficient estimates for functions in these classes and their subclasses are investigated.

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions of the from

$$f(z) = z + \sum_{n=2}^{+\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\Delta = \{z : |z| < 1\}$, and S be the subclass of $\mathcal{A}$ consisting of univalent functions. The generalized salagean differential operator is defined by

$$\begin{aligned} &D^0_\lambda f(z) = f(z) \\ &D^1_\lambda f(z) = (1-\lambda)f(z) + \lambda z f'(z) \\ &D^j_\lambda f(z) = D^1_\lambda(D^{j-1}_\lambda f(z)), \quad \lambda \geq 0. \quad (see[3]) \end{aligned}$$

In special case $\lambda = 1$ we get the classic Salagean differential operator [4]. A function $f(z) \in \mathcal{A}$ is in the class $\Omega_k(\alpha)$ if

$$Re\left\{\frac{z(D^j_\lambda f(z))' + D^j_\lambda f(z)}{f_k(z)}\right\} < \alpha,$$

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where $\alpha > 1, k \geq 1$ is a fixed positive integer and $f_k(z)$ is defined by

$$f_k(z) = \frac{1}{k}\sum_{\nu=0}^{k-1} \varepsilon^{-\nu} f(\varepsilon^{\nu} z), \quad (\varepsilon^k = 1, z \in \Delta).$$

A function $f(z) \in \mathcal{A}$ is in the class $\Omega'_k(\alpha)$ if and only if $zf'(z) \in \Omega_k(\alpha)$. If $f(z)$ is given by (1.1) we see that

$$D^j_\lambda f(z) = z + \sum_{n=2}^{+\infty}[1 + (n-1)\lambda]^j a_n z^n.$$

In this paper, we shall obtain some coefficient bounds for functions in the classes $\Omega_k(\alpha)$ and $\Omega'_k(\alpha)$ and their subclasses with positive coefficients. Note that other classes of univalent holomorphic functions have been studied by many authors. See [1] and [2].

2. MAIN RESULTS

Theorem 2.1. *Let $\alpha > 2$. If $f(z) \in \mathcal{A}$ satisfies*

$$\begin{aligned}
&\sum_{n=1}^{+\infty}[(nk+2)(1+nk\lambda)^j + |(nk+2)(1+nk\lambda)^j - 2\alpha|]|a_{nk+1}| \\
&+ \sum_{n=2}^{+\infty} 2(n+1)(1+(k-1)\lambda)^j \leq 2(\alpha - 2),
\end{aligned} \tag{2.1}$$

then $f(z) \in \Omega^k(\alpha)$.

Proof. Suppose that $\alpha > 2$ and $f(z) \in \mathcal{A}$, it is sufficient to show that

$$\left|\frac{z(D^j_\lambda f(z))' + D^j_\lambda f(z)}{f_k(z)}\right| < \left|\frac{z(D^j_\lambda f(z))' + D^j_\lambda f(z)}{f_k(z)} - 2\alpha\right|, \quad z \in \Delta.$$

By letting

$$A = |z(D^j_\lambda f(z))' + D^j_\lambda f(z)| - |z(D^j_\lambda f(z))' + D^j_\lambda f(z) - 2\alpha f_k(z)|$$

and

$$f_k(z) = z + \frac{1}{k}\sum_{k=2}^{+\infty} a_k \sum_{\nu=0}^{k-1} \varepsilon^{\nu(k-1)} z^k.$$

We have

$$\begin{aligned}
A = {}& |2z + \sum_{n=2}^{+\infty}(n+1)[1+(n-1)\lambda]^j a_n z^n| \\
& - |2z + \sum_{n=2}^{+\infty}(n+1)[1+(n-1)\lambda]^j a_n z^n - 2\alpha z - 2\alpha \sum_{n=2}^{+\infty} a_n b_n z^n|,
\end{aligned}$$

where

$$b_n = \frac{1}{k}\sum_{\nu=0}^{k-1} \varepsilon^{\nu(k-1)}, \quad (\varepsilon^k = 1). \tag{2.2}$$

Therefore, for $|z| = r < 1$, we have

$$A \leq 2r + \sum_{n=2}^{+\infty}(n+1)[1+(n-1)\lambda]^j r^n - [2(\alpha-1)r$$
$$- \sum_{n=2}^{+\infty} |(n+1)[1+(n-1)\lambda]^j - 2\alpha b_n||a_n|r^n]$$
$$< \{\sum_{n=2}^{+\infty}[(n+1)[1+(n-1)\lambda]^j + |(n+1)[1+(n-1)\lambda]^j - 2\alpha b_n|]|a_n| - 2(\alpha-2)\}r.$$

From (2.2), we know

$$b_n = \begin{cases} 0, & n \neq tk+1 \\ 1, & n = tk+1 \end{cases},$$

so we get

$$A < \{\sum_{n=1}^{+\infty}[(nk+2)(1+nk\lambda)^j + |(nk+2)(1+nk\lambda)^j - 2\alpha|]|\alpha_{nk+1}|$$
$$+ \sum_{n=2, n\neq tk+1}^{+\infty} 2(n+1)(1+(k-1)\lambda)^j|a_n| - 2(\alpha-2)\}r.$$

From (2.1), we know that $A < 0$. Thus we get the required result. □

Similarly, we have the following corollary.

Corollary 2.1. *Let $\alpha > 2$. If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=1}^{+\infty}(nk+2)[(nk+2)(1+nk\lambda)^j + |(nk+2)(1+nk\lambda)^j - 2\alpha|]|a_{nk+1}|$$
$$+ \sum_{n=2, n\neq tk+1}^{+\infty} 2(n+1)^2(1+(n-1)\lambda)^j|a_n| \leq 2(\alpha-2),$$

then $f(z) \in \Omega'_k(\alpha)$.

Now we define two subclasses of $\Omega_k(\alpha)$ and $\Omega'_k(\alpha)$ as follows:

$$\Omega_k^+(\alpha) = \{f(z) \in \Omega_k(\alpha) : f(z) = z + \sum_{n=2}^{+\infty} a_n z^n, with \ \ a_n \geq 0, (n \geq 2)\}$$
$${}^+\Omega'_k(\alpha) = \{f(z) \in \Omega'_k(\alpha) : f(z) = z + \sum_{n=2}^{+\infty} a_n z^n, with \ \ a_n \geq 0, (n \geq 2)\}.$$

Theorem 2.2. *Let $k \geq 2$, $2 < \alpha \leq k+1$ and $f(z) \in \mathcal{A}$, then $f(z) \in \Omega_k^+(\alpha)$ if and only if*

$$\sum_{n=2}^{+\infty}(n+1)(1+(n-1)\lambda)^j a_n - \alpha\sum_{t=1}^{+\infty} a_{tk+1} \leq 2(\alpha-2).$$

Proof. According to Theorem 2.1, we need only to prove the necessity. Since $f(z) = z + \sum_{n=2}^{+\infty} a_n z^n \in \Omega_k^+(\alpha)$, then $a_n \geq 0$ for $n \geq 2$ and

$$Re\{\frac{z(D_\lambda^j f(z))' + D_\lambda^j f(z)}{f_k(z)}\} < \alpha,$$

or equivalently

$$|\frac{z(D_\lambda^j f(z))' + D_\lambda^j f(z)}{f_k(z)}| < |\frac{z(D_\lambda^j f(z))' + D_\lambda^j f(z)}{f_k(z)} - 2\alpha|,$$

or

$$|z(D_\lambda^j f(z))' + D_\lambda^j f(z)| < |z(D_\lambda^j f(z))' + D_\lambda^j f(z) - 2\alpha f_k(z)|.$$

Hence

$$\begin{aligned} &|1 + \sum_{n=2}^{+\infty} n[1 + (n-1)\lambda]^j a_n z^{n-1} + 1 + \sum_{n=2}^{+\infty} (1 + (n-1)\lambda)^j a_n z^{n-1}| \\ &< |1 + \sum_{n=2}^{+\infty} n[1 + (n-1)\lambda]^j a_n z^{n-1} + 1 \\ &+ \sum_{n=2}^{+\infty} (1 + (n-1)\lambda)^j a_n z^{n-1} - 2\alpha - 2\alpha \sum_{t=1}^{+\infty} a_{tk+1} z^{tk}|. \end{aligned}$$

Since $a_n \geq 0$ for $n \geq 2$ and $\alpha > 2$, by setting $z \to 1^-$, we get

$$2 + \sum_{n=2}^{+\infty} (n+1)(1 + (n-1)\lambda)^j a_n \leq 2\alpha - 2 + 2\alpha \sum_{t=1}^{+\infty} a_{tk+1} - \sum_{n=2}^{+\infty} (n+1)(1 + (n-1)\lambda)^j a_n,$$

or

$$\sum_{n=2}^{+\infty} (n+1)(1 + (n-1)\lambda)^j a_n - \alpha \sum_{t=1}^{+\infty} a_{tk+1} \leq 2(\alpha - 2).$$

□

Similarly we have the following theorem for the class $^+\Omega'_k(\alpha)$.

Corollary 2.2. *Let $k \geq 2$, $2 < \alpha \leq k+1$ and $f(z) \in \mathcal{A}$, then $f(z)$ is in the class $^+\Omega'_k(\alpha)$ if and if*

$$\sum_{n=2}^{+\infty} (n+1)^2 (1 + (n-1)\lambda)^j a_n - \alpha \sum_{t=1}^{+\infty} (tk+1) a_{tk+1} \leq 2(\alpha - 2).$$

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