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SOME INEQUALITY RELATIONS INVOLVING MULTIVALENT FUNCTIONS

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Abstract. Let $f(z)$ be a multivalent function, i.e., analytic on the unit disk and of the form $f(z) = z^p + a_{p+1}z^{p+1} + \cdots, p = 2, 3 \dots$. In this work we give sufficient conditions (unfortunately not sharp) when the following implications hold:

$$\left|\arg\left[1+\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right]\right| < \frac{\alpha\pi}{2} \quad (z\in\mathbb{D}) \quad \Rightarrow \quad \left|\arg\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right| < \frac{\beta_1\pi}{2} \quad (z\in\mathbb{D})$$

and

$$\left|\arg\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right| < \frac{\beta_1\pi}{2} \quad (z\in\mathbb{D}) \quad \Rightarrow \quad \left|\arg\frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)}\right| < \frac{\beta_2\pi}{2} \quad (z\in\mathbb{D}).$$

1. Introduction

Let $\mathcal{H}(\mathbb{D})$ denote the class of all functions that are analytic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. For $n \in \mathbb{N}$ and $a \in \mathbb{C}$, let

$$\mathcal{H}[a,n] = \left\{f\in\mathcal{H}(\mathbb{D}) : f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \cdots\right\}.$$

Especially, let for a positive integer p, $\mathcal{A}_p$ be the subclass of $H(\mathbb{D})$ consisting of functions of the form $f(z) = z^p + a_{p+1}z^{p+1} + \cdots$ and $\mathcal{A} \equiv \mathcal{A}_1$. The functions in $\mathcal{A}$ that are one-to-one are called normalized univalent functions. For more details see [1, 3, 6].

A function f is said to be *multivalent* or *p-valent* in $\mathbb{D}$ if it is assumes no value more than p times in $\mathbb{D}$ and there is some ω_0 such that $f(z) = \omega_0$ has exactly p solutions in $\mathbb{D}$, when roots are counted in accordance with their multiplicities.

In this paper we will study the following two implications:

$$\left|\arg\left[1+\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right]\right| < \frac{\alpha\pi}{2} \quad (z\in\mathbb{D}) \quad \Rightarrow \quad \left|\arg\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right| < \frac{\beta_1\pi}{2} \quad (z\in\mathbb{D}) \tag{1.1}$$

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and

$$\left|\arg \frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right| < \frac{\beta_1 \pi}{2} \quad (z \in \mathbb{D}) \quad \Rightarrow \quad \left|\arg \frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)}\right| < \frac{\beta_2 \pi}{2} \quad (z \in \mathbb{D}), \tag{1.2}$$

and give sufficient conditions when they hold. They are part of a larger study (not yet completed) aiming to give sufficient conditions when

$$\left|\arg \left[1 + \frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right]\right| < \frac{\alpha\pi}{2} \quad (z \in \mathbb{D})$$

implies

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\beta\pi}{2} \quad (z \in \mathbb{D}).$$

For obtaining our main result we will use a method from the theory of differential subordinations. Valuable references on this topic are [2] and [3].

First we introduce the concept of subordination. Let $f, g \in \mathcal{A}$. Then we say that $f(z)$ is *subordinate* to $g(z)$, and write $f(z) \prec g(z)$, if there exists a function $\omega(z)$, analytic in the unit disc $\mathbb{D}$, such that $\omega(0) = 0$, $|\omega(z)| < 1$ and $f(z) = g(\omega(z))$ for all $z \in \mathbb{D}$. In particular, if $g(z)$ is univalent in $\mathbb{D}$ then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$.

The general theory of differential subordinations, as well as the theory of first-order differential subordinations, was introduced by Miller and Mocanu in [4] and [5]. Before we introduce term differential subordinations we will give this lemma:

Lemma 1.1 ([7]). *If $F : \mathbb{C}^n \to \mathbb{C}$ is analytic for each of the variables $z_i, 1 \le i \le n$, while other variables are considered as constants, than F is continuous and analytical (in sense of multiple variables).*

Further, if $\phi : \mathbb{C}^2 \to \mathbb{C}$ (where $\mathbb{C}$ is the complex plane) is analytic in a domain D, if $h(z)$ is univalent in $\mathbb{D}$, and if $p(z)$ is analytic in $\mathbb{D}$ with $(p(z), zp'(z)) \in D$ when $z \in \mathbb{D}$, then $p(z)$ is said to satisfy a first-order differential subordination if

$$\phi(p(z), zp'(z)) \prec h(z). \tag{1.3}$$

A univalent function $q(z)$ is said to be a *dominant* of the differential subordination (1.3) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.3). If $\widetilde{q}(z)$ is a dominant of (1.3) and $\widetilde{q}(z) \prec q(z)$ for all dominants of (1.3), then we say that $\widetilde{q}(z)$ is the *best dominant* of the differential subordination (1.3).

For the proof of implications (1.1) and (1.2) we will use a lemma from the theory of differential subordinations. It gives efficient tool for obtaining sufficient conditions (very often sharp, i.e., best possible) when certain differential inequality holds.

Lemma 1.2 (Theorem 2.3i(i), p.35, [3]). *Let $\Omega \subset \mathbb{C}$ and suppose that the function $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ satisfies $\psi(ix, y; z) \notin \Omega$ for all $x \in \mathbb{R}$, $y \le -(1 + x^2)/2$, and $z \in \mathbb{D}$. If $q \in H[1, 1]$ and $\psi(q(z), zq'(z); z) \in \Omega$ for all $z \in \mathbb{D}$, then* $\operatorname{Re} q(z) > 0$, $z \in \mathbb{D}$.

2. Implication (1.1)

In this section we will study implication (1.1).

Theorem 2.1. *Let $f \in \mathcal{A}_p$, $p \geq 2$, $0 < \beta_1 \leq 1$ and suppose that $f^{(k)}(z) \neq 0$ for all $z \in \mathbb{D} \setminus \{0\}$ and for all positive integer k. If*

$$\alpha \equiv \alpha(\beta_1) = \operatorname{arctg}\left[\frac{\beta_1}{1-\beta_1} \cdot \left(\frac{1-\beta_1}{1+\beta_1}\right)^{(1+\beta_1)/2} + \operatorname{tg}\frac{\beta_1 \pi}{2}\right],$$

then the following implication holds:

$$\left|\arg\left[1 + \frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right]\right| < \frac{\alpha\pi}{2} \quad (z \in \mathbb{D}) \quad \Rightarrow \quad \left|\arg\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right| < \frac{\beta_1\pi}{2} \quad (z \in \mathbb{D}).$$

Proof. Let choose $q^{\beta_1}(z) = \frac{zf^{(p)}(z)}{f^{(p-1)}(z)}$. Then we have

$$\frac{z\left[q^{\beta_1}(z)\right]'}{q^{\beta_1}(z)} = \frac{z\beta_1 q^{\beta_1 - 1}(z) q'(z)}{q^{\beta_1}(z)} = 1 + \frac{zf^{(p+1)}(z)}{f^{(p)}(z)} - q^{\beta_1}(z)$$

and

$$1 + \frac{zf^{(p+1)}(z)}{f^{(p)}(z)} = \frac{z\beta_1 q'(z)}{q(z)} + q^{\beta_1}(z).$$

Further, for the function

$$\psi(r, s; z) = \beta_1 \cdot \frac{s}{r} + r^{\beta_1},$$

we have

$$\psi(q(z), zq'(z); z) = \beta_1 \cdot \frac{zq'(z)}{q(z)} + q^{\beta_1}(z) \in \Omega \equiv \left\{\omega : |\arg\omega| < \frac{\alpha\pi}{2}\right\},$$

i.e.,

$$|\arg\psi(q(z), zq'(z); z)| < \frac{\alpha\pi}{2} \quad (z \in \mathbb{D}).$$

From Lemma 1.2 we realize that for proving

$$\left|\arg\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right| < \frac{\beta_1\pi}{2} \quad (z \in \mathbb{D})$$

it is enough to show that

$$\psi(ix, y; z) = \beta_1 \cdot \frac{y}{ix} + (ix)^{\beta_1} = -\beta_1 \cdot \frac{y}{x} \cdot i + (ix)^{\beta_1} \notin \Omega$$

for all real $x, y \leq -\frac{1+x^2}{2}$ ($n = 1$ in the Lemma 1.2) and for all $z \in \mathbb{D}$.

In the case when $x > 0$ we have

$$\begin{aligned} 0 < \arg\psi(ix, y; z) &= \operatorname{arctg}\left[\frac{-\beta_1 \frac{y}{x} + x^{\beta_1}\sin\frac{\beta_1\pi}{2}}{x^{\beta_1}\cos\frac{\beta_1\pi}{2}}\right] = \operatorname{arctg}\left[\frac{-\beta_1\frac{y}{x}}{x^{\beta_1}\cos\frac{\beta_1\pi}{2}} + \operatorname{tg}\frac{\beta_1\pi}{2}\right] \\ &\leq \operatorname{arctg}\left[\frac{\beta_1 \cdot \frac{1+x^2}{2x}}{x^{\beta_1}\cos\frac{\beta_1\pi}{2}} + \operatorname{tg}\frac{\beta_1\pi}{2}\right] \\ &= \operatorname{arctg}\left[\frac{\beta_1 \cdot (1+x^2)}{2x^{\beta_1+1}\cos\frac{\beta_1\pi}{2}} + \operatorname{tg}\frac{\beta_1\pi}{2}\right] \equiv \varphi(x). \end{aligned}$$

Similarly, for the case $x<0$,

$$|\arg\psi(ix,y;z)| = \arg\left(-\beta_1\cdot\frac{y}{|x|}\cdot i+(i|x|)^{\beta_1}\right)=\varphi(|x|).$$

It is easy to check that the function $\varphi(x)$, on the interval $(0,+\infty)$, attains its minimal value for $x_*=\sqrt{\frac{1+\beta_1}{1-\beta_1}}$, i.e.,

$$\inf\left\{|\arg\psi(ix,y;z)|: x,y\in\mathbb{R}, x\neq 0, y\leq -\frac{1+x^2}{2}\right\}=\varphi(x_*)=\alpha(\beta_1).$$

For $x=0$ we have

$$\lim_{|x|\to 0}|\arg\psi(ix,y;z)|=\lim_{x\to 0^+}\varphi(x)=\frac{\pi}{2}\geq\alpha(\beta_1).$$

This completes the proof of $\psi(ix,y;z)\notin\Omega$ for all real $x, y\leq -\frac{1+x^2}{2}$. □

3. Implication (1.2)

In this section we will study the implication (1.2) in a similar way as the implication (1.1).

Theorem 3.1. *Let $f\in\mathcal{A}_p$, $p\geq 2$, $0<\beta_2\leq 1$ and suppose that $f^{(k)}(z)\neq 0$ for all $z\in\mathbb{D}\setminus\{0\}$ and for all positive integer k. Also let x_* be the bigger, of the only two positive solutions of the equation*

$$2x^{\beta_2+1}\sin(\beta_2\pi/2)+\left(\beta_2x^2+\beta_2-x^2+1\right)x^{\beta_2}\cos(\beta_2\pi/2)+x^2-1=0,$$

and $\beta_1=\beta_1(\beta_2)\equiv\operatorname{arcctg}[h(x_*)]$ where

$$h(x)\equiv\frac{-1+x^{\beta_2}\cos\frac{\beta_2\pi}{2}}{\beta_2\frac{1+x^2}{2x}+x^{\beta_2}\sin\frac{\beta_2\pi}{2}}.$$

Then the following implication holds:

$$\left|\arg\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right|<\frac{\beta_1\pi}{2}\quad(z\in\mathbb{D})\quad\Rightarrow\quad\left|\arg\frac{zf^{(p-1)}}{f^{(p-2)}}\right|<\frac{\beta_2\pi}{2}\quad(z\in\mathbb{D}).$$

Proof. Let choose $q^{\beta_2}(z)=\frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)}$. Then we have

$$\frac{z\left[q^{\beta_2}(z)\right]'}{q^{\beta_2}(z)}=\frac{z\beta_2q^{\beta_2-1}(z)q'(z)}{q^{\beta_2}(z)}=1+\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}-q^{\beta_2}(z),$$

i.e.,

$$\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}=\frac{z\beta_2q'(z)}{q(z)}+q^{\beta_2}(z)-1.$$

Further, for the function

$$\psi(r,s;z)=\beta_2\cdot\frac{s}{r}+r^{\beta_2}-1,$$

we have

$$\psi(q(z),zq'(z);z)=\beta_2\cdot\frac{zq'(z)}{q(z)}+q^{\beta_2}(z)-1\in\Omega\equiv\left\{\omega:|\arg\omega|<\frac{\beta_1\pi}{2}\right\},$$

i.e.,

$$|\arg\psi(q(z),zq'(z);z)|<\frac{\beta_1\pi}{2}\quad(z\in\mathbb{D}).$$

From Lemma 1.2 we realize that for proving

$$\left|\arg\frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)}\right|<\frac{\beta_2\pi}{2}\quad(z\in\mathbb{D})$$

it is enough to show that

$$\psi(ix,y;z)=\beta_2\cdot\frac{y}{ix}+(ix)^{\beta_2}-1=-\beta_2\cdot\frac{y}{x}\cdot i+(ix)^{\beta_2}-1\notin\Omega$$

for all real $x,y\le-\frac{1+x^2}{2}$ ($n=1$ in the Lemma 1.2) and for all $z\in\mathbb{D}$.

In the case when $x>0$ we have

$$\operatorname{ctg}[\arg\psi(ix,y;z)]=\frac{-1+x^{\beta_2}\cos\frac{\beta_2\pi}{2}}{-\beta_2\frac{y}{x}+x^{\beta_2}\sin\frac{\beta_2\pi}{2}}\le h(x)$$

Similarly, for the case $x<0$,

$$|\operatorname{ctg}[\arg\psi(ix,y;z)]|=\left|\operatorname{ctg}\left[\arg\left(-\beta_2\cdot\frac{y}{|x|}\cdot i+(i|x|)^{\beta_2}-1\right)\right]\right|\le h(|x|).$$

Further, $h(x)$ is continuous on $(0,+\infty)$, $h(0)=0$, $\lim\limits_{x\to+\infty}h(x)>0$ and from

$$h'(x)=\frac{2\beta_2\left[2x^{\beta_2+1}\sin(\beta_2\pi/2)+\left(\beta_2x^2+\beta_2-x^2+1\right)x_2^{\beta}\cos(\beta_2\pi/2)+x^2-1\right]}{\left(2x^{\beta_2+1}\sin(\beta_2\pi/2)+\beta_2x^2+\beta_2\right)^2},$$

we receive $h'(0)<0$ and $\lim\limits_{x\to+\infty}h'(x)>0$. Therefore, $h(x)$ has at least one local minimum and at least one local maximum on $(0,+\infty)$. On the other hand, the nominator of $h(x)$ is an increasing function on $(0,+\infty)$ and its denominator is convex function on $(0,+\infty)$. Therefore, $h(x)$ has exactly one local minimum (at point x_{**}) and exactly one local maximum (at point $x_*>x_{**}$) on $(0,+\infty)$. So,

$$\sup\left\{|\arg\psi(ix,y;z)|:x>0,y\le-\frac{1+x^2}{2}\right\}=\operatorname{arcctg}[h(x_*)]=\beta_1(\beta_2).$$

In a similar way we can show that the same is true also for $x<0$.

For $x=0$ we have

$$\lim_{|x|\to0}|\arg\psi(ix,y;z)|=\lim_{x\to0^+}\operatorname{arcctg}[h(x)]=\frac{\pi}{2}\ge\beta_1(\beta_2).$$

This completes the proof of the theorem. □

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