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METHOD OF ALTERNATING PROJECTIONS IN A GENERAL CONSTRUCTION OF THE WEIGHTED BERGMAN KERNEL

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Abstract. In this article we will show how to derive the weighted Bergman kernel using the theorems of Halperin and Stone ([4, 15]) on alternating projections. This is generalization of what M. Skwarczyński did for regular Bergman kernels ([11, 12]).

1. Introduction

The Bergman kernel ([1, 5, 6, 10, 14]) has become a very important tool in geometric function theory, both in one and several complex variables. It turned out that not only classical Bergman kernel, but also weighted one can be useful, particularly from quantum theory point of view (look in Ref. [2, 3] and Ref [7]). Unfortunately, it is difficult to say anything about the regular or weighted kernel of a given domain. Maciej Skwarczyński used the theorems of Halperin and Stone ([4, 15]) to derive the Bergman kernel of a domain $D \subset \mathbb{C}^N$ by means of alternating projections. In this paper we will show how to obtain the weighted Bergman kernel using these theorems. We shall start from the definitions and basic facts used in this paper.

2. Definitions and notations

Let $D \subset \mathbb{C}^N$ be a domain, and let $W(D)$ be the set of weights on D, i.e., $W(D)$ is the set of all Lebesque measurable, real - valued, positive functions on D (we consider two weights as equivalent if they are equal almost everywhere with respect to the Lebesque measure on D). If $\mu \in W(D)$, we denote by $L^2(D, \mu)$ the space of all Lebesque measurable, complex-valued, μ-square integrable functions on D, equipped with the norm $||\cdot||_{D,\mu} := ||\cdot||_\mu$

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given by the scalar product

$$< f|g >_\mu := \int_D f(z)\overline{g(z)}\mu(z)dV, \quad f,g \in L^2(D,\mu).$$

The space $L^2_H(D,\mu) = H(D) \cap L^2(D,\mu)$ is called the **weighted Bergman space**, where $H(D)$ stands the space of all holomorphic functions on the domain D. For any $z \in D$ we define the evaluation functional E_z on $L^2_H(D,\mu)$ by the formula

$$E_z f := f(z), \quad f \in L^2_H(D,\mu).$$

Let us recall the definition [Def. 2.1] of admissible weight given in [9].

Definition 2.1 (Admissible weight). *A weight $\mu \in W(D)$ is called an admissible weight , an a-weight for short, if $L^2_H(D,\mu)$ is a closed subspace of $L^2(D,\mu)$ and for any $z \in D$ the evaluation functional E_z is continuous on $L^2_H(D,\mu)$. The set of all a-weights on D will be denoted by $AW(D)$.*

The definition of admissible weight provides us basically with existence and uniqueness of related Bergman kernel and completeness of the space $L^2_H(D,\mu)$. The concept of a-weight was introduced in [8], and in [9] several theorems concerning admissible weights are given. An illustrative one is :

Theorem 2.1. [9, Cor. 3.1] *Let $\mu \in W(D)$. If the function μ^{-a} is locally integrable on D for some $a > 0$ then $\mu \in AW(D)$.*

Now, let 's fix a point $t \in D$ and minimize the norm $||f||_\mu$ in the class $E_t = \{f \in L^2_H(D,\mu); f(t) = 1\}$. It can be proved in a similar way as in the classical case, that if μ is an admissible weight then there exists exactly one function minimizing the norm. Let us denote it by $\phi_\mu(z,t)$. **Weigted Bergman kernel function** $K_{D,\,\mu}$ is defined as follows:

$$K_{D,\,\mu}(z,t) = \frac{\phi_\mu(z,t)}{||\phi_\mu||^2_\mu}.$$

3. Theorems of Halperin and Stone and Bergman projections

3.1. **Theorems of Halperin and Stone.** We can reconstruct the weighted Bergman kernel using the properties of orthogonal projection and the classical results of functional analysis. Let us recall the theorems of Halperin and Stone. Let H be a Hilbert space, and $F_i,\ i = 1, 2, \dots k$ closed subspace of H.

Theorem 3.1 (Halperin, [4]). *Let $P_i : H \to F_i,\ i = 1, ..., k$ be an orthogonal projection on $F_i,\ i = 1, 2, \dots, k$ and let $P : H \to F$ be an orthogonal projection on $F := F_1 \cap F_2 \cap \dots F_k$. Then for any $x \in H$,*

$$(P_k \circ \dots \circ P_1)^n x \to Px.$$

Theorem 3.2 (Stone, [15]). *Consider the descending sequence $F_1 \Supset F_2 \Supset F_3 \dots$ of closed subspaces of H, and denote $P_i : H \to F_i,\ i = 1, 2, \dots$ an orthogonal projection on $F_i,\ i = 1, 2, \dots$. Then for any $x \in H$ we have*

$$P_n x \to Px$$

where P is an orthogonal projection on $F = \bigcap_{n=1}^{\infty} F_n$.

Prof. Maciej Skwarczyński applied the theorems of Halperin and Stone to reconstruct the Bergman kernel of a domain $\Omega \subset \mathbb{C}^N$. Let's follow him in the case of weighted Bergman kernels.

3.2. **The weighted Bergman projection.** By the definition of admissible weight it follows that $L^2_H(\Omega,\, \mu)$ is a closed subspace of $L^2(\Omega,\, \mu)$. The orthogonal projection P_Ω on $L^2_H(\Omega,\, \mu)$ is called the weighted Bergman projection. If $h \in L^2(\Omega,\, \mu)$ then $h = h_1 + h_2 \in L^2_H(\Omega,\, \mu) \oplus L^2_H(\Omega,\, \mu)^\perp$. Define the weighted Bergman projection by

$$(P_\Omega h)(t) = \int_\Omega h(z)\overline{K_{\Omega,\,\mu}(z,t)}\mu(z)dV$$

(then $(P_\Omega h)(t) = h_1(t)$ since $K_{\Omega,\,\mu} \in L^2_H(\Omega,\, \mu)$). So if we know the Bergman kernel, we know the Bergman projection P_Ω.

And conversely. Let $t \in U \subset \Omega$ and U be a ball or another domain, for which the weighted Bergman kernel is known (explicitly). Define $h(z) = \chi_U(z)K_{U,\,\nu}(z,t) \in L^2(\Omega,\, \mu)$ (we assume that $\nu(z) \le \mu(z)$ on U). We have for any $f \in L^2_H(\Omega,\, \mu)$ that

$$\begin{aligned}\int_\Omega f(z)\overline{(P_\Omega h)(z)}\mu(z)dV &= \textstyle\int_\Omega f(z)\overline{\chi_U(z)K_{U,\,\nu}(z,t)}\mu(z)dV \\ &= \int_U f(z)\overline{K_{U,\,\nu}(z,t)}\mu(z)dV = f(t).\end{aligned}$$

But the Bergman kernel is the only element in $L^2_H(\Omega,\, \mu)$ with the reproducing property, so

$$K_{\Omega,\,\mu}(z,t) = (P_\Omega h)(z),\ z \in \Omega,$$

for any $t \in \Omega$. So if we know the Bergman projection, we know the weighted Bergman kernel $K_{\Omega,\,\mu}$.

Assume first that $\Omega = \bigcup_{n=1}^{s} U_n$ where $U_n \subset \mathbb{C}^N$ is a domain, and let $\mu_n \in AW(U_n),\ n = 1, 2, \ldots, s$. We may define admissible weight on Ω on several ways. One one them is just to take $\mu(z)_{|U_i} = \mu_i(z)$ (we assume then that $\mu_i(z) = \mu_j(z)$ on $U_i \cap U_j$).

Theorem 3.3. *Denote $F_i = L^2_H(U_i,\, \mu_i), i = 1, 2, \ldots s$ the closed subspace of a Hilbert space $L^2(\Omega, \mu)$. Then orthogonal projection on F_i is given by*

$$(P_i f)(z) = \begin{cases} (P_{U_i} f_{U_i})(z) & \textit{for } z \in U_i \\ f(z) & \textit{for } z \in \Omega\backslash U_i \end{cases}$$

for $f \in L^2(\Omega,\, \mu)$ (we denote f_{U_i} the restriction of f to U_i). Moreover the Bergman projection on $L^2_H(\Omega,\, \mu)$ is given by

$$P_D f := \lim_{n\to\infty} (P_s \circ P_{s-1} \circ \ldots \circ P_1)^n f, \quad f \in L^2(\Omega,\, \mu).$$

Proof. We have that $P_i f \in L^2_H(U_i,\, \mu_i) = F_i$ by the very definition. Let us prove that for $f \in L^2(D,\, \mu)$ we have $f = P_i f + f_1 \in F_i \oplus F_i^\perp$, or $f - P_i f \in F_i^\perp$.

$$(f - P_i f)(z) = \begin{cases} f_{U_i}(z) - (P_{U_i} f_{U_i})(z) & \text{for } z \in U_i \\ 0 & \text{for } z \in \Omega\backslash U_i \end{cases}$$

So for any $g \in F_i$,

$$\int_{\Omega}(f - P_i f)(z)g(\bar{z})\mu(z)dV = \int_{U_i}(f_{U_i} - P_{U_i} f_{U_i})(z)g(\bar{z})\mu(z)dV = 0,$$

i.e., $f - P_i f \in F_i^{\perp}$ so P_i is a Bergman projection on F_i. Thus by the Halperin theorem

$$\lim_{n\to\infty}(P_s \circ P_{s-1} \circ \ldots P_1)^n f = P_\Omega f$$

where $P_\Omega f$ is a projection on $F = F_1 \cap F_2 \ldots \cap F_s$. But for $f \in L^2(\Omega, \mu)$ we have

$$f \in F_1 \cap F_2 \ldots \cap F_s = L^2_H(U_1,\, \mu_1) \cap L^2_H(U_2,\, \mu_2) \cap \ldots L^2_H(U_s,\, \mu_s)$$

means $f \in Hol(\Omega) \cap L^2(\Omega,\, \mu)$. And conversely, if $f \in L^2_H(\Omega,\, \mu)$ then for any $i = 1, 2, \ldots, s$ we have $f \in Hol(U_i)$ (since $\Omega = \bigcup_{i=1}^{s} U_i$) and $f \in L^2(U_i,\, \mu_i)$ - we used the definition of μ. □

Remark 3.1. *If $\Omega = \bigcup_{i=1}^{s} U_i$, where U_i is a ball or another domain, for which the weighted Bergman kernel is known, we may use this theorem to get $K_{\Omega,\,\mu}(z, w)$.*

Theorem 3.4. *Let $\Omega = \bigcup_{n=1}^{\infty} D_n$, $\quad D_1 \Subset D_2 \Subset D_3 \Subset \ldots$ and let $\mu \in AW(\Omega)$, $\mu_n \in AW(D_n)$. Assume moreover that $\mu_n(z) \leq \mu_m(z) \leq \mu(z)$ for $z \in D_n$ and $n \leq m$. Define $F_n = L^2_H(D_n,\, \mu_n)$ for $n \in \mathbb{N}$. Let P_n be the (Bergman) orthogonal projection on F_n. Then the (Bergman) projection on $L^2_H(\Omega,\, \mu)$ is given by*

$$P_\Omega f := \lim_{n\to\infty} P_n f, \quad f \in L^2(\Omega,\, \mu).$$

Proof. Look that if $f \in L^2(\Omega,\, \mu)$ then for any $n \in \mathbb{N}$

$$\int_{D_n} |f(z)|^2 \mu_n(z)dV \leq \int_{D_n} |f(z)|^2 \mu(z)dV \leq \int_{\Omega} |f(z)|^2 \mu(z)dV < \infty.$$

It is clear that $f \in Hol(\Omega) \iff f \in Hol(U_i)$, for any $i \in \mathbb{N}$. We can use Stone theorem to get desired result. □

Remark 3.2. *It is clear that any domain $\Omega \subset \mathbb{C}^N$ may be written as*

$$\Omega = \bigcup_{n=1}^{\infty} D_n, \quad D_1 \Subset D_2 \Subset D_3 \Subset \ldots$$

where $D_n = \bigcup_{i=1}^{s(n)} U_i$, where U_i is a ball (or another domain, for which the weighted Bergman kernel is known). Thus using this theorem we can reconstruct the weighted Bergman kernel of any domain $\Omega \subset \mathbb{C}^N$. In practice, this theorem is based on strong calculations, so in the most of cases we (probably) need to use the computer.

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