

CONSTRUCTION OF A MEASURE THAT MAPS EVERY CARLESON SET TO ZERO

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ABSTRACT. According to an important result by J. Shapiro (see [3]), the construction of such measure provides a corresponding singular inner cyclic vector for the shift operator on the Bergman space $L_a^t([0, 1])$.

1. INTRODUCTION

For $1 \leq t \leq \infty$, the classical Hardy space $H^t(D)$ is the collection of functions f that are analytic on D such that $\sup_{0 < r < 1} \int_{\partial D} |f(r\zeta)|^t dm(\zeta) < \infty$, and the Bergman space:

$$L_a^t(G) := \left\{ f : f \text{ is analytic in } G \text{ and } \int_G |f|^t dA < \infty \text{ (} dA \text{ is area measure)} \right\}.$$

Now, $L_a^t(G)$ is a Banach space with respect to the norm $\|f\|_{L_a^t(G)} = (\int_G |f|^t dA)^{1/t}$ and when $t = 2$, it is a Hilbert space with respect to the inner product

$$\langle f, g \rangle_{L_a^t(G)} = \int_G f \bar{g} dA.$$

An operator T on a Banach space $\mathfrak{F}$ is said to be cyclic if there exists h in $\mathfrak{F}$ such that $\{p(T)h : p \text{ is a polynomial}\}$ is dense in $\mathfrak{F}$. In this case h is called a cyclic vector for T on $\mathfrak{F}$. Since $p(M_z)h = M_{p(z)}.h = p.h$, the shift operator

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M_z is cyclic on $H^t(G)$ (respectively $L_a^t(G)$) if there exists h in $H^t(G)$ (respectively $L_a^t(G)$) such that $\{h.p : p \text{ is a polynomial}\}$ is dense in $H^t(G)$ (respectively $L_a^t(G)$).

Definition 1.1. A set E of ∂D is called a Carleson set if

- (i) E is closed;
- (ii) E is of Lebesgue measure zero; $|E| = 0$;
- (iii) $\sum_\nu |I_\nu| \log \frac{1}{|I_\nu|} < \infty$, where $\{I_\nu\}$ are the complementary arcs of E .

Similarly, one can define Carleson subsets of $[0, 1]$.

In the following example we will show that the Cantor set is a Carleson subset on $[0, 1]$.

Example 1. Let E be the Cantor set. It is well-known that

- (i) E is closed;
- (ii) E is of Lebesgue measure zero; and
- (iii) $[0, 1] \setminus E$ is the pairwise disjoint union $\bigcup_{k=0}^{\infty} \bigcup_{n=1}^{2^k} I_{n,k}$, where $I_{n,k}$ has length $\frac{1}{3^{k+1}}$.

Therefore,

$$\sum m(I_k) \log\left(\frac{1}{m(I_k)}\right) = \sum_0^{\infty} \frac{2^k}{3^{k+1}} \cdot \log(3^{k+1}) = \frac{1}{3} \sum_0^{\infty} (k+1) \cdot \left(\frac{2}{3}\right)^k \log(3) < \infty,$$

by the Ratio Test. Consequently, E is Carleson subset of $[0, 1]$.

Definition 1.2. A function S_μ defined and analytic in D is called singular inner function if it has the form $S_\mu(z) = \exp\left[-\int_{\partial D} \frac{\zeta+z}{\zeta-z} d\mu(\zeta)\right]$ ($z \in D$), where μ is a finite nonnegative Borel measure on ∂D that is singular with respect to m .

Definition 1.3. A function F defined and analytic in D is called an outer function if it has the form $F(z) = \chi \exp\left[\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log k(t) dt\right]$, where k is a nonnegative Lebesgue measurable function on ∂D , $\log(k(t)) \in L^1(m)$ and χ is a complex number of modulus 1.

2. MAIN RESULT

In the following proposition we give an essential condition for the cyclicity of the shift operator on the Bergman space of the disk which characterizes the nature of the cyclic vector and note that this result can be extended to any simply connected region.

Theorem 2.1. *If $f \in H^\infty(D)$, then f is a cyclic vector for M_z on $L_a^t(D)$ if and only if $f = S_\mu \cdot F$, where F is an outer function and S_μ is a singular inner function with $\mu(E) = 0$ for every Carleson set $E \subseteq \partial D$.*

Proof. Assume that f is a bounded cyclic vector. Then $f = S_\mu \cdot F$ where F is an outer function and S_μ is a singular inner function. By Beurling's Theorem $F \cdot H^\infty(D)$ is dense in $H^t(D)$ and therefore is dense in $L_a^t(D)$. Since f is a cyclic vector, it follows that $S_\mu \cdot L_a^t(D)$ is dense in $L_a^t(D)$.

Now, J. Robert (see [3]) and B. Korenblum (see [1] and [2]) proved independently that this implies that $\mu(E) = 0$ for every Carleson set $E \subseteq \partial D$.

Conversely, suppose $f = S_\mu \cdot F$ where F is an outer function and $\mu(E) = 0$ for every Carleson set $E \subseteq \partial D$. Now, by Beurling's Theorem, $F \cdot P$ is dense in $H^t(D)$ and therefore is dense in $L_a^t(D)$. Moreover, since $\mu(E) = 0$ for every Carleson set $E \subseteq \partial D$, Korenblum in [1] proved that $S_\mu \cdot H^\infty(D)$ is dense in $L_a^t(D)$. Therefore $\{f \cdot p : p \in P\}$ is dense in $L_a^t(D)$ and hence f is a cyclic vector for M_z on $L_a^t(D)$. $\square$

In the next example we construct a measure on the interval $[0, 1]$ which is singular with respect to Lebesgue measure and sends every Carleson set to zero.

Example 2. *Divide the interval $[0, 1]$ into five equal subintervals and then construct A_0 by removing two nonconsecutive intervals without removing any from either end. Then divide each of the remaining intervals in A_0 into nine intervals and similarly remove four nonconsecutive intervals from each without removing any from either end to construct A_1 . Continue this process by dividing each interval in A_{k-1} into $2^{2^k+1} + 1$ intervals then remove 2^{2^k} nonconsecutive intervals where none of the removed intervals is from either end to construct A_k , (see Figure 1).*

Let $A = \bigcap_{k=0}^{\infty} A_k$; observe that $m(A) = \lim_{k \rightarrow \infty} m(A_k) = 0$.

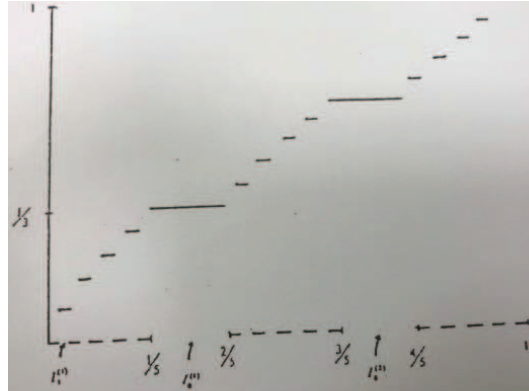


FIGURE 1

Now let $I_k^{(j)}$ be the j^{th} interval removed from the first remaining interval in A_{k-1} , $j = 1, 2, \dots, n_k$ where $n_k = 2^{2^k}$; so $I_k^{(j)} = \left(\frac{2j-1}{\prod_{i=0}^k (2n_i+1)}, \frac{2j}{\prod_{i=0}^k (2n_i+1)} \right)$. Let $f(x) = \frac{j}{\prod_{i=0}^k (n_i+1)}$ whenever $x \in I_k^{(j)}$, and define f in a similar fashion on each of the intervals removed. Then f is continuous, monotonically increasing on $[0, 1]$, constant on each of the intervals $I_k^{(j)}$ and $\dot{f} = 0$ a.e. λ (Lebesgue measure on $[0, 1]$). Let μ be the positive Borel measure with support contained in $[0, 1]$ such that $\mu([a, b]) = f(b) - f(a)$ whenever $0 \leq a \leq b \leq 1$. Since $\dot{f} = 0$ a.e., it follows that $\mu \perp \lambda$.

Consider the first interval in each A_k ; that is $I_k^{(1)} = \left(\frac{1}{\prod_{i=0}^k (2n_i+1)}, \frac{2}{\prod_{i=0}^k (2n_i+1)} \right)$. Now

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{f(x)}{x \cdot \log \frac{1}{x}} &= \lim_{k \rightarrow \infty} \left(\frac{1}{\prod_{i=0}^k (n_i+1)} \cdot \frac{\prod_{i=0}^k (2n_i+1)}{\log(\prod_{i=0}^k (2n_i+1))} \right) \\ &= \lim_{k \rightarrow \infty} \left(\frac{2^k \cdot \prod_{i=0}^k (n_i + \frac{1}{2})}{\prod_{i=0}^k (n_i+1) (\sum_{i=0}^k \log(2^{2^i+1} + 1))} \right) \\ &= \lim_{k \rightarrow \infty} \left(\frac{2^k}{\log(2) \sum_{i=0}^k (2^i + 1)} \right) < \infty. \end{aligned}$$

Because of the self-similarity in the construction of the set A and the way that the function f is defined, this condition holds for every point in the interval $[0, 1]$, which means that

$$\omega_\mu(\delta) = O(\delta \cdot \log(1/\delta)); \quad \omega_\mu(\delta) = \sup\{\mu(I) : \lambda(I) < \delta, I \subseteq [0, 1]\}.$$

Therefore, by [3, Theorem 6.2.1], S_μ is cyclic and hence by Theorem 2.1, $\mu(E) = 0$ for every Carleson set $E \subseteq [0, 1]$.

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