

APPLICATIONS OF SUMUDU TRANSFORM OF TWO VARIABLE FUNCTIONS WITH HERMITE POLYNOMIAL TO PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. The present article gives the review on development of Sumudu transform and its applications. Theorems on Sumudu transform of two variables are proved in this paper. The theorem on Double Sumudu transform for any n using Hermite polynomial is proved. Using these theorems one can find Sumudu transform of homogeneous and non homogeneous functions of order n . Partial differential equations like, one dimensional heat equation, Poisson equation, second order hyperbolic equations are solved using these theorems.

1. INTRODUCTION

In 1993, an Australian mathematician Watugula [1] introduced Sumudu Transform and solved the problems in control engineering. He defined both power series as well as Integral definition of Sumudu transform. Weerakoon [2] in 1994 has shown the applications of Sumudu transform to the partial differential equations. In 1998, Watugula [3] gave properties of Sumudu transform and Weerakoon [4] developed complex integral formula for Sumudu transform. The development of Sumudu transform was then done by M. Asiru [5] and [6]. Both 1st and 2nd kind of integral equations using convolution product were solved in his paper. In 2001, M. Asiru proved general properties of

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Sumudu transform and showed applications of Sumudu transform to Abel's integral equations and Integro-differential equation. Watugala [7] in 2002, defined Sumudu transform for function of two variables and showed new direction to the mathematicians to work on Sumudu transform. Asiru [8] solve discrete dynamic systems using Sumudu transform in 2003, he obtained solution of first, second and third order dynamic systems. The detail properties like scale-preserving property, shifting and limit properties with analytical investigations of Sumudu transform, its existence for positive and negative of t are given by Belgacem and Karaballi [9]. The complex inversion formula for sumudu transform is proved and applied to solution of linear as well as partial differential equation by Belgacem and Karaballi [10]. H. Eltayeb and A Kalicman with H. Gadain, A. Atan [11–14] made the major work on Sumudu transform of Single and double variables. In their papers, authors have given the applications, properties as well as convolution theorem of Sumudu transform. In 2010, the relationship between Laplace transform and Sumudu transform is unidirectional that means $S_2(f(t)) \implies L_2(f(t))$ this was proved with example by H. Eltayeb and Kilicman [15]. Eltayeb and Kilicman [16] used locally integrable concept and found Sumudu transform of higher order derivative and solved differential equations involving Heaviside function and Dirac delta function. J Tchuenche and N Mbare [17] gave applications of Double Sumudu transforms in 2007 and derived relation between double Sumudu and double Laplace transform. The validity of double Sumudu transform was checked by applying it to Kermack-Mackendrich Von Foerster type model. The concept of Laplace transform and Double Laplace transform with properties and applications was gave by L. Debnath [18]. Kiwane and S Sonawane [21] solved the same example given by H. Eltayeb and Kilicman with small change in boundary conditions and proved that Sumudu transform exists for same problem.

In this article author has proved theorems on Sumudu transform of two variables. This theorems help to find Sumudu transform of homogenous and non homogenous function of any order and used it for solution of partial differential equations. In the first section, Sumudu transform of one variable and useful results are proved. Sumudu transform of two variables, theorems on Sumudu transform and applications to partial differential equations are given

in the second and last section respectively. Let us consider the set A as,

$$A = \left\{ y : |y(t)| < M e^{\frac{|t|}{\alpha_j}}, t \in (-1)^j \times [0, \infty), j = 1, 2; M, \alpha_1, \alpha_2 > 0 \right\}$$

The constant M is finite number and α_1, α_2 are finite or may be infinite numbers.

Definition 1.1. (Sumudu transform [1]) Let $y(t) \in A$, then the Sumudu transform is:

$$S[y(t)] = \int_0^\infty e^{-t} y(vt) dt = \frac{1}{v} \int_0^\infty e^{-\frac{t}{v}} y(t) dt = Y(v), \alpha_1 < v < \alpha_2,$$

and inverse Sumudu transform [4] is:

$$y(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{vt} Y\left(\frac{1}{v}\right) \frac{dv}{v}; \gamma \geq 0.$$

Now we will prove some results on Sumudu transform required for next section.

Lemma 1.1. Let $S[y(t)] = Y(v)$ and $y(t) = \frac{1}{t^{3/2}} e^{-w^2/4t}$, $w > 0$ then

$$(1.1) \quad \frac{Y(\sqrt{v})}{\sqrt{v}} = \frac{t^{-3/2}}{2\sqrt{\pi}} S \left[\int_0^\infty e^{-\frac{w^2}{4t}} w y(w) dw \right].$$

Proof. Using Sumudu transform table [10] one will get,

$$S \left[\frac{w}{t^{3/2}} e^{-w^2/4t} \right] = \frac{1}{v} e^{-w/\sqrt{v}}.$$

Multiply both sides by $y(w)$ and integrating with limits 0 to ∞ . One will get,
 $\frac{t^{-3/2}}{2\sqrt{\pi}} S \left[\int_0^\infty e^{-\frac{w^2}{4t}} w y(w) dw \right] = \frac{1}{v} \int_0^\infty e^{-w/\sqrt{v}} y(w) dw = \frac{Y(\sqrt{v})}{\sqrt{v}}.$ $\square$

Lemma 1.2. Let $S[y(t)] = Y(v)$ and $y(t) = \frac{1}{\sqrt{\pi} t^{1/2}} e^{-w^2/4t}$, $w > 0$ then

$$(1.2) \quad Y(\sqrt{v}) = \frac{t^{-1/2}}{\sqrt{\pi}} S \left[\int_0^\infty e^{-\frac{w^2}{4t}} y(w) dw \right].$$

Proof. Using Sumudu transform table [10] one will get,

$$S \left[\frac{1}{\sqrt{\pi} t^{1/2}} e^{-w^2/4t} \right] = \frac{1}{\sqrt{v}} e^{-w/\sqrt{v}}.$$

Multiply both sides by $y(w)$ and integrating with limits 0 to ∞ . One will get,

$$\frac{t^{-1/2}}{\sqrt{\pi}} S \left[\int_0^\infty e^{-\frac{w^2}{4t}} y(w) dw \right] = \frac{1}{\sqrt{v}} \int_0^\infty e^{-w/\sqrt{v}} y(w) dw = Y(\sqrt{v}).$$

□

Sumudu transform exists for continuous, exponentially ordered and absolutely integrable functions, so we consider all such functions in this article.

Consider the definition of Double Sumudu transform given by J. Tchuente and N Mbare [17], let $u(x, t)$ be a function of two variables in the first quadrant of the XOY plane. Then Sumudu transform of two variables is given by a following equation.

$$S_2 [u(x, t)] = \frac{1}{v_1 v_2} \int_0^\infty \int_0^\infty e^{-\left(\frac{x}{v_1} + \frac{t}{v_2}\right)} u(x, t) dx dt = U(v_1, v_2) .$$

Some of the properties of Double Sumudu transform are the following:

- (1) $S_2 \left[\frac{\partial u(x, t)}{\partial x} \right] = \frac{1}{v_1} Y(v_1, v_2) - \frac{1}{v_1} S[u(0, t)] ;$
- (2) $S_2 \left[\frac{\partial u(x, t)}{\partial t} \right] = \frac{1}{v_2} Y(v_1, v_2) - \frac{1}{v_2} S[u(x, 0)] ;$
- (3) $S_2 \left[\frac{\partial^2 u(x, t)}{\partial x^2} \right] = \frac{1}{v_1^2} Y(v_1, v_2) - \frac{1}{v_1^2} Y(0, v_2) - \frac{1}{v_1} S[u_x(0, t)] ;$
- (4) $S_2 \left[\frac{\partial^2 u(x, t)}{\partial t^2} \right] = \frac{1}{v_2^2} Y(v_1, v_2) - \frac{1}{v_2^2} Y(0, v_2) - \frac{1}{v_2} S[u_y(x, 0)] ;$
- (5) $S_2 \left[\frac{\partial^2 u(x, t)}{\partial x \partial t} \right] = \frac{1}{v_1 v_2} Y(v_1, v_2) - \frac{1}{v_1 v_2} S[u(x, 0)] - \frac{1}{v_1 v_2} S[u(0, t)] + \frac{1}{v_1 v_2} u(0, 0) .$

2. THEOREMS ON SUMUDU TRANSFORM OF FUNCTION OF TWO VARIABLES

Theorem 2.1. *Suppose that:*

- i) $S[y(t)] = Y(v),$
- ii) $S[u(t)] = \frac{Y(\sqrt{v})}{\sqrt{v}},$
- iii) $S\left[\frac{y(t)}{t}\right] = G(v), t \neq 0,$
- iv) $S[t^n y(t)] = G_n(v), n = 1, 2, 3$

then:

- (1) $S_2 \left(\frac{u\left[\frac{xy}{x+y}\right]}{(x+y)^{3/2}} \right) = \frac{2\sqrt{\pi}}{\sqrt{v_1 v_2}} \frac{1}{\sqrt{v_1} + \sqrt{v_2}} G\left(\frac{\sqrt{v_1 v_2}}{\sqrt{v_1} + \sqrt{v_2}}\right) ;$
- (2) $S_2 \left(\frac{xy u\left[\frac{xy}{x+y}\right]}{(x+y)^{3/2}} \right) = \frac{\sqrt{\pi}}{2} \frac{1}{\sqrt{v_1} + \sqrt{v_2}} G_1\left(\frac{\sqrt{v_1 v_2}}{\sqrt{v_1} + \sqrt{v_2}}\right) ;$
- (3) $S_2 \left(\frac{x^2 y^2 u\left[\frac{xy}{x+y}\right]}{(x+y)^{\frac{3}{2}}} \right) = \frac{\sqrt{\pi} \sqrt{v_1 v_2}}{8\sqrt{v_1} + \sqrt{v_2}}$

$$\left[\sqrt{v_1 v_2} G_1\left(\frac{\sqrt{v_1 v_2}}{\sqrt{v_1} + \sqrt{v_2}}\right) + (\sqrt{v_1} + \sqrt{v_2}) G_2\left(\frac{\sqrt{v_1 v_2}}{\sqrt{v_1} + \sqrt{v_2}}\right) + G_3\left(\frac{\sqrt{v_1 v_2}}{\sqrt{v_1} + \sqrt{v_2}}\right) \right] .$$

Proof. (1) Using equation (1.1) and condition (ii) given in the theorem, one can get,

$$(2.1) \quad u(t) = \frac{1}{2\sqrt{\pi}} \left[\int_0^\infty t^{-\frac{3}{2}} e^{-w^2/4t} w y(w) dw \right].$$

Replace $\frac{1}{t} = \frac{1}{x} + \frac{1}{y}$ and multiplying both sides of equation by $\frac{1}{v_1 v_2} \frac{1}{(xy)^{3/2}} e^{-\frac{x}{v_1}} e^{-\frac{y}{v_2}}$ and integrating with respect to x and y in the interval $(0, \infty)$ and $(0, \infty)$.

$$\begin{aligned} & \frac{1}{v_1 v_2} \int_0^\infty \int_0^\infty \frac{1}{(xy)^{3/2}} \frac{u \left[\frac{xy}{x+y} \right]}{\left(\frac{1}{x} + \frac{1}{y} \right)^{3/2}} e^{-\frac{x}{v_1} - \frac{y}{v_2}} dx dy \\ &= \frac{1}{2\sqrt{\pi}} \frac{1}{v_1 v_2} \int_0^\infty \int_0^\infty \frac{1}{(xy)^{3/2}} e^{-\frac{x}{v_1} - \frac{y}{v_2}} \left[\int_0^\infty e^{-\frac{w^2}{4} \left(\frac{1}{x} + \frac{1}{y} \right)} w y(w) dw \right] dx dy. \end{aligned}$$

Changing the order of integration by using Fubini's theorem

$$\begin{aligned} S_2 \left(\frac{u \left[\frac{xy}{x+y} \right]}{(x+y)^{3/2}} \right) &= \frac{4\pi}{2\sqrt{\pi}} \int_0^\infty \left[\frac{1}{v_1 v_2} e^{-\frac{w}{\sqrt{v_1}} - \frac{w}{\sqrt{v_2}}} \right] \frac{y(w)}{w} dw \\ &= \frac{2\sqrt{\pi}}{\sqrt{v_1 v_2}} \frac{1}{\sqrt{v_1} + \sqrt{v_2}} G \left(\frac{\sqrt{v_1 v_2}}{\sqrt{v_1} + \sqrt{v_2}} \right). \end{aligned}$$

(2) Replace $\frac{1}{t} = \frac{1}{x} + \frac{1}{y}$ in equation (2.1) and multiplying equation by $\frac{1}{v_1 v_2} \frac{1}{(xy)^{1/2}} e^{-\frac{x}{v_1}} e^{-\frac{y}{v_2}}$ both sides and integrating with respect to x and y in the interval $(0, \infty)$ and $(0, \infty)$.

$$\begin{aligned} S_2 \left(\frac{xy u \left[\frac{xy}{x+y} \right]}{(x+y)^{3/2}} \right) &= \frac{\pi}{2\sqrt{\pi}} \int_0^\infty \left[\frac{1}{\sqrt{v_1 v_2}} e^{-\frac{w}{\sqrt{v_1}} - \frac{w}{\sqrt{v_2}}} \right] w y(w) dw \\ &= \frac{\sqrt{\pi}}{2(\sqrt{v_1} + \sqrt{v_2})} G_1 \left(\frac{\sqrt{v_1 v_2}}{\sqrt{v_1} + \sqrt{v_2}} \right). \end{aligned}$$

(3) Replace $\frac{1}{t} = \frac{1}{x} + \frac{1}{y}$ in equation (2.1) and multiplying both sides of equation by $\frac{1}{v_1 v_2} (xy)^{1/2} e^{-\frac{x}{v_1}} e^{-\frac{y}{v_2}}$ and integrating with respect to x and y in the interval $(0, \infty)$ and $(0, \infty)$.

$$S_2 \left(\frac{x^2 y^2 u \left[\frac{xy}{x+y} \right]}{(x+y)^{3/2}} \right) = \frac{\pi}{8\sqrt{\pi}} \int_0^\infty \left[[\sqrt{v_1} + w] [\sqrt{v_2} + w] e^{-\frac{w}{\sqrt{v_1}} - \frac{w}{\sqrt{v_2}}} \right] w y(w) dw.$$

Simple calculation give the required result.

□

Theorem 2.2. Suppose that

- i) $S[y(t)] = Y(v)$,
- ii) $S[u(\frac{1}{t})] = Y(\sqrt{v})$,
- iii) $S\left[\frac{y(t)}{t^2}\right] = H_2(v)$, $t \neq 0$,
- iv) $S[t^n y(t)] = G_n(v)$, $n = 1, 2$,

then:

$$(1) S_2\left(\frac{u\left[\frac{x+y}{xy}\right]}{(x+y)^{1/2}}\right) = \frac{\sqrt{\pi}}{\sqrt{v_1} + \sqrt{v_2}} Y\left(\frac{\sqrt{v_1 v_2}}{\sqrt{v_1} + \sqrt{v_2}}\right);$$

$$(2) S_2\left(\frac{(xy)^{-1} u\left[\frac{x+y}{xy}\right]}{(x+y)^{1/2}}\right) = \frac{4\sqrt{\pi}}{\sqrt{v_1 v_2}} \frac{1}{\sqrt{v_1} + \sqrt{v_2}} H_2\left(\frac{\sqrt{v_1 v_2}}{\sqrt{v_1} + \sqrt{v_2}}\right);$$

$$(3) S_2\left(\frac{xy u\left[\frac{x+y}{xy}\right]}{(x+y)^{\frac{1}{2}}}\right) = \frac{\sqrt{\pi} \sqrt{v_1 v_2}}{4(\sqrt{v_1} + \sqrt{v_2})}$$

$$\left[\sqrt{v_1 v_2} Y\left(\frac{\sqrt{v_1 v_2}}{\sqrt{v_1} + \sqrt{v_2}}\right) + (\sqrt{v_1} + \sqrt{v_2}) G_1\left(\frac{\sqrt{v_1 v_2}}{\sqrt{v_1} + \sqrt{v_2}}\right) + G_2\left(\frac{\sqrt{v_1 v_2}}{\sqrt{v_1} + \sqrt{v_2}}\right) \right].$$

Proof. i) Using equation (1.2) and condition (ii) given in the theorem, one can get,

$$u\left(\frac{1}{t}\right) = \frac{1}{\sqrt{\pi}} \left[\int_0^\infty t^{-\frac{1}{2}} e^{-w^2/4t} y(w) dw \right].$$

Replace $\frac{1}{t} = \frac{1}{x} + \frac{1}{y}$ and multiplying equation by $\frac{1}{v_1 v_2} \frac{1}{(xy)^{1/2}} e^{-\frac{x}{v_1}} e^{-\frac{y}{v_2}}$ both sides and integrating with respect to x and y in the interval $(0, \infty)$ and $(0, \infty)$.

$$\begin{aligned} & \frac{1}{v_1 v_2} \int_0^\infty \int_0^\infty \frac{1}{(xy)^{1/2}} \frac{u\left[\frac{x+y}{xy}\right]}{\left(\frac{1}{x} + \frac{1}{y}\right)^{1/2}} e^{-\frac{x}{v_1} - \frac{y}{v_2}} dx dy \\ &= \frac{1}{\sqrt{\pi}} \frac{1}{v_1 v_2} \int_0^\infty \int_0^\infty \frac{1}{(xy)^{1/2}} e^{-\frac{x}{v_1} - \frac{y}{v_2}} \left[\int_0^\infty e^{-\frac{w^2}{4}\left(\frac{1}{x} + \frac{1}{y}\right)} y(w) dw \right] dx dy. \end{aligned}$$

Changing the order of integration by using Fubini's theorem

$$\begin{aligned} S_2\left(\frac{u\left[\frac{x+y}{xy}\right]}{(x+y)^{1/2}}\right) &= \frac{\pi}{\sqrt{\pi}} \int_0^\infty \left[\frac{1}{\sqrt{v_1 v_2}} e^{-\frac{w}{\sqrt{v_1}} - \frac{w}{\sqrt{v_2}}} \right] y(w) dw \\ &= \frac{\sqrt{\pi}}{\sqrt{v_1} + \sqrt{v_2}} Y\left(\frac{\sqrt{v_1 v_2}}{\sqrt{v_1} + \sqrt{v_2}}\right) \end{aligned}$$

Simiral as we proved in theorem 2.1, follows the proof of (ii) and (iii). $\square$

Hermite Polynomial [19] is defined as:

$$H_{e_n}(t) = t^n \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k n!}{K! (n-2k)! 2^k t^{2k}},$$

here $\left[\frac{n}{2}\right] = \begin{cases} \frac{n}{2}, & \text{if } n \text{ even} \\ \frac{n-1}{2}, & \text{if } n \text{ odd} \end{cases}$ and its Sumudu transform, one may write using the duality relation with Laplace transform [6] and Laplace transform of Hermite polynomial [20].

$$(2.2) \quad S_2 \left(\frac{1}{x^{\left(\frac{n+1}{2}\right)} y^{\left(\frac{m+1}{2}\right)}} e^{-\frac{w}{2} \left(\frac{1}{x} + \frac{1}{y}\right)} H_{e_n} \left(\sqrt{\frac{w}{x}} \right) H_{e_m} \left(\sqrt{\frac{w}{y}} \right) \right) \\ = \frac{2^{\left(\frac{n+m}{2}\right)} \pi}{v_1^{\left(\frac{n+1}{2}\right)} v_2^{\left(\frac{m+1}{2}\right)}} e^{-\sqrt{2w} \left(\frac{1}{\sqrt{v_1}} + \frac{1}{\sqrt{v_2}}\right)}$$

Theorem 2.3. Suppose that

- i) $S[y(t)] = Y(v)$,
- ii) $S[tu\left(\frac{t^2}{2}\right)] = \bar{Y}(v)$,

then:

$$S_2 \left(\frac{1}{x^{\left(\frac{n+1}{2}\right)} y^{\left(\frac{m+1}{2}\right)}} \int_0^\infty e^{-\frac{w}{2} \left(\frac{1}{x} + \frac{1}{y}\right)} H_{e_n} \left(\sqrt{\frac{w}{x}} \right) H_{e_m} \left(\sqrt{\frac{w}{y}} \right) y(w) dw \right) \\ = \frac{2^{\left(\frac{n+m}{2}\right)} \pi}{v_1^{\left(\frac{n+1}{2}\right)} v_2^{\left(\frac{m+1}{2}\right)}} \frac{\sqrt{v_1 v_2}}{\sqrt{v_1} + \sqrt{v_2}} \bar{Y} \left(\frac{\sqrt{v_1 v_2}}{\sqrt{v_1} + \sqrt{v_2}} \right).$$

Proof. Consider equation (2.2), multiply it by $y(w)$ and integrate it with respect to w in $(0, \infty)$, one can get

$$S_2 \left(\frac{1}{x^{\left(\frac{n+1}{2}\right)} y^{\left(\frac{m+1}{2}\right)}} \int_0^\infty e^{-\frac{w}{2} \left(\frac{1}{x} + \frac{1}{y}\right)} H_{e_n} \left(\sqrt{\frac{w}{x}} \right) H_{e_m} \left(\sqrt{\frac{w}{y}} \right) y(w) dw \right) \\ = \frac{2^{\left(\frac{n+m}{2}\right)} \pi}{v_1^{\left(\frac{n+1}{2}\right)} v_2^{\left(\frac{m+1}{2}\right)}} \int_0^\infty e^{-\sqrt{2w} \left(\frac{1}{\sqrt{v_1}} + \frac{1}{\sqrt{v_2}}\right)} y(w) dw \\ = \frac{2^{\left(\frac{n+m}{2}\right)} \pi}{v_1^{\left(\frac{n+1}{2}\right)} v_2^{\left(\frac{m+1}{2}\right)}} \frac{\sqrt{v_1 v_2}}{\sqrt{v_1} + \sqrt{v_2}} \bar{Y} \left(\frac{\sqrt{v_1 v_2}}{\sqrt{v_1} + \sqrt{v_2}} \right).$$

After using Substitution $2w = t^2$ and using the condition (ii), we have the result. $\square$

3. APPLICATIONS TO PARTIAL DIFFERENTIAL EQUATIONS

Example 1. Let us solve Poisson equation with function $u(x, t)$, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = f(x, t)$; $x > 0, t > 0$ with the conditions,

$$u(0, t) = 0 = u(x, 0), u_x(0, t) = 0 = u_t(x, 0).$$

Let $S_2[u(x, t)] = U(v_1, v_2)$ and $S_2[f(x, t)] = F(v_1, v_2)$. Applying Double Sumudu transform with given conditions, we will get:

$$U(v_1, v_2) = \left[\frac{v_1^2 v_2^2}{v_1^2 + v_2^2} \right] F(v_1, v_2).$$

Let $f(x, t) = \frac{(x^2 + t^2)^2 - 6xt(x^2 + t^2)}{\sqrt{xt}(x+t)^4}$. Consider $y(t) = t$, and $n = 4, m = 0$ and $n = 0, m = 4$, in equation (2.2):

$$(3.1) \quad S_2 \left[\frac{8t^{7/2}}{x^{1/2}(x+t)^4} - \frac{8t^{5/2}}{x^{1/2}(x+t)^3} + \frac{t^{3/2}}{x^{1/2}(x+t)^2} \right] = \frac{\pi(v_1 v_2)^{3/2}}{v_1^2 (\sqrt{v_1} + \sqrt{v_2})^4},$$

$$(3.2) \quad S_2 \left[\frac{8x^{7/2}}{t^{1/2}(x+t)^4} - \frac{8x^{5/2}}{t^{1/2}(x+t)^3} + \frac{x^{3/2}}{t^{1/2}(x+t)^2} \right] = \frac{\pi(v_1 v_2)^{3/2}}{v_2^2 (\sqrt{v_1} + \sqrt{v_2})^4}.$$

Adding equations (3.1) and equations (3.2)

$$S_2 \left[\frac{(x^2 + t^2)^2 - 6xt(x^2 + t^2)}{\sqrt{xt}(x+t)^4} \right] = F(v_1, v_2) = \left[\frac{v_1^2 + v_2^2}{v_1^2 v_2^2} \right] \frac{\pi(v_1 v_2)^{3/2}}{(\sqrt{v_1} + \sqrt{v_2})^4}.$$

From this one will get, $U(v_1, v_2) = \left[\frac{v_1^2 v_2^2}{v_1^2 + v_2^2} \right] F(v_1, v_2) = \frac{\pi(v_1 v_2)^{3/2}}{(\sqrt{v_1} + \sqrt{v_2})^4}$, so $u(x, t) = \frac{4(xt)^{3/2}}{3(x+t)^2}$, by using theorem 2.1 and $y(t) = t^2$.

Example 2. Consider second order hyperbolic differential equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = f(x, t); x > 0, t > 0 \text{ with the conditions,}$$

$$u(0, t) = f(t), u(x, 0) = 0, u_x(0, t) = 0 = u_t(x, 0).$$

Let $S_2[u(x, t)] = U(v_1, v_2)$, $S[u(0, t)] = F(v_2)$ and $S_2[f(x, t)] = F(v_1, v_2)$.
Applying Double Sumudu transform with given conditions, we have:

$$U(v_1, v_2) = \left[\frac{v_1^2 v_2^2}{v_1^2 - v_2^2} \right] F(v_1, v_2) + \left[\frac{v_2^2}{v_1^2 - v_2^2} \right] F(v_2).$$

Let $f(x, t) = \frac{8(t^4 - x^4) + 12(x + t)(x^3 - t^3) + 3(t - x)(x + t)^3}{2x^{3/2}t^{3/2}(x + t)^3}$ and $f(t) = 1$.

Let $y(t) = 1$, and $n = 5, m = 0$ and $n = 0, m = 5$ in equation (2.2):

$$(3.3) \quad S_2 \left[\frac{4t^{5/2}}{x^{3/2}(x + t)^3} - \frac{6t^{3/2}}{x^{3/2}(x + t)^2} + \frac{3t^{1/2}}{2x^{3/2}(x + t)} \right] = \frac{\pi(v_1 v_2)^{3/2}}{v_1^2 (\sqrt{v_1} + \sqrt{v_2})^4}$$

$$(3.4) \quad S_2 \left[\frac{4x^{5/2}}{t^{3/2}(x + t)^3} - \frac{6x^{3/2}}{t^{3/2}(x + t)^2} + \frac{3x^{1/2}}{2t^{3/2}(x + t)} \right] = \frac{\pi(v_1 v_2)^{3/2}}{v_2^2 (\sqrt{v_1} + \sqrt{v_2})^4}$$

subtracting equation (3.3) and (3.4)

$$\begin{aligned} S_2 \left[\frac{8(t^4 - x^4) + 12(x + t)(x^3 - t^3) + 3(t - x)(x + t)^3}{2x^{3/2}t^{3/2}(x + t)^3} \right] \\ = \left[\frac{v_1^2 - v_2^2}{v_1^2 v_2^2} \right] \frac{\pi(v_1 v_2)^{1/2}}{(\sqrt{v_1} + \sqrt{v_2})^2} \\ \Rightarrow \left[\frac{v_1^2 v_2^2}{v_1^2 - v_2^2} \right] F(v_1, v_2) = \frac{\pi(v_1 v_2)^{1/2}}{(\sqrt{v_1} + \sqrt{v_2})^2} = S_2 \left[\frac{(xt)^{1/2}}{2(x + t)} \right]. \end{aligned}$$

Using theorem 2.2, and $y(t) = H(t)$, one we have:

$$u(x, t) = \frac{(xt)^{1/2}}{2(x + t)} - H(t - x).$$

Example 3. Consider inhomogeneous diffusion problem, $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = q(x, t)$; $x > 0, t > 0$ with the conditions:

$$u(0, t) = 0, u(x, 0) = f(x), u_x(0, t) = 0.$$

Let $S_2[u(x, t)] = U(v_1, v_2)$, $S[u(x, 0)] = F(v_1)$ and $S_2[q(x, t)] = Q(v_1, v_2)$.
Applying Double Sumudu transform with given conditions, we have:

$$U(v_1, v_2) = \left[\frac{v_1^2 v_2}{v_1^2 - v_2} \right] Q(v_1, v_2) + \left[\frac{v_1^2}{v_1^2 - v_2} \right] F(v_1).$$

Let $q(x, t) = \frac{4(20t^2+x^3)(x+t)-3(x+t)^2(2x^2+10t)-48t^2}{4x^{3/2}t^{1/2}(x+t)^4}$ and $f(x) = 0$. Let $y(t) = 1$, and $n = 5, m = 1$ and $n = 1, m = 3$ in equation (2.2)

$$(3.5) \quad S_2 \left[\frac{12t^{5/2}}{x^{3/2}(x+t)^4} - \frac{20t^{3/2}}{x^{3/2}(x+t)^3} + \frac{15t^{1/2}}{2x^{3/2}(x+t)^2} \right] = \frac{\pi}{v_1^2(\sqrt{v_1} + \sqrt{v_2})^2},$$

$$(3.6) \quad S_2 \left[\frac{4x^{3/2}}{t^{1/2}(x+t)^3} - \frac{3x^{1/2}}{t^{1/2}(x+t)^2} \right] = \frac{\pi}{v_2(\sqrt{v_1} + \sqrt{v_2})^2}.$$

Subtracting equation (3.5) and (3.6)

$$\begin{aligned} S_2 \left[\frac{4(20t^2+x^3)(x+t)-3(x+t)^2(2x^2+10t)-48t^2}{4x^{3/2}t^{1/2}(x+t)^4} \right] \\ = \left[\frac{v_1^2 - v_2}{v_1^2 v_2} \right] \frac{\pi}{(\sqrt{v_1} + \sqrt{v_2})^2} \\ \Rightarrow \left[\frac{v_1^2 v_2}{v_1^2 - v_2} \right] Q(v_1, v_2) = \frac{\pi}{(\sqrt{v_1} + \sqrt{v_2})^2} = S_2 \left[\frac{2(xt)^{1/2}}{(x+t)^2} \right], \end{aligned}$$

$$\text{so, } u(x, t) = \frac{2(xt)^{1/2}}{(x+t)^2}.$$

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REFERENCES

- [1] G.K. WATUGALA: *Sumudu Transform: A new Integral Transform to solve Differential Equations and Control Engineering Problems*, Int. J. Math. Educ. Sci. Technol., **24** (1993), 35–43.
- [2] S. WEERAKOON: *Application of Sumudu Transform to Partial Differential Equations*, Int. J. Math. Educ. Sci. Technol., **25** (1994), 277–283.
- [3] G.K. WATUGALA: *Sumudu Transform: A new Integral Transform to solve Differential Equations and Control Engineering Problems*, Math. Engr. Indust., **6** (1998), 319–329.
- [4] S. WEERAKOON: *Complex Inversion formula for Sumudu Transform*, Int. J. Math. Educ. Sci. Technol., **29** (1998), 618–621.

- [5] M. A. ASIRU: *Sumudu Transform and Solution of Integral Equations of Convolution type*, Int. J. Math. Educ. Sci. Technol., **32** (2001), 906–910.
- [6] M.A. ASIRU: *Further Properties of the Sumudu Transform and its Applications*, Int. J. Math. Educ. Sci. Technol., **33** (2002), 441–449.
- [7] G.K. WATUGALA: *The Sumudu Transform for functions of two variables*, Math. Engr. Indust., **8** (2002), 293–302.
- [8] M. A. ASIRU: *Application of the Sumudu Transform to Discrete Dynamic Systems*, Int. J. Math. Educ. Sci. Technol., **34** (2003), 944–950.
- [9] F.B.M. BELGACEM, A.A. KARABALLI, S. KALLA: *Analytical investigations of the Sumudu transform and applications to integral production equations*, Math. Probl. in Engr., **3** (2003), 103–118.
- [10] F.B.M. BELGACEM, A.A. KARABALLI: *Sumudu transform fundamental properties investigations and applications*, J. Appl. Math. Stoch. Anal., **1** (2006), 1–23.
- [11] E.HASSAN, A.KILICMAN: *On some applications of a new integral transform*, Int. J. Math. Anal., **4** (2010), 123–132.
- [12] E.HASSAN, A.KILICMAN: *On Double Sumudu Transform and Double Laplace Transform*, Malaysian J. Math. Sci., **4** (2010), 17–30.
- [13] A KILICMAN., H.E. GADAIN: *An Application of Double Laplace Transform and Double Sumudu Transform*, Lobachevskii J. Math., **347** (2010), 848–862.
- [14] A KILICMAN., H.E. GADAIN: *On the Applications of Laplace and Sumudu Transforms*, J. Franklin I., **347** (2010), 848–862.
- [15] A. KILICMAN, H. ELTYAB, ATAN: *A note on the Comparison between Laplace and Sumudu transforms*, B Iran Math. Soc., **37** (2011), 131–141.
- [16] A. KILICMAN, H. ELTAYEB: *Some Remarks on the Sumudu and Laplace Transforms and Applications to Differential Equations*, International Scholarly Research Network, ISRN Appl. Math., (2012), doi:10.5402/2012/591517.
- [17] J.. TCHUENCHE, N MBARE: *An Application of the Double Sumudu Transform*, Appl. Math. Sci., **1** (2007), 31–39.
- [18] L. DEBNATH: *The Double Laplace Transform and Their Properties with Applications to Functional, Integral and partial differential Equations*, Int. J. Appl. Comput. Math., **2** (2016), 223–241.
- [19] N.N. LEBEDEV: *Special Functions and their Applications*, edited by Siverman, R. A. Englewood, Cliffs N.J.: Prentice Hall, 1965.
- [20] G.E.ROBERTS, H. KAUFMAAN: *Table of Laplace Transforms*, Philadelphia: W. B. Saunders Co.,1996.
- [21] S.B. KIWNE, S. M SONAWANE: *On the Applications of Laplace and Aboodh Transforms in Engineering field*, Int. Int. J. Pure Appl. Math., **15** (2019), 1079–1101.

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