Advances in Mathematics: Scientific Journal **9** (2020), no.1, 119–143
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.1.12

ON UNIQUENESS OF L-SHARING OF DIFFERENTIAL POLYNOMIALS OF MEROMORPHIC FUNCTIONS

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ABSTRACT. In this paper, we shall study the uniqueness problems of differential polynomials of meromorphic functions sharing 1 value. Here we prove two uniqueness theorems which extend and improve recent results of H.P. Waghamore and N.H. Sannappala [10].

1. INTRODUCTION

Throughout the article we study the uniqueness of differential polynomials of f and g, where f and g are non-constant meromorphic functions in whole complex plane. Here we use the standard definitions, theorems and notations of Nevanlinna's value distribution theory (see [3]). The Nevanlinna characteristic function is denoted by $T(r,f)$ and $S(r,f)$ is small quantity define by $o(T(r,f)) = S(r,f)$, as $r \to \infty$ and $r \notin E$ where $E \subseteq \mathbb{R}^+$ and measure of E is finite. The greatest common divisor of positive real number shall be denoted by $GCD(q_1, q_2, ..., q_p)$ where $q_1, q_2, ..., q_p$ are positive integers. Let $a \in \mathbb{C} \setminus \{0\}$ and we say that f and g share value a CM(Counting Multiplicities) if $f - a$ and $g - a$ have same zeros with same multiplicities. We say that f and g share a IM (Ignoring Multiplicities) if $f - a$ and $g - a$ have the same zeros. Now we define, $\Theta(a,f) = 1 - \lim_{r\to\infty} sup\frac{\overline{N}(r,a;f)}{T(r,f)}$. where $a \in \mathbb{C} \cup \{\infty\}$. $E_l(a,f)$ is the set of all a-points of f where an a-point with multiplicities m is counted m times if

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2010 *Mathematics Subject Classification.* 30D35.

Key words and phrases. Non-constant meromorphic function, l-Sharing, Uniqueness.

$m \leq l$ and $l+1$ times if $m > l$. If $E_l(a, f) = E_l(a, g)$, then we say that f and g share the value a with weight l. We also use the notation $N(r, a; f|p)$ to denote the counting function of $f - a$ where m fold zeros is counted m times if $m \leq p$ and p times if $m > p$, where p is an integer.

Definition 1.1. *[4] Let f is a nonconstant meromorphic function and $a \in \mathbb{C} \cup \{\infty\}$, the counting function of a-points of f with multiplicities at least $p(\in \mathbb{Z}^+)$ is denoted by $N(r, a; f \mid\geq p)$ and $\overline{N}(r, a; f \mid\geq p)$ is the corresponding reduced counting function. Similarly we can define $N(r, a; f \mid\leq p)$ and $\overline{N}(r, a; f \mid\leq p)$.*

Definition 1.2. *[4] The counting function of a-points of f, where an a-point of multiplicities m is counted m times if $m \leq p$ and p times if $m > p$ is denoted by $N_p(r, a; f)$, where $p \in \mathbb{Z}^+ \cup \{\infty\}$. Then we can write:*

$$N_p(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2) + ... + \overline{N}(r, a; f \mid\geq p) \,.$$

Definition 1.3. *[4] Let f and g be two nonconstant meromorphic functions those share the value 1 IM. Let z_0 be a 1-point of f and g with multiplicity p and q respectively. The counting function of those 1-points of f and g, where $p > q$ is denoted by $\overline{N}_L(r, 1; f)$, and the counting function of those 1-points of f and g, where $p = q \geq k$ is denoted by $\overline{N}_E^{(k}(r, 1; f)$ ($k \geq 2$ is an integer), where each point in those counting functions is counted only once. Similarly we can define $\overline{N}_L(r, 1; g)$ and $\overline{N}_E^{(k}(r, 1; g)$.*

Definition 1.4. *[4] Let f and g be two nonconstant meromorphic functions those share the value a IM. The reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of corresponding a-points of g is denote by $\overline{N}_*(r, a; f; g)$. So we claim that $\overline{N}_*(r, a; f; g) = \overline{N}_*(r, a; g; f)$ and $\overline{N}_*(r, a; f; g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.*

In 2001, Fang and Hong [2] deduce the following theorem,

Theorem 1.1. *[2] Let f and g be two transcendental entire functions and $n(\geq 11)$ be an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 CM then $f \equiv g$.*

In 2006, Lahiri and Pal proved the following result:

Theorem 1.2. *[6] Let f and g be two nonconstant meromorphic functions and $n(\geq 14)$ be an integer. Let $F^{@} = f^n(f^3-1)f'$ and $G^{@} = g^n(g^3-1)g'$. If $E_{3)}(1, F^{@}) = E_{3)}(1, G^{@})$ then $f \equiv g$.*

In 2015, Chao Meng established the following result:

Theorem 1.3. *[7] Let f and g be two non-constant meromorphic functions, $n(\geq 12)$ a positive integer. If $E_2(1, f^n(f^3-1)f') = E_2(1, g^n(g^3-1)g')$ and f and g share ∞ IM, then $f \equiv g$.*

Let $Q(z) = \sum_{i=0}^{p} a_i z^i$, where $a_0(\neq 0), a_1, a_2, ..., a_{p-1}, a_p(\neq 0)$ are complex constants and $i \in \mathbb{Z}^+$.
We assume that $F = [f^n Q(f)]^{(k)}$ and $G = [g^n Q(g)]^{(k)}$.

In 2018, H.P. Waghamore and N.H. Sannappala [10] proved the results:

Theorem 1.4. *[10] Let f and g be two non-constant meromorphic functions whose zeros and poles are multiplicities at least m and $n > p + k + \frac{1}{m}(3k+4)$, where m, n, p are positive integers, and $\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n}$. If F and G share $(1,2)$ and f and g share ∞ IM, then one of the following two cases holds:*

i) *$f = tg$ for a constant t such that $t^{\chi} = 1$, where $\chi = GCD(n+p, ..., n+p-i, ..., n+1, n)$ and $a_{p-i} \neq 0$ for some $i = 0, 1, ..., p$.*

ii) *f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where $R(f,g) = f^n Q(f) - g^n Q(g)$.*

Theorem 1.5. *[10] Let f and g be two non-constant meromorphic functions whose zeros and poles are multiplicities at least m and $n > p + k + \frac{1}{m}(3k+4)$, where m, n, p are positive integers. If $f^n Q(f) f'$ and $g^n Q(g) g'$ share $(1,2)$ and f and g share ∞ IM, then one of the following two cases holds:*

i) *$f = tg$ for a constant t such that $f^{\chi} = 1$, where $\chi = GCD(n+p, ..., n+p-i, ..., n+1, n)$ and $a_{p-i} \neq 0$ for some $i = 0, 1, ..., p$.*
ii) *f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where $R(f,g) = f^{n+1} \sum_{i=0}^{p} \frac{a_{p-i} f^{m-i}}{n+p+1-i} - g^{n+1} \sum_{i=0}^{p} \frac{a_{p-i} g^{m-i}}{n+p+1-i}$.*

In their paper [10], the authors posed the open problems that:

i) n can be still reduced, and
ii) $(1,2)$ sharing can be replaced by $(1,l)$,$(l \geq 0)$ sharing?

The solution of this problems is given in the section Main results, theorem 3.1 and theorem 3.2.

2. LEMMAS

To prove our results following lemmas will be needed. Let F_1 and G_1 be two non-constant meromorphic functions defined in $\mathbb{C}$. We denote by H_1 the function as follows : $H_1 = (\frac{F_1''}{F_1'} - \frac{2F_1'}{F_1-1}) - (\frac{G_1''}{G_1'} - \frac{2G_1'}{G_1-1})$.

Lemma 2.1. *[12] Let f be a non-constant meromorphic function, where $a_0, a_1, a_2, ..., a_n (\neq 0)$ are complex constants and $i \in \mathbb{Z}^+$. Then*

$$T(r, \sum_{i=0}^{n} a_i f^i) = nT(r, f) + S(r, f)\,.$$

Lemma 2.2. *[13] Let f be a nonconstant meromorphic function, and $p, k \in \mathbb{Z}^+$. Then: $N(r, 0; f^{(k)}) \leq k\overline{N}(r, \infty; f) + N(r, 0; f) + S(r, f)$.*

Lemma 2.3. *[14] Let f be a nonconstant meromorphic function, and $p, k \in \mathbb{Z}^+$. Then, $N(r, 0; f^{(k)}|p) \leq k\overline{N}(r, \infty; f) + N(r, 0; f|p+k) + S(r, f)$.*

Lemma 2.4. *[1] Let F_1 and G_1 be two non-constant meromorphic functions sharing $(1, 2)$,$(\infty, 0)$ and $H_1 \not\equiv 0$, then, $T(r, F_1) \leq N(r, 0; F_1|2) + N(r, 0; G_1|2) + \overline{N}(r, \infty; F_1) + \overline{N}(r, \infty; G_1) + \overline{N}_*(r, \infty; F_1; G_1) - m(r, 1; G_1) - N_E^{(3}(r, 1; F_1) - \overline{N}_L(r, 1; G_1) + S(r, F_1) + S(r, G_1)$.*
We can deduce same result for $T(r, G)$.

Lemma 2.5. *[11] Let F_1 and G_1 be two non-constant meromorphic functions sharing $(1, 1)$,$(\infty, 0)$ and $H_1 \not\equiv 0$, then $T(r, F_1) \leq N(r, 0; F_1|2) + N(r, 0; G_1|2) + \frac{2}{3}\overline{N}(r, \infty; F_1) + \overline{N}(r, \infty; G_1) + \overline{N}_*(r, \infty; F_1, G_1) + \frac{1}{2}\overline{N}(r, 0; F_1) + S(r, F_1) + S(r, G_1)$.*
We can deduce same result for $T(r, G)$.

Lemma 2.6. *[11] Let F_1 and G_1 be two non-constant meromorphic functions sharing $(1, 0)$,$(\infty, 0)$ and $H_1 \not\equiv 0$, then, $T(r, F_1) \leq N(r, 0; F_1|2) + N(r, 0; G_1|2) + 3\overline{N}(r, \infty; F_1) + 2\overline{N}(r, \infty; G_1) + \overline{N}_*(r, \infty; F_1; G_1) + 2\overline{N}(r, 0; F_1) + \overline{N}(r, 0; G_1) + S(r, F_1) + S(r, G_1)$. We can deduce same result for $T(r, G)$.*

Lemma 2.7. *[5] Let f and g be two non-constant meromorphic functions and $\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n}$ for all integers $n \geq 3$. Now if $f^n(af + b) = g^n(ag + b)$ then $f = g$, where a and b are two finite non-zero complex constants.*

Lemma 2.8. *[10] Let F_1 and G_1 be two non-constant meromorphic functions, whose zeros and poles are of multiplicities at least m, where m is positive integer. Let $n, k \in \mathbb{Z}^+$ and if there exists two non-zero constants α and β such* *that* $\overline{N}(r,0;F_1) = \overline{N}(r,0;G_1-\alpha)$ *and* $\overline{N}(r,0;G_1) = \overline{N}(r,0;F_1-\beta)$ *then* $n > p+\frac{3}{m}(k+1)$.

Lemma 2.9. *[9] Let* f *and* g *be two non-constant meromorphic functions and let* $n(\geq 1)$,$k(\geq 1)$, *and* $m(\geq 1)$ *be integers. Then* $FG \neq 1$, *where* F, G *are as define above.*

Lemma 2.10. *[8] Let* f *and* g *be two nonconstant meromorphic functions and* $n+p \geq 6$ *is a positive integer then* $f^nQ(f)f'g^nQ(g)g' \neq 1$.

3. Main results

Theorem 3.1. *Let* f *and* g *be two non-constant meromorphic functions whose zeros and poles are multiplicities at least* m, *where* m, n, p, k *are positive integers, and* $\Theta(\infty,f)+\Theta(\infty,g) > \frac{4}{n}$. *If* $E_l(1,F) = E_l(1,G)$ *and* f *and* g *share* ∞ *IM. Then for the one of the following conditions:*

i) $l \geq 2$; *a)* $m \geq 2$ *and* $n > p+k+\frac{1}{m}(3k+7)$, *b)* $m=1$ *and* $n > p+4k+6$;
ii) $l=1$; *a)* $m \geq 2$ *and* $n > \frac{3p}{2}+k+\frac{1}{m}(4k+8)$, *b)* $m=1$ *and* $n > \frac{3p}{2}+5k+7$;
iii) $l=0$; *a)* $m \geq 2$ *and* $n > 4p+k+\frac{1}{m}(9k+13)$, *b)* $m=1$ *and* $n > 4p+10k+12$;

one of the following results hold:

i) $f = tg$ *for a constant* t *such that* $t^\chi = 1$, *where* $\chi = GCD(n+p,...,n+p-i,...,n+1,n)$ *and* $a_{p-i} \neq 0$ *for some* $i = 0,1,...,p$.
ii) f *and* g *satisfy the algebraic equation* $R(f,g) \equiv 0$, *where* $R(f,g) = f^nQ(f) - g^nQ(g)$.

Proof. First we defined two functions $F^* = \sum_{i=0}^{p} \frac{a_{p-i}(n+p-i)!}{(n-k+1+p-i)!} f^{n-k+1+p-i}$ and $G^* = \sum_{i=0}^{p} \frac{a_{p-i}(n+p-i)!}{(n-k+1+p-i)!} g^{n-k+1+p-i}$. Then from lemma 2.1 we have

$$T(r,F^*) = (n-k+1+p)T(r,f) + S(r,f). \tag{3.1}$$

Since $(F^*)' = F$, we deduce, $m(r, \frac{1}{F^*}) = m(r, \frac{1}{F}) + S(r, f)$. By Nevanlinna's first fundamental theorem, we get

$$\begin{aligned} T(r, F^*) &\leq \overline{N}(r, \infty; F) + N(r, 0; F^*) - N(r, 0; F) + S(r, f) \\ &\leq T(r, F) + N(r, 0; f) + \sum_{i=0}^{p} N(r, 0; f - \mu_i) - \sum_{i=0}^{p} N(r, 0; f - \nu_i) \\ &- k\overline{N}(r, \infty; f) + S(r, f), \end{aligned} \tag{3.2}$$

where μ_i and $\nu_i (i = 1, 2, ..., p)$ are roots of algebraic equations $\sum_{i=0}^{p} \frac{a_{p-i}(n+p-i)!}{(n-k+1+p-i)!} z^{p-i} = 0$ and $\sum_{i=0}^{p} a_{p-i} z^{p-i} = 0$ respectively. Also we use the result for $m(\geq 2)$, $m(= 1)$

$$\overline{N}_*(r, \infty; f; g) \leq \overline{N}(r, \infty; f), \tag{3.3}$$

$$\overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + \overline{N}_*(r, \infty; f; g) \leq N(r, \infty; f) + N(r, \infty; g), \tag{3.4}$$

respectively. As we assume that zeros and poles of f and g are of multiplicities at least $m(\geq 2)$, then

$$\overline{N}(r, \infty; f) \leq \frac{1}{m} N(r, \infty; f) \leq \frac{1}{m} T(r, f), \tag{3.5}$$

$$\overline{N}(r, 0; f) \leq \frac{1}{m} N(r, 0; f) \leq \frac{1}{m} T(r, f), \tag{3.6}$$

$$\overline{N}(r, \infty; f) \leq \frac{1}{m} N(r, \infty; f) \leq \frac{1}{m} T(r, f), \tag{3.7}$$

$$\overline{N}(r, 0; f) \leq \frac{1}{m} N(r, 0; f) \leq \frac{1}{m} T(r, f). \tag{3.8}$$

Now F and G are transcendental meromorphic functions that share (1,l) and f, g share $(\infty, 0)$. We discuss the following two cases separately.

Case 1. We assume that $H_1 \not\equiv 0$. Now we study the following subcases.

Subcase 1.1 If $l \geq 2$, then, using lemma 2.4, we obtain

$$\begin{aligned} T(r, F) &\leq N(r, 0; F|2) + N(r, 0; G|2) + \overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) \\ &+ \overline{N}_*(r, \infty; F; G) + S(r, F) + S(r, G). \end{aligned} \tag{3.9}$$

Now from equation (3.2) and (3.9) we have:

$$
\begin{aligned}
& T(r,F^*) \\
\leq\ & N(r,0;F|2)+N(r,0;G|2)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G) \\
+\ & \overline{N}_*(r,\infty;F;G)+N(r,0;f)+\sum_{i=1}^{p}N(r,0;f-\mu_i) \\
-\ & \sum_{i=1}^{p}N(r,0;f-\nu_i)-k\overline{N}(r,\infty;f)+S(r,f)+S(r,g) \\
\leq\ & N(r,0;[f^nQ(f)]^{(k)}|2)+N(r,0;[g^nQ(g)]^{(k)}|2)+\overline{N}(r,\infty;f) \\
+\ & \overline{N}(r,\infty;g)+\overline{N}_*(r,\infty;f;g)+N(r,0;f)+\sum_{i=1}^{p}N(r,0;f-\mu_i) \\
-\ & \sum_{i=1}^{p}N(r,0;f-\nu_i)-k\overline{N}(r,\infty;f)+S(r,f)+S(r,g) \\
\leq\ & (k+2)\overline{N}(r,0;f)+k\overline{N}(r,\infty;g)+(k+2)\overline{N}(r,0;g) \\
+\ & \sum_{i=1}^{p}N(r,0;g-\nu_i)+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}_*(r,\infty;f;g) \\
+\ & N(r,0;f)+\sum_{i=1}^{p}N(r,0;f-\mu_i)+S(r,f)+S(r,g)\,.
\end{aligned}
\tag{3.10}
$$

Subsubcase 1.1.1 If $m \geq 2$, then we deduce with help of inequalities (3.1),(3.3),(3.5) - (3.8) and (3.10) that,

$$
\begin{aligned}
& (n-k+1+p)T(r,f) \\
\leq\ & \frac{k+2}{m}T(r,f)+\frac{k}{m}T(r,g)+\frac{k+2}{m}T(r,f)+pT(r,f)+\frac{2}{m}T(r,f) \\
+\ & \frac{1}{m}T(r,g)+T(r,f)+pT(r,f)+S(r,f)+S(r,g) \\
\leq\ & [p+1+\frac{1}{m}(k+4)]T(r,f)+[p+\frac{1}{m}(2k+3)]T(r,g) \\
& +S(r,f)+S(r,g)\,.
\end{aligned}
\tag{3.11}
$$

Similarly we can show that:

$$
\begin{aligned}
& (n-k+1+p)T(r,g) \\
\leq\ & [p+1+\frac{1}{m}(k+4)]T(r,g)+[p+\frac{1}{m}(2k+3)]T(r,f) \\
+\ & S(r,f)+S(r,g)\,. \qquad (3.12)
\end{aligned}
$$

Adding (3.11) and (3.12) we have: $(n-k+1+p)[T(r,f)+T(r,g)] \leq [2p+1+\frac{1}{m}(3k+7)][T(r,f)+T(r,g)]+S(r,f)+S(r,g)$, which implies that $n \leq p+k+\frac{1}{m}(3k+7)$, but $n > p+k+\frac{1}{m}(3k+7)$, a contradiction.

Subsubcase 1.1.2 If $m=1$, then using inequalities (3.1),(3.4) and (3.10), we have:

$$
\begin{aligned}
& (n-k+1+p)T(r,f) \\
\leq\ & (k+2)T(r,f)+kT(r,g)+(k+2)T(r,f)+pT(r,f)+T(r,f) \\
+\ & T(r,g)+T(r,f)+pT(r,f)+S(r,f)+S(r,g) \\
& \leq (p+k+4)T(r,f)+(p+2k+3)T(r,g)+S(r,f)+S(r,g)\,. \qquad (3.13)
\end{aligned}
$$

Similarly we can show,

$$
\begin{aligned}
(n-k+1+p)T(r,g) \ \leq\ & (p+k+4)T(r,g)+(p+2k+3)T(r,f) \\
+\ & S(r,f)+S(r,g)\,. \qquad (3.14)
\end{aligned}
$$

Adding (3.13) and (3.14) we can deduce that $n \leq p+4k+6$ which is contradiction as $n > p+4k+6$.

Subcase 1.2 If $l=1$, then, we obtain from lemma 2.5

$$
\begin{aligned}
T(r,F) \ \leq\ & N(r,0;F|2)+N(r,0;G|2)+\frac{3}{2}\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G) \\
+\ & \overline{N}_*(r,\infty;F;G)+\frac{1}{2}\overline{N}(r,0;F)+S(r,F)+S(r,G), \qquad (3.15)
\end{aligned}
$$

Now using (3.2) and (3.15) we have:

$$
\begin{aligned}
& T(r,F^*) \\
\leq\ & T(r,F)+N(r,0;f)+\sum_{i=1}^{p}N(r,0;f-\mu_i)-\sum_{i=1}^{p}N(r,0;f-\nu_i) \\
-\ & k\overline{N}(r,\infty;f)+S(r,f)
\end{aligned}
$$

$$\begin{aligned}
&\leq && N(r,0;F|2)+N(r,0;G|2)+\frac{3}{2}\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)\\
&+ && \overline{N}_*(r,\infty;F;G)+\frac{1}{2}\overline{N}(r,0;F)+N(r,0;f)+\sum_{i=1}^{p}N(r,0;f-\mu_i)\\
&- && \sum_{i=1}^{p}N(r,0;f-\nu_i)-k\overline{N}(r,\infty;f)+S(r,f)+S(r,g)\\
&\leq && N(r,0;[f^nQ(f)]^{(k)}|2)+N(r,0;[g^nQ(g)]^{(k)}|2)+\frac{3}{2}\overline{N}(r,\infty;F)\\
&+ && \overline{N}(r,\infty;G)+\overline{N}_*(r,\infty;F;G)+\frac{1}{2}\overline{N}(r,0;[f^nQ(f)]^{(k)})+N(r,0;f)\\
&+ && \sum_{i=1}^{p}N(r,0;f-\mu_i)-\sum_{i=1}^{p}N(r,0;f-\nu_i)-k\overline{N}(r,\infty;f)\\
&+ && S(r,f)+S(r,g)\\
&\leq && (k+2)\overline{N}(r,0;f)+k\overline{N}(r,\infty;g)+(k+2)\overline{N}(r,0;g)\\
&+ && \sum_{i=1}^{p}N(r,0;g-\nu_i)+\frac{3}{2}\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}_*(r,\infty;f;g)\\
&+ && \frac{1}{2}k\overline{N}(r,\infty;f)+\frac{1}{2}(k+1)\overline{N}(r,0;f)+\frac{1}{2}\sum_{i=1}^{p}N(r,0;f-\nu_i)\\
&+ && N(r,0;f)+\sum_{i=1}^{p}N(r,0;f-\mu_i)+S(r,f)+S(r,g),
\end{aligned} \tag{3.16}$$

Subsubcase 1.2.1 If $m\geq 2$, then we deduce with help of inequalities (3.1),(3.3),(3.5) - (3.8) and (3.16) that

$$\begin{aligned}
& && (n-k+1+p)T(r,f)\\
&\leq && [\frac{3p}{2}+1+\frac{1}{2m}(4k+10)]T(r,f)+[p+\frac{1}{m}(2k+3)]T(r,g)\\
&+ && S(r,f)+S(r,g)\,.
\end{aligned} \tag{3.17}$$

Similarly we can show,

$$\begin{aligned}
& && (n-k+1+p)T(r,g)\\
&\leq && [\frac{3p}{2}+1+\frac{1}{2m}(4k+10)]T(r,g)+[p+\frac{1}{m}(2k+3)]T(r,f)\\
&+ && S(r,f)+S(r,g)\,.
\end{aligned} \tag{3.18}$$

Adding (3.17) and (3.18), we have $(n-k+1+p)[T(r,f)+T(r,g)] \leq [\frac{5p}{2}+1+\frac{1}{m}(4k+8)][T(r,f)+T(r,g)]+S(r,f)+S(r,g)$, which implies that $n \leq \frac{3p}{2}+k+\frac{1}{m}(4k+8)$, but $n > \frac{3p}{2}+k+\frac{1}{m}(4k+8)$, a contradiction.

Subsubcase 1.2.2 If $m=1$, then using inequalities (3.1),(3.4) and (3.16), we have

$$\begin{aligned}
& (n-k+1+p)T(r,f) \\
\leq\ & (k+2)T(r,f)+kT(r,g)+(k+2)T(r,g)+pT(r,g)+\frac{1}{2}T(r,f) \\
+\ & T(r,f)+T(r,g)+\frac{1}{2}kT(r,f)+\frac{1}{2}(k+1)T(r,f)+\frac{1}{2}pT(r,f) \\
+\ & T(r,f)+pT(r,f)+S(r,f)+S(r,g) \\
& \leq (\frac{3p}{2}+2k+5)T(r,f)+(p+2k+3)T(r,g)+S(r,f)+S(r,g)\,. \qquad (3.19)
\end{aligned}$$

Similarly we can show,

$$\begin{aligned}
(n-k+1+p)T(r,g) \ \leq\ & (\frac{3p}{2}+2k+5)T(r,g)+(p+2k+3)T(r,f) \\
+\ & S(r,f)+S(r,g)\,. \qquad (3.20)
\end{aligned}$$

Adding (3.19) and (3.20) we can deduce that $n \leq \frac{3p}{2}+5k+7$ which is contradiction as $n > \frac{3p}{2}+5k+7$.

Subcase 1.3 If $l=0$, then using lemma 2.6, we obtain

$$\begin{aligned}
& T(r,F) \\
\leq\ & N(r,0;F|2)+N(r,0;G|2)+3\overline{N}(r,\infty;F)+2\overline{N}(r,\infty;G) \\
& +\overline{N}_*(r,\infty;F;G)+2\overline{N}(r,0;F)+\overline{N}(r,0;G)+S(r,F)+S(r,G)\,. \qquad (3.21)
\end{aligned}$$

Now using (3.2) and (3.21)

$$\begin{aligned}
& T(r,F^*) \\
\leq\ & T(r,F)+N(r,0;f)+\sum_{i=1}^{p}N(r,0;f-\mu_i)-\sum_{i=1}^{p}N(r,0;f-\nu_i) \\
-\ & k\overline{N}(r,\infty;f)+S(r,f)
\end{aligned}$$

$$
\begin{aligned}
&\leq && N(r,0;F|2)+N(r,0;G|2)+3\overline{N}(r,\infty;F)+2\overline{N}(r,\infty;G)\\
&+ && \overline{N}_*(r,\infty;F;G)+2\overline{N}(r,0;F)+\overline{N}(r,0;G)+N(r,0;f)\\
&+ && \sum_{i=1}^{p}N(r,0;f-\mu_i)-\sum_{i=1}^{p}N(r,0;f-\nu_i)-k\overline{N}(r,\infty;f)+S(r,f)+S(r,g)\\
&\leq && N(r,0;[f^nQ(f)]^{(k)}|2)+N(r,0;[g^nQ(g)]^{(k)}|2)+3\overline{N}(r,\infty;F)\\
&+ && 2\overline{N}(r,\infty;G)+\overline{N}_*(r,\infty;F;G)+2\overline{N}(r,0;[f^nQ(f)]^{(k)})\\
&+ && \overline{N}(r,0;[g^nQ(g)]^{(k)})+N(r,0;f)+\sum_{i=1}^{p}N(r,0;f-\mu_i)\\
&- && \sum_{i=1}^{p}N(r,0;f-\nu_i)-k\overline{N}(r,\infty;f)+S(r,f)+S(r,g)\\
&\leq && (k+2)\overline{N}(r,0;f)+k\overline{N}(r,\infty;g)+(k+2)N(r,0;g)+\sum_{i=1}^{p}N(r,0;g-\nu_i)\\
&+ && 3\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+\overline{N}_*(r,\infty;f;g)+2k\overline{N}(r,\infty;f)\\
&+ && 2(k+1)N(r,0;f)+2\sum_{i=1}^{p}N(r,0;f-\nu_i)+k\overline{N}(r,\infty;g)\\
&+ && (k+1)N(r,0;g)+\sum_{i=1}^{p}N(r,0;g-\nu_i)+N(r,0;f)\\
&&& +\sum_{i=1}^{p}N(r,0;f-\mu_i)+S(r,f)+S(r,g)\,. \qquad (3.22)
\end{aligned}
$$

Subsubcase 1.3.1 If $m\geq 2$,then we deduce with help of inequalities (3.1),(3.3),(3.5) - (3.8) and (3.22) that

$$
\begin{aligned}
& && (n-k+1+p)T(r,f)\\
&\leq && [3p+1+\frac{1}{m}(5k+8)]T(r,f)+[2p+\frac{1}{m}(4k+5)]T(r,g)\\
&+ && S(r,f)+S(r,g)\,. \qquad (3.23)
\end{aligned}
$$

Similarly we can show,

$$
\begin{aligned}
& && (n-k+1+p)T(r,g)\\
&\leq && [3p+1+\frac{1}{m}(5k+8)]T(r,g)+[2p+\frac{1}{m}(4k+5)]T(r,f)\\
&+ && S(r,f)+S(r,g)\,. \qquad (3.24)
\end{aligned}
$$

Adding (3.23) and (3.24) we have, $(n-k+1+p)[T(r,f)+T(r,g)] \leq [5p+1+\frac{1}{m}(9k+13)][T(r,f)+T(r,g)]+S(r,f)+S(r,g)$. Which implies that $n \leq 4p+k+\frac{1}{m}(9k+13)$, but $n > 4p+k+\frac{1}{m}(9k+13)$, a contradiction.
Subsubcase 1.3.2 If $m=1$,then using inequalities (3.1),(3.4) and (3.22), we have

$$\begin{aligned}
& (n-k+1+p)T(r,f) \\
\leq\ & (k+2)T(r,f)+kT(r,g)+(k+2)T(r,g)+pT(r,g) \\
+\ & 2T(r,f)+T(r,g) \\
+\ & T(r,f)+T(r,g)+2kT(r,f)+2(k+1)T(r,f)+2pT(r,f)+kT(r,f) \\
+\ & (k+1)T(r,g)+pT(r,g)+T(r,f)+pT(r,f)+S(r,f)+S(r,g)
\end{aligned}$$

$$\leq (3p+5k+8)T(r,f)+(2p+4k+5)T(r,g)+S(r,f)+S(r,g)\,. \tag{3.25}$$

Similarly we can show,

$$\begin{aligned}
(n-k+1+p)T(r,g) \ \leq\ & (3p+5k+8)T(r,g)+(2p+4k+5)T(r,f) \\
+\ & S(r,f)+S(r,g)\,.
\end{aligned} \tag{3.26}$$

Adding (3.25) and (3.26) we can deduce that $n \leq 4p+10k+12$ which is contradiction as $n > 4p+10k+12$.
Case 2 We assume that $H_1 \equiv 0$. Then we can write for our functions F and G, $(\frac{F''}{F'}-\frac{2F'}{F-1})-(\frac{G''}{G'}-\frac{2G'}{G-1})=0$, Now after two times integration of the equation, we have

$$\frac{1}{F-1}=\frac{C}{G-1}+D, \tag{3.27}$$

where C and D are complex constants. Now we can say from (3.27) that F and G share 1 CM, that is F and G share 1 with weight $l(\geq 2)$. Now we study the following subcases.
Subcase 2.1 Let $D \neq 0$ and $C=D$. Then from (3.27) we have

$$\frac{1}{F-1}=\frac{DG}{G-1}\,. \tag{3.28}$$

If $D=-1$, then from (3.28), we obtain $FG=1$. Then by lemma 2.9, we get a contradiction. If $D \neq -1$, we have, $\frac{1}{G}=\frac{1}{D(F-\frac{D-1}{D})}$ and then $\overline{N}(r,\frac{D-1}{D};F)=$

$\overline{N}(r,0;G)$. Now using the second fundamental theorem of Nevanlinna

$$\begin{aligned} & T(r,F) \\ \leq\ & \overline{N}(r,0;F)+\overline{N}(r,\frac{D-1}{D};F)+\overline{N}(r,\infty;F)+S(r,F) \\ \leq\ & \overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;F)+S(r,F)+S(r,G), \end{aligned} \tag{3.29}$$

and using (3.3) and (3.29) we have

$$\begin{aligned} & T(r,F^*) \\ \leq\ & T(r,F)+N(r,0;f)+\sum_{i=1}^{p}N(r,0;f-\mu_i)-\sum_{i=1}^{p}N(r,0;f-\nu_i) \\ -\ & k\overline{N}(r,\infty;f)+S(r,f) \\ \leq\ & \overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;F)+N(r,0;f)+\sum_{i=1}^{p}N(r,0;f-\mu_i) \\ -\ & \sum_{i=1}^{p}N(r,0;f-\nu_i)-k\overline{N}(r,\infty;f)+S(r,f)+S(r,g) \\ \leq\ & (k+1)\overline{N}(r,0;f)+k\overline{N}(r,\infty;g)+(k+1)\overline{N}(r,0;g)+\sum_{i=1}^{p}N(r,0;g-\nu_i) \\ & +\overline{N}(r,\infty;f)+N(r,0;f)+\sum_{i=1}^{p}N(r,0;f-\mu_i)+S(r,f)+S(r,g)\,. \end{aligned} \tag{3.30}$$

Subsubcase 2.1.1 If $m\geq 2$, then we deduce with help of inequalities (3.1),(3.3),(3.5) - (3.8) and (3.30) that

$$\begin{aligned} (n-k+1+p)T(r,f) &\leq [p+1+\frac{1}{m}(k+2)]T(r,f) \\ &+[p+\frac{1}{m}(2k+1)]T(r,g)+S(r,f)+S(r,g)\,. \end{aligned} \tag{3.31}$$

Similarly we can show,

$$\begin{aligned} (n-k+1+p)T(r,g) &\leq [p+1+\frac{1}{m}(k+2)]T(r,g) \\ &+[p+\frac{1}{m}(2k+1)]T(r,f)+S(r,f)+S(r,g)\,. \end{aligned} \tag{3.32}$$

Adding (3.31) and (3.32) we have $(n-k+1+p)[T(r,f)+T(r,g)]\leq[2p+1+\frac{1}{m}(3k+3)][T(r,f)+T(r,g)]+S(r,f)+S(r,g)$, Which implies that $n\leq p+k+\frac{1}{m}(3k+3)$, but $n>p+k+\frac{1}{m}(3k+7)$, a contradiction.

Subsubcase 2.1.2 If $m = 1$, then using inequalities (3.1),(3.4) and (3.32), we have:

$$\begin{aligned} (n-k+1+p)T(r,f) &\leq (p+k+3)T(r,f) + (p+2k+1)T(r,g) \\ &+ S(r,f) + S(r,g)\,. \end{aligned} \tag{3.33}$$

Similarly we can show,

$$\begin{aligned} (n-k+1+p)T(r,g) &\leq (p+k+3)T(r,g) + (p+2k+1)T(r,f) \\ &+ S(r,f) + S(r,g)\,. \end{aligned} \tag{3.34}$$

Adding (3.33) and (3.34) we can deduce that $n \leq p+4k+3$ which is contradiction as $n > p+4k+6$.

Subcase 2.2 Let $D \neq 0$ and $C \neq D$, then from (3.27), $G = \frac{\frac{C+D+1}{D} - F}{\frac{1}{D}+1-F}$. So, $\overline{N}(r, \frac{C+D+1}{D}; F) = \overline{N}(r, 0; G)$ and proceeding similarly as case 2.1, we obtain a contradiction.

Subcase 2.3 Let $D = 0$ and $C \neq 0$. Then $F = \frac{G+C-1}{C}$ and $G = CF - (C-1)$. If $C \neq 1$, then we have $\overline{N}(r, \frac{C-1}{C}; F) = \overline{N}(r, 0; G)$ and $\overline{N}(r, 1-C; F) = \overline{N}(r, 0; F)$ and proceeding similarly as case 2.1 we attain a contradiction. Thus $C = 1$, which implies $F = G$ i.e

$$[f^n Q(f)]^{(k)} = [g^n Q(g)]^{(k)}, \tag{3.35}$$

Now integrating equation (3.35) we have:

$$[f^n Q(f)]^{(k-1)} = [g^n Q(g)]^{(k-1)} + b_{k-1}, \tag{3.36}$$

where b_{k-1} is constant. If $b_{k-1} \neq 0$, then by lemma 2.8, we have $n < p + \frac{1}{m}(3k+3)$ which is contradiction for both the cases $m \geq 2$ or $m = 1$ as $n > p + k + \frac{1}{m}(3k+7)$ or $n > p+k+6$ respectively. Now repeating the process up to k-times we have

$$[f^n Q(f)] = [g^n Q(g)]\,. \tag{3.37}$$

Now if $p = 1$, then from equation (3.37) and lemma 2.7 we have $f = g$. Suppose $p \geq 2$ and let $h = \frac{f}{g}$. If h is constant then putting $f = hg$ in equation (3.37) we get

$$\sum_{i=0}^{p} a_{p-i} g^{n+p-i}(h^{n+p-i} - 1) = 0\,, \tag{3.38}$$

which implies $h^\chi = 1$, where $\chi = GCD(n+p, n+p-1, ..., n+p-i, ..., n+1, n)$, $i = 0, 1, ..., p$. If h is not constant, then we can show that f and g satisfy the algebraic equation $R(f, g) = 0$ and from (3.38), we have, $R(f, g) = f^n Q(f) - g^n Q(g)$.

This complete the proof of theorem. □

Remark 3.1. *It is observed at the theorem 3.1 that the value of n is continuously decreasing for the increasing value of m when k is fixed, in any case according to $l(\geq 0)$.*

Theorem 3.2. *Let f and g be two non-constant meromorphic functions whose zeros and poles are multiplicities at least m, where m, n, p, k are positive integers. If $E_l(1, f^n Q(f) f') = E_l(1, g^n Q(g) g')$ and f and g share ∞ IM. Then for one of the following conditions:*

i) *$l \geq 2$; a) $m \geq 2$ and $n > p + 1 + \frac{8}{m}$, b) $m = 1$ and $n > p + 8$*
ii) *$l = 1$; a) $m \geq 2$ and $n > \frac{3p}{2} + k + \frac{1}{m}(4k+8)$, b) $m = 1$ and $n > \frac{3p}{2} + 5k + 7$*
iii) *$l = 0$; a) $m \geq 2$ and $n > 4p + k + \frac{1}{m}(9k + 13)$, b) $m = 1$ and $n > 4p + 10k + 12$*

one of the following results hold

i) *$f = tg$ for a constant t such that $t^\chi = 1$, where $\chi = GCD(n + p, ..., n + p - i, ..., n + 1, n)$ and $a_{p-i} \neq 0$ for some $i = 0, 1, ..., p$.*
ii) *f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\phi, \psi) = \phi^{n+1} \sum_{i=0}^{p} \frac{a_{p-i}}{n+p+1-i} \phi^{m-i} - \psi^{n+1} \sum_{i=0}^{p} \frac{a_{p-i}}{n+p+1-i} \psi^{m-i}$.*

Proof. Let $X = f^n Q(f) f'$ and $Y = g^n Q(g) g'$. We define two functions: $X^* = \sum_{i=0}^{p} \frac{a_{p-i}}{n+p+1-i} f^{n+p+1-i}$ and $Y^* = \sum_{i=0}^{p} \frac{a_{p-i}}{n+p+1-i} g^{n+p+1-i}$. From lemma 2.1, we have

$$T(r, X^*) = (n + p + 1) T(r, f) + S(r, f) . \tag{3.39}$$

Since $(X^*)' = X$, we deduce $m(r, \frac{1}{X^*}) = m(r, \frac{1}{X}) + S(r, f)$. By Nevanlinna's first fundamental theorem, we get

$$\begin{aligned} T(r, X^*) &\leq \overline{N}(r, \infty; X) + N(r, 0; X^*) - N(r, 0; X) + S(r, f) \\ &\leq T(r, X) + N(r, 0; f) + \sum_{i=1}^{p} N(r, 0; f - \mu_i) - \sum_{i=1}^{p} N(r, 0; f - \nu_i) \\ &- N(r, 0; f') + S(r, f) , \end{aligned} \tag{3.40}$$

where μ_i and $\nu_i (i = 1, 2, ..., p)$ are roots of algebraic equations $\sum_{i=1}^{p} \frac{a_{p-i}}{(n+p+1-i)} z^{p-i} = 0$ and $\sum_{i=1}^{p} a_{p-i} z^{p-i} = 0$ respectively.
Also we use the result for $m(\geq 2)$, $m(=1)$

$$\overline{N}_*(r, \infty; f; g) \leq \overline{N}(r, \infty; f), \tag{3.41}$$

$$\overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + \overline{N}_*(r, \infty; f; g) \leq N(r, \infty; f) + N(r, \infty; g), \tag{3.42}$$

respectively. As we assume that zeros and poles of f and g are of multiplicities at least $m(\geq 2)$, then

$$\overline{N}(r, \infty; f) \leq \frac{1}{m} N(r, \infty; f) \leq \frac{1}{m} T(r, f), \tag{3.43}$$

$$\overline{N}(r, 0; f) \leq \frac{1}{m} N(r, 0; f) \leq \frac{1}{m} T(r, f), \tag{3.44}$$

$$\overline{N}(r, \infty; f) \leq \frac{1}{m} N(r, \infty; f) \leq \frac{1}{m} T(r, f), \tag{3.45}$$

$$\overline{N}(r, 0; f) \leq \frac{1}{m} N(r, 0; f) \leq \frac{1}{m} T(r, f) \,. \tag{3.46}$$

Now X and Y are transcendental meromorphic functions that share (1,l) and f, g share $(\infty, 0)$. We discuss the following two cases separately.
Case 1 We assume that $H_1 \not\equiv 0$. Now we study the following subcases,
Subcase 1.1 If $l \geq 2$, then using lemma 2.4, we obtain

$$\begin{aligned} T(r, X) &\leq N(r, 0; X|2) + N(r, 0; Y|2) + \overline{N}(r, \infty; X) + \overline{N}(r, \infty; Y) \\ &+ \overline{N}_*(r, \infty; X; Y) + S(r, X) + S(r, Y) \,. \end{aligned} \tag{3.47}$$

Now from inequalities (3.40) and (3.47)and lemma 2.2 we have:

$$\begin{aligned} & T(r, X^*) \\ \leq\ & N(r, 0; X|2) + N(r, 0; Y|2) + \overline{N}(r, \infty; X) + \overline{N}(r, \infty; Y) \\ +\ & \overline{N}_*(r, \infty; X; Y) + N(r, 0; f) + \sum_{i=1}^{p} N(r, 0; f - \mu_i) - \sum_{i=1}^{p} N(r, 0; f - \nu_i) \\ -\ & N(r, 0; f') + S(r, f) + S(r, g) \end{aligned}$$

$$\begin{aligned}
&\leq N(r,0;[f^nQ(f)f']|2)+N(r,0;[g^nQ(g)g']|2)+\overline{N}(r,\infty;f)\\
&+ \overline{N}(r,\infty;g)+\overline{N}_*(r,\infty;f;g)+N(r,0;f)+\sum_{i=1}^{p}N(r,0;f-\mu_i)\\
&- \sum_{i=1}^{p}N(r,0;f-\nu_i)-N(r,0;f')+S(r,f)+S(r,g)\\
&\leq 2\overline{N}(r,0;f)+2\overline{N}(r,0;g)+\sum_{i=1}^{p}N(r,0;g-\nu_i)+N(r,0;g)\\
&+ \overline{N}(r,\infty;g)+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}_*(r,\infty;f;g)+N(r,0;f)\\
&\quad +\sum_{i=1}^{p}N(r,0;f-\mu_i)+S(r,f)+S(r,g)\,. \qquad (3.48)
\end{aligned}$$

Subsubcase 1.1.1 If $m\geq 2$, then we deduce with help of inequalities (3.39),(3.41),(3.43) - (3.46) and (3.48) that

$$\begin{aligned}
&(n+p+1)T(r,f)\\
\leq\ &(\frac{2}{m}+\frac{2}{m}+p+1)T(r,f)+(\frac{2}{m}+\frac{2}{m}+p+1)T(r,g)+S(r,f)+S(r,g)\\
&\leq [p+1+\frac{4}{m}]T(r,f)+[p+1+\frac{4}{m}]T(r,g)+S(r,f)+S(r,g)\,. \qquad (3.49)
\end{aligned}$$

Similarly we can show that

$$\begin{aligned}
(n+p+1)T(r,f) &\leq [p+1+\frac{4}{m}]T(r,f)+[p+1+\frac{4}{m}]T(r,g)\\
&+ S(r,f)+S(r,g)\,. \qquad (3.50)
\end{aligned}$$

Adding (3.49) and (3.50) we have $(n+p+1)[T(r,f)+T(r,g)]\leq 2[p+1+\frac{4}{m}][T(r,f)+T(r,g)]+S(r,f)+S(r,g)$. Which implies that $n\leq p+\frac{8}{m}+1$, but $n>p+\frac{8}{m}+1$, a contradiction.

Subsubcase 1.1.2 If $m=1$, then using inequalities (3.39),(3.42) and (3.48), we have

$$\begin{aligned}
(n+p+1)T(r,f) &\leq 2T(r,f)+2T(r,g)+pT(r,g)+T(r,g)+T(r,g)\\
&+ T(r,f)+T(r,g)+pT(r,f)+S(r,f)+S(r,g)\\
&\leq (p+4)T(r,f)+(p+5)T(r,g)+S(r,f)+S(r,g)\,. \qquad (3.51)
\end{aligned}$$

Similarly we can show,

$$(n+p+1)T(r,g)\leq (p+4)T(r,g)+(p+5)T(r,f)+S(r,f)+S(r,g)\,. \qquad (3.52)$$

Adding (3.51) and (3.52) we can deduce that $n \leq p+8$ which is contradiction as $n > p+8$.
Subcase 1.2 If $l=1$, then we obtain from lemma 2.5

$$\begin{aligned} T(r,X) &\leq N(r,0;X|2)+N(r,0;Y|2)+\frac{3}{2}\overline{N}(r,\infty;X)+\overline{N}(r,\infty;Y) \\ &+ \overline{N}_*(r,\infty;X;Y)+\frac{1}{2}\overline{N}(r,0;X)+S(r,X)+S(r,Y)\,. \end{aligned} \tag{3.53}$$

Now using (3.40) and (3.53) and lemma 2.2

$$\begin{aligned} & T(r,X^*) \\ \leq\ & T(r,X)+N(r,0;f)+\sum_{i=1}^{p}N(r,0;f-\mu_i)-\sum_{i=1}^{p}N(r,0;f-\nu_i) \\ -\ & N(r,0;f')+S(r,f) \\ \leq\ & N(r,0;X|2)+N(r,0;Y|2)+\frac{3}{2}\overline{N}(r,\infty;X)+\overline{N}(r,\infty;Y) \\ +\ & \overline{N}_*(r,\infty;X;Y)+\frac{1}{2}\overline{N}(r,0;X)+N(r,0;f)+\sum_{i=1}^{p}N(r,0;f-\mu_i) \\ -\ & \sum_{i=1}^{p}N(r,0;f-\nu_i)-N(r,0;f')+S(r,f)+S(r,g) \\ \leq\ & N(r,0;[f^nQ(f)f']|2)+N(r,0;[g^nQ(g)g']|2)+\frac{3}{2}\overline{N}(r,\infty;X) \\ +\ & \overline{N}(r,\infty;Y)+\overline{N}_*(r,\infty;X;Y)+\frac{1}{2}\overline{N}(r,0;[f^nQ(f)f']) \\ +\ & N(r,0;f)+\sum_{i=1}^{p}N(r,0;f-\mu_i)-\sum_{i=1}^{p}N(r,0;f-\nu_i) \\ -\ & N(r,0;f')+S(r,f)+S(r,g) \\ \leq\ & 2\overline{N}(r,0;f)+2\overline{N}(r,0;g)+\sum_{i=1}^{p}N(r,0;g-\nu_i)+N(r,0;g)+\overline{N}(r,\infty;g) \\ +\ & \frac{3}{2}\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}_*(r,\infty;f;g)+\frac{1}{2}\overline{N}(r,0;f) \\ +\ & \frac{1}{2}\sum_{i=1}^{p}N(r,0;f-\nu_i)+\frac{1}{2}N(r,0;f)+\frac{1}{2}\overline{N}(r,\infty;f)+N(r,0;f) \end{aligned}$$

$$+\sum_{i=1}^{p} N(r,0;f-\mu_i)+S(r,f)+S(r,g)\,. \tag{3.54}$$

Subsubcase 1.2.1 If $m\geq 2$, then we deduce with help of inequalities (3.39),(3.41),(3.43) - (3.46) and (3.54) that

$$\begin{aligned}(n+p+1)T(r,f) &\leq [\frac{3p}{2}+1+\frac{11}{2m}+\frac{3}{2}]T(r,f)\\ &+[p+\frac{4}{m}+1]T(r,g)+S(r,f)+S(r,g)\,. \end{aligned}\tag{3.55}$$

Similarly we can show,

$$\begin{aligned}(n+p+1)T(r,g) &\leq [\frac{3p}{2}+1+\frac{11}{2m}+\frac{3}{2}]T(r,g)\\ &+[p+\frac{4}{m}+1]T(r,f)+S(r,f)+S(r,g)\,. \end{aligned}\tag{3.56}$$

Adding (3.55) and (3.56) we have $(n+p+1)[T(r,f)+T(r,g)]\leq[\frac{5p}{2}+\frac{19}{2m}+\frac{5}{2}][T(r,f)+T(r,g)]+S(r,f)+S(r,g)$. Which implies that $n\leq\frac{3p}{2}+\frac{19}{2m}+\frac{3}{2}$, but $n>\frac{3p}{2}+\frac{19}{2m}+\frac{3}{2}$, a contradiction.

Subsubcase 1.2.2 If $m=1$, then using inequalities (3.39),(3.42) and (3.54), we have

$$(n+p+1)T(r,f)\leq(\frac{3p}{2}+6)T(r,f)+(p+3)T(r,g)+S(r,f)+S(r,g)\,. \tag{3.57}$$

Similarly we can show,

$$(n+p+1)T(r,g)\leq(\frac{3p}{2}+6)T(r,g)+(p+3)T(r,f)+S(r,f)+S(r,g)\,. \tag{3.58}$$

Adding (3.57) and (3.58) we can deduce that $n\leq\frac{3p}{2}+8$ which is contradiction as $n>\frac{3p}{2}+8$.

Subcase 1.3 If $l=0$, then using lemma 2.6, we obtain

$$\begin{aligned}T(r,X) &\leq N(r,0;X|2)+N(r,0;Y|2)+3\overline{N}(r,\infty;X)+2\overline{N}(r,\infty;Y)\\ &+ \overline{N}_*(r,\infty;X;Y)+2\overline{N}(r,0;X)+\overline{N}(r,0;Y)+S(r,X)+S(r,Y)\,. \end{aligned}\tag{3.59}$$

Now using (3.40),(3.59) and lemma 2.2

$$\begin{aligned}&T(r,X^*)\\ \leq\;& T(r,X)+N(r,0;f)+\sum_{i=1}^{p}N(r,0;f-\mu_i)-\sum_{i=1}^{p}N(r,0;f-\nu_i)\\ -\;& N(r,0;f')+S(r,f)\end{aligned}$$

$$
\begin{aligned}
&\leq N(r,0;X|2)+N(r,0;Y|2)+3\overline{N}(r,\infty;X)+2\overline{N}(r,\infty;Y)\\
&+\overline{N}_*(r,\infty;X;Y)+2\overline{N}(r,0;X)+\overline{N}(r,0;Y)+N(r,0;f)\\
&+\sum_{i=1}^{p}N(r,0;f-\mu_i)-\sum_{i=1}^{p}N(r,0;f-\nu_i)-N(r,0;f')\\
&+S(r,f)+S(r,g)\\
&\leq N(r,0;[f^nQ(f)f']|2)+N(r,0;[g^nQ(g)g']|2)+3\overline{N}(r,\infty;X)\\
&+2\overline{N}(r,\infty;Y)+\overline{N}_*(r,\infty;X;Y)+2\overline{N}(r,0;[f^nQ(f)f'])\\
&+\overline{N}(r,0;[g^nQ(g)g'])+N(r,0;f)+\sum_{i=1}^{p}N(r,0;f-\mu_i)\\
&-\sum_{i=1}^{p}N(r,0;f-\nu_i)-N(r,0;f')+S(r,f)+S(r,g)\\
&\leq 2\overline{N}(r,0;f)+2\overline{N}(r,0;g)+\sum_{i=1}^{p}N(r,0;g-\nu_i)+N(r,0;g)\\
&+\overline{N}(r,\infty;g)+3\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+\overline{N}_*(r,\infty;f;g)\\
&+2\overline{N}(r,0;f)+2\sum_{i=1}^{p}N(r,0;f-\nu_i)+2N(r,0;f)+2\overline{N}(r,\infty;f)\\
&+\overline{N}(r,0;g)+\sum_{i=1}^{p}N(r,0;f-\nu_i)+N(r,0;g)
\end{aligned}
$$

$$
+\overline{N}(r,\infty;g)+N(r,0;f)+\sum_{i=1}^{p}N(r,0;f-\mu_i)+S(r,f)+S(r,g)\,. \tag{3.60}
$$

Subsubcase 1.3.1 If $m\geq 2$, then we deduce with help of inequalities (3.39),(3.41),(3.43) - (3.46) and (3.60) that

$$
\begin{aligned}
&(n+p+1)T(r,f)\\
&\leq[3p+\frac{10}{m}+3]T(r,f)+[2p+\frac{7}{m}+2]T(r,g)++S(r,f)+S(r,g)\,.
\end{aligned} \tag{3.61}
$$

Similarly we can show,

$$
\begin{aligned}
&(n+p+1)T(r,g)\\
&\leq[3p+\frac{10}{m}+3]T(r,g)+[2p+\frac{7}{m}+2]T(r,f)++S(r,f)+S(r,g)\,.
\end{aligned} \tag{3.62}
$$

Adding (3.61) and (3.62) we have $(n+p+1)[T(r,f)+T(r,g)] \leq [5p+\frac{17}{m}+5][T(r,f)+T(r,g)]+S(r,f)+S(r,g)$. Which implies that $n \leq 4p+\frac{17}{m}+4$, but $n > 4p+\frac{17}{m}+4$, a contradiction.

Subsubcase 1.3.2 If $m=1$, then using inequalities (3.39),(3.42) and (3.60), we have

$$\begin{aligned}
& (n+p+1)T(r,f) \\
\leq\ & 2T(r,f)+2T(r,g)+pT(r,g)+T(r,g)+T(r,g)+2T(r,f)+T(r,g) \\
+\ & T(r,f)+T(r,g)+2T(r,f)+2pT(r,f)+2T(r,f)+2T(r,f) \\
+\ & T(r,g)+pT(r,g)+T(r,g)+T(r,g)+T(r,f)+pT(r,f) \\
+\ & S(r,f)+S(r,g) \\
\leq\ & (3p+12)T(r,f)+(2p+9)T(r,g)+S(r,f)+S(r,g)\,. \qquad (3.63)
\end{aligned}$$

Similarly we can show,

$$(n+p+1)T(r,g) \leq (3p+12)T(r,g)+(2p+9)T(r,f)+S(r,f)+S(r,g)\,. \qquad (3.64)$$

Adding (3.63) and (3.64) we can deduce that $n \leq 4p+20$ which is contradiction as $n > 4p+20$.

Case 2 We assume that $H_1 \equiv 0$.Then we can write for our functions X and Y, $(\frac{X''}{X'}-\frac{2X'}{X-1})-(\frac{Y''}{Y'}-\frac{2Y'}{Y-1})=0$, Now after two times integration of the equation, we have

$$\frac{1}{X-1}=\frac{C}{Y-1}+D, \qquad (3.65)$$

Where C and D are complex constants. Now we can say from (3.65) that X and Y share 1 CM, that is X and Y share 1 with weight $l(\geq 2)$. Now we study the following subcases.

Subcase 2.1 Let $D \neq 0$ and $C=D$. Then from (3.65) we have

$$\frac{1}{X-1}=\frac{DY}{Y-1}\,. \qquad (3.66)$$

If $D=-1$, then from (3.66), we obtain $XY=1$. Then by lemma 2.10, we get a contradiction. If $D \neq -1$, we have $\frac{1}{Y}=\frac{1}{D(X-\frac{D-1}{D})}$ and then $\overline{N}(r,\frac{D-1}{D};X)=\overline{N}(r,0;Y)$. Using the second fundamental theorem of Nevanlinna

$$\begin{aligned}
T(r,X) &\leq \overline{N}(r,0;X)+\overline{N}(r,\frac{D-1}{D};X)+\overline{N}(r,\infty;X)+S(r,X) \\
&\leq \overline{N}(r,0;X)+\overline{N}(r,0;Y)+\overline{N}(r,\infty;X)+S(r,X)+S(r,Y)\,. \qquad (3.67)
\end{aligned}$$

Using (3.40) and (3.67) we have

$$\begin{aligned}
& T(r,X^*) \\
\leq\ & T(r,X)+N(r,0;f)+\sum_{i=1}^{p}N(r,0;f-\mu_i)-\sum_{i=1}^{p}N(r,0;f-\nu_i) \\
-\ & N(r,0;f')+S(r,f) \\
\leq\ & \overline{N}(r,0;X)+\overline{N}(r,0;Y)+\overline{N}(r,\infty;X)+N(r,0;f)+\sum_{i=1}^{p}N(r,0;f-\mu_i) \\
-\ & \sum_{i=1}^{p}N(r,0;f-\nu_i)-N(r,0;f')+S(r,f)+S(r,g) \\
\leq\ & \overline{N}(r,0;f)+\overline{N}(r,0;g)+\sum_{i=1}^{p}N(r,0;g-\nu_i)+N(r,0;g')+N(r,0;f) \\
& +\sum_{i=1}^{p}N(r,0;f-\mu_i)+S(r,f)+S(r,g)\,. \qquad (3.68)
\end{aligned}$$

Subsubcase 2.1.1 If $m \geq 2$, then we deduce with help of inequalities (3.39),(3.41),(3.43) - (3.46) and (3.68) that

$$\begin{aligned}
& (n+p+1)T(r,f) \\
\leq\ & [p+\frac{1}{m}+1]T(r,f)+[p+\frac{2}{m}+1]T(r,g)+S(r,f)+S(r,g)\,. \qquad (3.69)
\end{aligned}$$

Similarly we can show,

$$\begin{aligned}
& (n+p+1)T(r,g) \\
\leq\ & [p+\frac{1}{m}+1]T(r,g)+[p+\frac{2}{m}+1]T(r,f)+S(r,f)+S(r,g)\,. \qquad (3.70)
\end{aligned}$$

Adding (3.69) and (3.70) we have $(n+p+1)[T(r,f)+T(r,g)] \leq [2p+\frac{3}{m}+2][T(r,f)+T(r,g)]+S(r,f)+S(r,g)$. Which implies that $n \leq p+\frac{3}{m}+1$, but $n > p+\frac{8}{m}+1$, a contradiction.

Subsubcase 1.3.2 If $m=1$, then using inequalities (3.39),(3.42) and (3.68), we have

$$\begin{aligned}
& (n+p+1)T(r,f) \\
\leq\ & (p+2)T(r,f)+(p+3)T(r,g)+S(r,f)+S(r,g)\,. \qquad (3.71)
\end{aligned}$$

Similarly we can show,

$$\begin{aligned} & (n+p+1)T(r,g) \\ \leq\ & (p+2)T(r,g)+(p+3)T(r,f)+S(r,f)+S(r,g)\,. \end{aligned} \tag{3.72}$$

Adding (3.71) and (3.72) we can deduce that $n \leq p+4$ which is contradiction as $n > p+8$.

Subcase 2.2 Let $D \neq 0$ and $C \neq D$, then from (3.65), $Y = \frac{\frac{C+D+1}{D}-X}{\frac{1}{D}+1-X}$. So, $\overline{N}(r, \frac{C+D+1}{D}; X) = \overline{N}(r,0;Y)$ and proceeding similarly as case 2.1, we attain a contradiction.

Subcase 2.3 Let $D = 0$ and $C \neq 0$. Then $X = \frac{Y+C-1}{C}$ and $Y = CX-(C-1)$. If $C \neq 1$, then we have $\overline{N}(r, \frac{C-1}{C}; X) = \overline{N}(r,0;Y)$ and $\overline{N}(r,1-C;Y) = \overline{N}(r,0;X)$ and proceeding similarly as subcase 2.1, we attain a contradiction. Thus $C = 1$, which implies $X = Y$ i.e $[f^nQ(f)f'] = [g^nQ(g)g']$. Now we can write $X^* = Y^* + c$, where c is constant, then it follows

$$T(r,f) = T(r,g)+S(r,g)\,. \tag{3.73}$$

Suppose that $c \neq 0$. By the second fundamental theorem and lemma 2.10 we have

$$\begin{aligned} T(r,Y^*) &= \overline{N}(r,0;Y^*)+\overline{N}(r,0;Y^*+c)+\overline{N}(r,\infty;Y^*)+S(r,g) \\ &\leq \overline{N}(r,0;Y^*)+\overline{N}(r,0;X^*)+\overline{N}(r,\infty;Y^*)+S(r,g) \\ &\leq \overline{N}(r,0;g)+\overline{N}(r,0;\sum_{i=0}^{p}\frac{a_{p-i}(n+p+1)}{a_p(n+p+1-i)}g^{p-i})+\overline{N}(r,0;f) \\ &+ \overline{N}(r,0;\sum_{i=0}^{p}\frac{a_{p-i}(n+p+1)}{a_p(n+p+1-i)}f^{p-i})+\overline{N}(r,\infty;g) \\ &+ S(r,f)+S(r,g)\,. \end{aligned} \tag{3.74}$$

Subsubcase 2.3.1 If $m \geq 2$, then using (3.39), (3.43)-(3.46) and (3.74) we have

$$(n+p+1)T(r,g) \leq (p+\frac{2}{m})T(r,g)+(p+\frac{1}{m})T(r,f)+S(r,f)+S(r,g)\,. \tag{3.75}$$

Similarly we can show

$$(n+p+1)T(r,f) \leq (p+\frac{2}{m})T(r,f)+(p+\frac{1}{m})T(r,g)+S(r,f)+S(r,g)\,. \tag{3.76}$$

Adding (3.75) and (3.76) we have $(n+p+1)(T(r,f)+T(r,g)) \leq (2p+\frac{3}{m})(T(r,f)+T(r,g))+S(r,f)+S(r,g)$. Which implies that $n \leq p+\frac{3}{m}-1$ which is contradiction as $n > p+\frac{8}{m}+1$.
Subsubcase 2.3.2 If $m=1$ then using (3.39) and (3.74) we can show $n \leq p+2$ which is contradiction as $n > p+8$. That is for all m , we arrive at a contradiction. Now we claim that $c=0$. Therefore $X^* = Y^*$, that is

$$f^{n+1}\sum_{i=0}^{p}\frac{a_p}{n+p+1-i}f^{p-i} = g^{n+1}\sum_{i=0}^{p}\frac{a_p}{n+p+1-i}g^{p-i}\,. \tag{3.77}$$

Let $h=\frac{f}{g}$. If h is constant, then, substituting $f=gh$ into (3.77), we deduce,

$$g^{n+1}\sum_{i=0}^{p}\frac{a_{p-i}}{n+p+1-i}g^{p-i}(h^{n+p+1-i}-1)=0\,, \tag{3.78}$$

which implies $h^{\chi}=1$, where $\chi=(n+p+1, n+p, n+p-1, ..., n+p+1-i, ..., n+1)$ and $a_{p-i}\neq 0$ for some $i=0,1,...,p$. Thus $f=tg$, for a constant t, such that $t^{\chi}=1$, where χ is previously defined. If h is not constant, then by (3.78) f and g satisfy the algebraic equation $R(f,g)=0$, where, $R(\phi,\psi)=\phi^{n+1}\sum_{i=0}^{p}\frac{a_{p-i}\phi^{m-i}}{n+p+1-i}-\psi^{n+1}\sum_{i=0}^{p}\frac{a_{p-i}\psi^{m-i}}{n+p+1-i}$.

This complete the proof of the theorem. □

Remark 3.2. *Let $Q(f)=f^3-1$ and $m=1$ in theorem 3.2, then it will reduce to theorem 1.4.*

Remark 3.3. *In the theorem 3.2 for every case according to $l(\geq 0)$, the value of n is continuously decreasing for the increasing value of m.*

ACKNOWLEDGMENT

The authors are grateful to referees for their careful reading and effective suggestions.

REFERENCES

[1] A. BANERJEE: *Uniqueness of meromorphic functions sharing two sets with finite weight*, Port. Math. (N.S.), **65**(2008), 81 – 93.

[2] M.L. FANG, W.HONG: *A unicity theorem for entire functions concerning differential polynomials*, Indian J. Pure Appl. Math., **32**(9) (2001), 1343 – 1348.

[3] W.K. HAYMAN: *Meromorphic functions,* Oxford Mathematical Monographs Clarendon Press, Oxford, 1964.

[4] I. LAHIRI: *Weighted value sharing and uniqueness of meromorphic functions*, Complex Var. Theory Appl., **46** (2001), 241–253.

[5] I. LAHIRI: *On a question of Hong Xun Yi,* Arch. Math., (Brno) **46**(2) (2002), 119–128.

[6] I. LAHIRI, R. PAL: *Non-linear differential polynomials sharing* 1*-points,* Bull. Korean Math. Soc., **43**(1) (2006), 161–168.

[7] C. MENG, X.LI: *The uniqueness of meromorphic functions whose differential polynomials share some values*, J. Applied Math. Inform., **33**(5-6) (2015), 475 – 484.

[8] H.P. WAGHAMORE, T.ADAVISWAMY: *Uniqueness of Entire and Meromorphic functions with their Nonlinear differentiaal polynomials*, Global Journal of Mathematical Sciences: Theory and Practical, **3**(3) (2011), 209–215.

[9] H.P. WAGHAMORE, T. ADAVISWAMY: *Weighted sharing and uniqueness of meromophic functions*, Tamkang J. Math., **45**(1) (2014), 1–12.

[10] H.P. WAGHAMORE, N.H. SANNAPPALA: *Uniqueness of meromorphic functions with regard to multiplicity*, Khayyam J. Math., **4**(2) (2018), 155–166.

[11] P. SAHOO: *Meromorphic functions that share fixed points with finite weights,* Bull. Math. Anal. Appl., **2**(2010), 106–118.

[12] C.C. YANG:*On deficiencies of differential polynomials*, II. Math. Z., **125** (1972), 107–112.

[13] H.X. YI, C.C. YANG: *Uniqueness Theory of Meromorphic functions*, Science Press, Beijing, 1995.

[14] J.L. ZHANG, L.Z. YANG: *Some results related to a conjecture of R. Bruck*, JIPAM. J. Inequal. Pure Appl. Math., **8**(18) (2007), 11.

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