ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **9** (2020), no.1, 145–150
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.1.13

A NOTE ON PROPERTIES OF EXPONENTIAL POWER DISTRIBUTION

A. A. OLOSUNDE

ABSTRACT. The most common generalization of the normal, Kotz-symmetric and double exponential distribution functions was the exponential power distribution. This distribution had been found useful in modelling real life data as studied in the literature. The present study found it necessary to fill the void in the literature by presenting some properties which characterized exponential power distribution and further made it useful in applications.

1. INTRODUCTION

The random variable X has the univariate exponential power distribution if it can be expressed as

$$f(x;\mu,\sigma,\beta) = \frac{1}{\sigma\Gamma\left(1+\frac{1}{2\beta}\right)2^{1+\frac{1}{2\beta}}}\exp\left\{-\frac{1}{2}\left|\frac{x-\mu}{\sigma}\right|^{2\beta}\right\}, \tag{1.1}$$

where the parameters $\mu \in \Re$ and $\sigma \in (0,\infty)$ are respectively scale and location parameters $\beta \in (0,-\infty)$ is the shape parameter which regulates the tails of the distribution such that when $\beta = 1$ the density (1.1) is normal; but for $\beta = 1/2$ we have double exponential distribution; $\beta \to \infty$, we have uniform distribution; and when $\beta < 1$, the density function has heavier tail than the normal distribution and can be useful in providing robustness against outliers,

2010 *Mathematics Subject Classification.* 60E05,62H10,6207.
Key words and phrases. Exponential power distribution, characterization, pivotal quantity, entropy, log-concave.

it should be noted that the shape parameter β determines the kurtosis of the distribution. The density function (1.1) has been used in robust inference [3] where the parameters of the distribution were estimated via moments. Also, applications of this distribution had been found in [6] in modelling poultry feeds data. For brevity we denote the density function (1.1) as $EPD(\mu, \sigma, \beta)$, where μ, σ, and β are as defined above. In addition, its central moment estimated by [2] are:

$$E(X) = \mu;$$

$$E\,|X - E(X)| = \frac{\sigma 2^{\frac{1}{2\beta}} \Gamma(\frac{1}{\beta})}{\Gamma(\frac{1}{2\beta})};$$

and

$$Var(X) = \frac{\sigma^2 2^{\frac{2}{2\beta}} \Gamma\left(\frac{3}{2\beta}\right)}{\Gamma\left(\frac{1}{2\beta}\right)};$$

$$E(X - E(X))^3 = 0;$$

$$E(X - E(X))^4 = \frac{\sigma^4 2^{\frac{4}{2\beta}} \Gamma(\frac{5}{2\beta})}{\Gamma(\frac{1}{2\beta})};$$

and Kurtosis

$$= \frac{\Gamma(\frac{5}{2\beta})\Gamma(\frac{1}{2\beta})}{\Gamma^2(\frac{3}{2\beta})}.$$

The results indicate that the sample mean $\overline{X}$ is the estimate of the true mean μ while the shape parameter can be numerically obtained from the estimate of the kurtosis. Substituting shape parameter estimate into $Var(X)$ we estimate the scale parameter σ.

2. Some Characterizations

Proposition 2.1. *Let X be a random variable with density function* (1.1)*, then*
$$\left|\frac{X-\mu}{\sigma}\right|^{\beta} \sim \Gamma(\tfrac{1}{2\beta}, 2).$$

In the Proposition 2.1 the parameters: $-\infty < \mu < \infty$, $\sigma > 0$, and $\beta > 0$ are location, scale and shape parameters respectively for the density function

(1.1), also $\Gamma(.)$ is a gamma function. Values for $\Gamma(.)$ for various β can be obtained from [1]. Note that the complementary incomplete gamma function is defined by

$$\Gamma(a,x)=\int_x^\infty t^{a-1}\exp(-t)dt\,.$$

Proof. By transformation techniques, we have that:

$$f_Y(y)=|\frac{d}{dy}g^{-1}(y)|f_X(g^{-1}(y))=\Gamma(\frac{1}{2\beta},2),\ y>0\,.$$

the pdf (1.1) is a three parameter family, $\underline{\theta}=(\mu,\sigma,\beta)$. □

The following Corollary 2.1 can be deduce from above Proposition 2.1 .

Corollary 2.1. *A random variable X having a density function* (1.1) *then,*

$$Z=\sqrt{\frac{\beta}{n}}\left|\frac{x-\mu}{\sigma}\right|^{\beta}\sim EPD(0,\frac{1}{n},\beta)\,, \tag{2.1}$$

and

$$\left|\frac{x-\mu}{\sigma}\right|^{\beta}\sim\Gamma(\frac{1}{2\beta},2)$$

are pivotal quantities.

Remark 2.1. *The proposition 2.1 and the Corollary 2.1 are useful in deriving the pivotal quantity which can be used to derive optimum confidence interval for the parameter μ of density function* (1.1)*, since it is independent of μ.*

Proposition 2.2. *Let X and Y be two independent random variables with each having a density function* (1.1) *denoted by $\phi(.)$ with distribution function $\Phi(.)$. Then:*

i. *$V=max\,\{X,Y\}$ has distribution function $\Phi_V(v)=\Phi(v)^2$ and density function $\phi_V(v)=2\phi(v)\Phi(v)$.*
ii. *$W=min\,\{X,Y\}$ has distribution function $\Phi_W(w)=1-[1-\Phi(w)]^2$ and density function $\phi_W(w)=2\phi(w)[1-\Phi(w)]$.*

Proof. (i.)

$$\begin{aligned}\Phi_V(v)&=P(V\le v)=P(max\,\{X,Y\}\le v)\\&=P(X\le v)P(Y\le v)=[\Phi(v)]^2\,.\end{aligned}$$

differentiate the distribution function to derive the density function by using chain rule

$$\phi_V(v) = \frac{d}{dv}\Phi_V(v) = 2\Phi(v)\phi(v)\,.$$

(ii.) in a similar manner we have:

$$\begin{aligned}\Phi_W(w) &= P(W \leq w) = P(min\,\{X, Y\} \leq w) = 1 - P(min\,\{X, Y\} \geq w)\\ &= 1 - P(X \geq w)P(Y \geq w) = 1 - (1 - \Phi(w))^2\,.\end{aligned}$$

differentiate the distribution function to derive the density function by using chain rule

$$\frac{d}{dv}\Phi_W(w) = 2\phi_W(w)(1 - \Phi(w))\,.$$

□

Remark 2.2. *The Proposition 2.2 is useful in deriving the skewed version of the symmetric distribution for exponential power distribution with skewing parameter equal to unity. The skewed version has been found useful in modelling financial/income data etc.*

Proposition 2.3. *A random variable X defined as in* (1.1) *above has the single entropy*

$$\ln(2\sigma\beta^{1/\beta}\Gamma(1 + \frac{1}{\beta})) + \frac{1}{\beta}\,.$$

Proof. Given $H(X) = -\int_{\Re} f(x)\ln f(x)dx$, then let $f(x)$ be (1.1) we have that:

$$H(X) = E\left\{\frac{|x-\mu|^{2\beta}}{\beta\sigma^\beta}\right\} - E\left\{\ln\left(2\sigma^\beta\beta^{1/\beta}\Gamma\left(1 + \frac{1}{\beta}\right)\right)\right\}\,.$$

and the result (2.1) follow immediately after. □

Remark 2.3. *Proposition 2.3 is useful in diagnosing exponential power within the the normal distribution [5], it can be also useful in goodness-of-fit test.*

Proposition 2.4. *Let $Y \sim N(0,1)$, then $Z = |Y|$ has half normal distribution, further, let X has exponential power with shape parameter β, then:*

$$E(h(X)) = E(h(Y)) = E(h(Z))\,,$$

if and only if $h(.)$ is an even function and $E(|h(X)|) < \infty$.

Proof. Since $Z = |Y|$, it is clear that $E(h(X)) = E(h(Z))$, now

$$E(h(X)) = \int_{-\infty}^{0} h(x)f(x)dx + \int_{0}^{\infty} h(x)f(x)dx = 2\int_{0}^{\infty} h(x)f(x)dx = E(h(Z)).$$

□

In [7] the theory and applications of log-concave version of the density function (1.1) was given. Furthermore, then we have the followings;

Proposition 2.5. *Let X be random variable with the probability density function* (1.1)*, Then*

i. *the density function* (1.1) *is log-convex, when $0 < \beta < \frac{1}{2}$; but log-concave when $\beta \geq \frac{1}{2}$;*
ii. *the distribution function of* (1.1) *is log-convex, when $0 < \beta < \frac{1}{2}$; but log-concave when $\beta \geq \frac{1}{2}$; and*
iii. *the hazard function of* (1.1) *is log-convex, when $0 < \beta < \frac{1}{2}$; but log-concave when $\beta \geq \frac{1}{2}$.*

Proof. To proof the given proposition 2.5 it is sufficient to proof that if $(\ln g(x))'' < 0$ then it is log-concave otherwise log-convex. Given the density function (1.1), then the distribution function is obtained after some algebra as if $x \leq \mu$

$$F(x) = \frac{\Gamma(\frac{1}{2\beta}, (\frac{(\mu-x)}{\sigma})^{2\beta})}{2\Gamma(\frac{1}{2\beta})}, \tag{2.2}$$

and if $x > \mu$ then,

$$F(x) = 1 - \frac{\Gamma(\frac{1}{2\beta}, (\frac{(x-\mu)}{\sigma})^{2\beta})}{2\Gamma(\frac{1}{2\beta})}. \tag{2.3}$$

Also, the hazard rate function defined by $h(x) = f(x)/[1-F(x)]$ It is immediate from equation (1.1), (2.2) and (2.3) that the hazard rate function is given by

$$h(x) = \frac{2\beta \exp\left\{-\left|\frac{(x-\mu)}{\sigma}\right|^{2\beta}\right\}}{\sigma\left\{2\Gamma(\frac{1}{2\beta}) - \Gamma\left(\frac{1}{2\beta}, [\frac{(\mu-x)}{\sigma}]^{2\beta}\right)\right\}}, \tag{2.4}$$

if $x \leq \mu$

$$h(x) = \frac{2\beta \exp\left\{-\left|\frac{(x-\mu)}{\sigma}\right|^{2\beta}\right\}}{\sigma\Gamma\left(\frac{1}{2\beta}, [\frac{(\mu-x)}{\sigma}]^{2\beta}\right)}, \tag{2.5}$$

if $x > \mu$. Twice differentiating the natural logarithm of the functions (1.1), (2.2), (2.3), (2.4)and (2.5) the results in Proposition 2.5 follows immediately. It is noteworthy to mention that the complementary incomplete gamma function encountered during calculations that is $\Psi(y) = dlog\Gamma(x)/dx$ is a special function which [4] have presented their properties. $h(x)$ is an increasing function of x for $\beta \geq 1/2$. $\square$

REFERENCES

[1] M. ABRAMOWITZ, I. A. STEGUN : *Handbook of Mathematical Functions*, 9th ed., Dover Publications, Inc. New York, 1963.

[2] G. AGRO : *Maximum Likelihood Estimation for the Exponential Power Distribution*, Communications in Statistics (simulation and computation), **24**(2) (1995) 523–536.

[3] G. E. P. BOX: *A note on the Region of Kurtosis,* Biometrika, **40** (1953), 465–468.

[4] I. S. GRADSHTEYN, I. M. RYZHIK: *Table of integrals, series, and products,* 6th ed. San Diego: Academic Press, 2000.

[5] S. KULLBACK: *Information theory and statistics,* John Wiley, New York, 1959.

[6] A. A. OLOSUNDE: *On exponential power distribution and poultry feeds data: a case study*, Journal of Iran Statistical Society, **12**(2) (2013), 253–269.

[7] A. A. OLOSUNDE: *On log-concavity of exponential power distribution and its application,* Missouri Journal of Mathematical Sciences (To appear).

DEPARTMENT OF MATHEMATICS
OBAFEMI AWOLOWO UNIVERSITY, ILE-IFE, NIGERIA
E-mail address: aolosunde@oauife.edu.ng