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NUMERICAL QUENCHING VERSUS BLOW-UP FOR A NONLINEAR PARABOLIC EQUATION WITH NONLINEAR BOUNDARY OUTFLUX

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ABSTRACT. In this paper, we study numerical approximations for positive solutions of a semilinear heat equations, $u_t = u_{xx} + u^p$, in a bounded interval $(0,1)$, with a nonlinear flux boundary condition at the boundary $u_x(0,t) = 0$, $u_x(1,t) = -u^{-q}(1,t)$. By a semi-discretization using finite difference method, we get a system of ordinary differential equations which is expected to be an approximation of the original problem. We obtain some conditions under which the positive solution of our system quenches or blows up in a finite time and estimate its semidiscrete blow-up and quenching time. We also estimate the semidiscrete blow-up and quenching rate. Finally, we give some numerical results to illustrate our analysis.

1. INTRODUCTION

In this paper, we consider the following initial-boundary value problem:

$$\begin{cases} u_t = u_{xx} + u^p, & 0 < x < 1,\ 0 < t < \infty, \\ u_x(0,t) = 0, & 0 < t < \infty, \\ u_x(1,t) = -u^{-q}(1,t), & 0 < t < \infty, \\ u(x,0) = u_0(x), & 0 \le x \le 1, \end{cases} \tag{1.1}$$

where $p, q > 0$ and $u_0(x)$ is a positive function with $u_0'(0) = 0$, $u_0'(1) = -u_0^{-q}(1)$. Physically, (1.1) can be treated as a heat conduction model that incorporates

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the effects of reaction and nonlinear outflux. Mathematically, (1.1) is a combination of the following two problems:

$$
\begin{cases}
u_t = u_{xx} + u^p, & 0 < x < 1,\ t > 0, \\
u_x(0,t) = 0, & t > 0, \\
u_x(1,t) = 0, & t > 0, \\
u(x,0) = u_0(x), & 1 \le x \le 0,
\end{cases}
\tag{1.2}
$$

and

$$
\begin{cases}
u_t = u_{xx}, & 0 < x < 1,\ t > 0, \\
u_x(0,t) = 0, & t > 0, \\
u_x(1,t) = -u^{-q}(1,t), & t > 0, \\
u(x,0) = u_0(x), & 1 \le x \le 0.
\end{cases}
\tag{1.3}
$$

Problems (1.2) and (1.3) has been widely analyzed (see [5–8] and the references cited therein). In particular, it is well known that if $p > 1$ all positive solutions of problem (1.2) blow up in finite time (see [5]) and that the rate near the blow-up time T_b is $(T_b - t)^{\frac{-1}{p-1}}$, [2].

Regarding problem (1.3), Keng Deng and Mingxi Xu in [8] considered a nonlinear diffusion equation $(\psi(u))_t = u_{xx}$, $0 < x < 1$ with a singular boundary condition $u_x(1,t) = -g(u(1,t))$, they proved finite time quenching for the solution. They are also established results on the quenching set and rate. Moreover in [7] the authors was shown that u quenches in finite time for all u_0, and the only quenching point is $x = 1$. They estimated the quenching rate by $(T_q - t)^{\frac{1}{2(q+1)}}$.

Let us give the two following definitions.

Definition 1.1. *We say that the classical solution u of (1.1) quenches in a finite time if there exists a finite time T_q such that $\min\limits_{0\le x\le 1} u(x,t) > 0$ for $t \in [0, T_q)$ but $\lim\limits_{t\to T_q^-} \min\limits_{0\le x\le 1} u(x,t) = 0$.*

Definition 1.2. *We say that the classical solution u of (1.1) blows up in a finite time if there exists a finite time T_b such that $\|u(t)\|_\infty < \infty$ for $t \in [0, T_b)$ but $\lim\limits_{t\to T_b} \|u(t)\|_\infty = \infty$.*

From now on, we denote by T_q and T_b the quenching time and the blow up time respectively of problem (1.1).

Concerning problem (1.1), K. Deng and C. L Zhao [10] established criteria for finite time blow-up and quenching, they are discussed to the blow-up and quenching sets and obtained the blow-up and quenching rates. They also characterized the sets of stationary states and analyzed their instability in [9].

Here, our objective is the numerical study of (1.1). To the best of our knowledge, very few works are concerned with the numerical study of this kind of problem. For previous works on numerical study we refer to ([1, 11–16] and the references therein). Here we give some assumptions under which the solution of a semidiscrete form of (1.1) quenches or blows up in a finite time depends upon certain conditions on the initial data and estimate its semidiscrete quenching or blow-up time. We show that the rate estimate near blow-up time is the same as (1.3), but the one near quenching time is different that (1.3). We also prove that, under suitable assumptions on the initial datum, the semidiscrete quenching or blow-up time converges to the theoretical one when the mesh size goes to zero.

The paper is written in the following manner. In the next Section, we present a semidiscrete scheme of (1.1). In section 3, we give some properties concerning our semidiscrete scheme. In Section 4, under some conditions, we prove that the solution of the semidiscrete form blows up in a finite time, study the convergence of semidiscrete blow-up time and estimate the semidiscrete rate near the blow up time. In Section 5, under some conditions, we prove that the solution of the semidiscrete form quenches in a finite time, study the convergence of semidiscrete quenching time and estimate the semidiscrete rate near the quenching time. Finally, in the last section, we give some numerical experiments.

2. The Semidiscrete Problem

Let I be a positive integer, we set $h = 1/I$, and we define the grid , $x_i = ih$, $i = 0, \ldots, I$. Let T be a positive real such that $[0, T]$ is a time interval on which the solution u of the continuous problem is defined. We approximate the solution u of the problem (1.1) by the solution $U_h = (U_0(t), U_1(t), \ldots, U_I(t))^T$

of the semidiscrete equations

$$\frac{dU_i(t)}{dt} = \delta^2 U_i(t) + U_i^p(t), \quad 1 \le i \le I-1,\ t > 0, \tag{2.1}$$

$$\frac{dU_0(t)}{dt} = \delta^2 U_0(t) + U_0^p(t), \quad t > 0, \tag{2.2}$$

$$\frac{dU_I(t)}{dt} = \delta^2 U_I(t) - \frac{2}{h} U_I^{-q}(t) + U_I^p(t), \quad t > 0, \tag{2.3}$$

$$U_i(0) = \varphi_i > 0, \quad i = 0, \ldots, I, \tag{2.4}$$

where

$$\delta^2 U_i(t) = \frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{h^2}, \quad i = 1, \ldots, I-1,$$

$$\delta^2 U_0(t) = \frac{2U_1(t) - 2U_0(t)}{h^2}, \quad \delta^2 U_I(t) = \frac{2U_{I-1}(t) - 2U_I(t)}{h^2}.$$

3. Properties of the semidiscrete Problem

In this section, we give some important results which will be used later.

Definition 3.1. *A function $U_h \in \mathcal{C}^1([0,T], \mathbb{R}^{I+1})$ is an upper solution of* (2.1)-(2.4) *if*

$$\begin{aligned}
&\frac{dU_i(t)}{dt} - \delta^2 U_i(t) \ge U_i^p(t), \quad i = 0, \ldots, I-1, \quad t \in [0,T], \\
&\frac{dU_I(t)}{dt} - \delta^2 U_I(t) + \frac{2}{h} U_I^{-q}(t) \ge U_I^p(t), \quad t \in [0,T], \\
&U_i^0 \ge \varphi_i, \quad i = 0, \ldots, I.
\end{aligned}$$

On the other hand, we say that $U_h \in \mathcal{C}^1([0,T], \mathbb{R}^{I+1})$ is a lower solution of (2.1)-(2.4) *if these inequalities are reversed.*

Lemma 3.1. *Let $a_h(t) \in C^0([0,T], \mathbb{R}^{I+1})$ and $V_h(t) \in C^1([0,T], \mathbb{R}^{I+1})$ such that*

$$\begin{aligned}
&\frac{dV_i(t)}{dt} - \delta^2 V_i(t) + a_i(t) V_i(t) \ge 0, \quad 0 \le i \le I,\ t \in [0,T], \\
&V_i(0) \ge 0, \quad 0 \le i \le I.
\end{aligned}$$

Then we have $V_i(t) \ge 0,\ 0 \le i \le I,\ t \in [0,T]$.

Proof. Define the vector $Z_h(t) = e^{\lambda t} V_h(t)$ where λ is such that

$$a_i(t) - \lambda > 0 \quad \text{for} \quad 0 \leq i \leq I,\ t \in [0, T].$$

Let $m = \min\{Z_i(t) : 0 \leq i \leq I, 0 \leq t \leq T\}$. Since for $i \in \{0, \ldots, I\}$, $Z_i(t)$ is a continuous function on the compact $[0, T]$, there exists $i_0 \in \{0, \ldots, I\}$ and $t_0 \in [0, T]$ such that $m = Z_{i_0}(t_0)$. We observe that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \leq 0, \tag{3.1}$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} \geq 0, 1 \leq i_0 \leq I - 1, \tag{3.2}$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_1(t_0) - 2Z_0(t_0)}{h^2} \geq 0 \quad if \quad i_0 = 0, \tag{3.3}$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_{I-1}(t_0) - 2Z_I(t_0)}{h^2} \geq 0 \quad if \quad i_0 = I. \tag{3.4}$$

Moreover, by a straightforward computation, we get

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + (a_{i_0}(t_0) - \lambda) Z_{i_0}(t_0) \geq 0. \tag{3.5}$$

Using (3.1)-(3.4), we deduce from (3.5) that $(a_{i_0}(t_0) - \lambda) Z_{i_0}(t_0) \geq 0$, which implies that $Z_{i_0}(t_0) \geq 0$. We deduce that $V_h(t) \geq 0$ for $t \in [0, T]$ and the proof is complete. ☐

Another form of the maximum principle for semidiscrete equations is the comparison lemma below.

Lemma 3.2. *Let $g \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $W_h(t), V_h(t) \in C^1([0, T], \mathbb{R}^{I+1})$ such that*

$$\begin{aligned}
&\frac{dV_i(t)}{dt} - \delta^2 V_i(t) + g(V_i(t), t) < \frac{dW_i(t)}{dt} - \delta^2 W_i(t) + g(W_i(t), t)\,, \\
&1 \leq i \leq I - 1,\ t \in (0, T)\,, \\
&\frac{dV_0(t)}{dt} - \delta^2 V_0(t) + g(V_0(t), t) < \frac{dW_0(t)}{dt} - \delta^2 W_0(t) + g(W_0(t), t) \quad t \in (0, T), \\
&\frac{dV_I(t)}{dt} + \frac{2}{h} V_I^{-q} - \delta^2 V_I(t) + g(V_I(t), t) < \frac{dW_I(t)}{dt} + \frac{2}{h} W_I^{-q} - \delta^2 W_I(t) + g(W_I(t), t) \\
&V_i(0) < W_i(0)\,,\ i = 0, \ldots, I.
\end{aligned}$$

Then $V_i(t) < W_i(t)$, $0 \leq i \leq I$, $t \in (0, T)$.

Lemma 3.3. *Let $g \in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$ and $W_h(t), V_h(t) \in C^1([0,T],\mathbb{R}^{I+1})$ such that*

$$\frac{dV_i(t)}{dt} - \delta^2 V_i(t) + g(V_i(t),t) \le \frac{dW_i(t)}{dt} - \delta^2 W_i(t) + g(W_i(t),t)\,,$$
$$1 \le i \le I-1\,, t \in (0,T)\,,$$
$$\frac{dV_0(t)}{dt} - \delta^2 V_0(t) + g(V_0(t),t) \le \frac{dW_0(t)}{dt} - \delta^2 W_0(t) + g(W_0(t),t)\,,$$
$$t \in (0,T)\,,$$
$$\frac{dV_I(t)}{dt} + \frac{2}{h}V_I^{-q} - \delta^2 V_I(t) + g(V_I(t),t) \le \frac{dW_I(t)}{dt} + \frac{2}{h}W_I^{-q} - \delta^2 W_I(t) + g(W_I(t),t)$$
$$V_i(0) \le W_i(0) \quad i = 0,\ldots,I.$$

Then $V_i(t) \le W_i(t)$, $0 \le i \le I$, $t \in (0,T)$.

Lemma 3.4. *Let $T > 0$ and U_h be a solution of semidiscrete problem (2.1)-(2.4) and assume that the initial data at (2.4) verifies $\varphi_i > \varphi_{i+1}$, $0 \le i \le I-1$. Then, for $i = 0,\ldots,I-1$ and $t \in (0,T]$ we have:*

$$U_i(t) > U_{i+1}(t).$$

Proof. Introduce the vector Z_h such that $Z_i(t) = U_{i+1}(t) - U_i(t)$ for $t \in (0,T]$, $i = 0,\ldots,I-1$. Let t_0 be the first $t > 0$ such that $Z_i(t) < 0$ for $t \in [0,t_0)$ but $Z_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0,\ldots,I-1\}$. Without loss of generality, we suppose that i_0 is the smallest integer checking the inequality above. We observe that

$$\frac{d}{dt}Z_{i_0}(t_0) = \lim_{k\to 0}\frac{Z_{i_0}(t_0) - Z_{i_0}(t_0-k)}{k} \ge 0, \quad 0 \le i_0 \le I-1,$$
$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0-1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0+1}(t_0)}{h^2} < 0, \quad 1 \le i_0 \le I-2,$$
$$\delta^2 Z_{i_0}(t_0) = \frac{Z_1(t_0) - 3Z_0(t_0)}{h^2} < 0, \quad i_0 = 0,$$
$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{I-2}(t_0) - 3Z_{I-1}(t_0)}{h^2} < 0, \quad i_0 = I-1.$$

Moreover, by a straightforward computation, we get

$$\frac{d}{dt}Z_{i_0}(t_0) - \delta^2 Z_{i_0}(t_0) - (U_{i_0+1}^p - U_{i_0}^p) > 0, \quad 0 \le i_0 \le I-2,$$
$$\frac{d}{dt}Z_{I-1}(t_0) - \delta^2 Z_{I-1}(t_0) + \frac{2}{h}U_I^{-q}(t_0) - (U_I^p(t_0) - U_{I-1}(t_0)) > 0, \quad i_0 = I-1.$$

But these inequalities contradict (2.1)-(2.3) and this proof is complete. □

Theorem 3.1. *Assume that the problem (1.1) has a solution $u \in C^{4,1}([0,1] \times [0,T])$ and the initial condition φ_h at (2.4) verifies*

$$\|\varphi_h - u_h(0)\|_\infty = \circ(1), \quad as \quad h \to 0 \tag{3.6}$$

where $u_h(t) = (u(x_0,t),...,u(x_I,t))^T$, $t \in [0,T]$. Then, for h small enough, the semidiscrete problem (2.1)-(2.4) has a unique solution $U_h \in C^1([0,T],\mathbb{R}^{I+1})$ such that $\max\limits_{t\in[0,T]}(\|U_h(t) - u_h(t)\|_\infty) = O(\|\varphi_h - u_h(0)\|_\infty + h^2) \quad as \quad h \to 0$.

Proof. Let $K > 0$ be such that

$$\| u(.,t)\|_\infty \le K \quad \text{for } t \in [0,T]. \tag{3.7}$$

Then the semidiscrete problem (2.1)-(2.4) has for each h, a unique solution $U_h \in C^1([0,T],\mathbb{R}^{I+1})$. Let $t(h) \le T$ be the greatest value of $t > 0$ such that

$$\|U_h(t) - u_h(t)\|_\infty < 1. \tag{3.8}$$

The relation (3.6) implies $t(h) > 0$ for h small enough. Using the triangle inequality, we obtain

$$\|U_h(t)\|_\infty \le \|u(.,t)\|_\infty + \|U_h(t) - u_h(t)\|_\infty \quad \text{for} \quad t \in (0,t(h)),$$

which implies that

$$\|U_h(t)\|_\infty \le 1 + K \quad \text{for} \quad t \in (0,t(h)). \tag{3.9}$$

Let $e_h(t) = U_h(t) - u_h(t)$ be the discretization error. Using the Taylor's expansion, we have for $t \in (0,t(h))$

$$\begin{aligned}
&\frac{de_i}{dt} = \delta^2 e_i(t) + p\zeta_i^{p-1}(t)e_i(t) + O(h^2), \quad i = 0,...,I-1,\\
&\frac{d}{dt}e_I(t) = \delta^2 e_I(t) + p\zeta_I^{p-1}(t)e_I(t) + \frac{2}{h}q\mu_I^{-q-1}(t)e_I(t) + O(h^2),
\end{aligned}$$

where $\zeta_i(t)$ is the intermediate value between $U_i(t)$ and $u(x_i,t)$ for $i = 0,\dots,I$ and $\mu_I(t)$ the one between $U_I(t)$ and $u(x_I,t)$. Using (3.7) and (3.9), there exist L et λ positive constants such that

$$\begin{aligned}
&\frac{d}{dt}e_i - \delta^2 e_i \le L|e_i(t)| + \lambda h^2, \quad 0 \le i \le I-1,\\
&\frac{d}{dt}e_I - \delta^2 e_I \le \frac{L}{h}|e_I| + \lambda h^2.
\end{aligned}$$

Now, we consider the function $Z \in \mathcal{C}^{4,1}([0,1],[0,T])$ such that

$$Z(x,t) = e^{(\alpha+1)t+c(-x^2+1)}(\|\varphi_h - u_h(0)\|_\infty + \lambda h^2), \quad 0 \le i \le I.$$

A simple computational give

$$\frac{dZ(x_i,t)}{dt} - \delta^2 Z(x_i,t) > L|z(x_i,t)|\lambda h^2, \quad 0 \le i \le I-1,$$
$$\frac{dZ_I}{dt} - \delta^2 Z_I > \frac{L}{h}|Z(x_I,t)| + \lambda h^2.$$

From lemma (3.2), we obtain $Z_i(t) > e_i(t)$, for $t \in (0, t(h))$, $i = 0, ..., I$. By analogy, we also prove that $Z_i(t) > -e_i(t)$, for $t \in (0, t(h)$, $i = 0, ..., I$. Hence we have $Z_i(t) > |e_i(t)|$, for $t \in (0, t(h)), i = 0, ..., I$. We deduce that

$$\|U_h(t) - u_h(t)\|_\infty \le (\|\varphi_h - u_h(0)\|_\infty + \lambda h^2)e^{(\alpha+1)t+c}, \quad \text{for} \quad t \in (0, t(h)).$$

Next we prove that $t(h) = T$. Suppose that $T > t(h)$, from (3.8), we obtain

$$1 = \|U_h(t(h)) - u_h(t(h))\|_\infty \le (\|\varphi_h - u_h(0)\|_\infty + \lambda h^2)e^{(\alpha+1)T+c}. \tag{3.10}$$

Since $(\|\varphi_h - u_h(0)\|_\infty + \lambda h^2)e^{(\alpha+1)T+c} \to 0$ as $h \to 0$, we deduce from (3.10) that $1 \le 0$, which is impossible. □

4. Blow up, blow up rate

In this section, under some assumptions, we show that the solution U_h of (2.1)-(2.4) blows up in a finite time. We assume that the initial data is a positive function and verifies.

$$(u_0(x))_{xx} + u_0^p(x) \ge 0 \text{ for } x \in [0,1]. \tag{4.1}$$

Lemma 4.1. *Let U_h be a solution of (2.1)-(2.4) and the initial data at (2.4) verifies*

$$\delta^2 \varphi_i + \varphi_i^p > 0, \quad 0 \le i \le I. \tag{4.2}$$

Then, $\frac{dU_i(t)}{dt} > 0$ for $0 \le i \le I$, $t \in (0, T_b^h)$.

Proof. Consider the vector $Z_h(t)$ such that $Z_i(t) = \frac{dU_i(t)}{dt}$, $0 \le i \le I$, $t \in [0, T_b^h)$. Let t_0 be the first $t \in (0, T_b^h)$ such that $Z_i(t) > 0$ for $t \in [0, t_0)$, but $Z_{i_0}(t_0) = 0$

for a certain $i_0 \in \{0, \ldots, I\}$. Without loss of generality, we suppose that i_0 is the smallest integer checking the inequality above. We observe that:

$$\frac{d}{dt} Z_{i_0}(t_0) = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \leq 0, \quad 0 \leq i_0 \leq I,$$
$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0-1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0+1}(t_0)}{h^2} > 0, \quad 1 \leq i_0 \leq I - 1,$$
$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_1(t_0) - 2Z_0(t_0)}{h^2} > 0, \quad i_0 = 0,$$
$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_{I-1}(t_0) - 2Z_I(t_0)}{h^2} > 0, \quad i_0 = I.$$

Moreover, by a straightforward computation, we get

$$\frac{d}{dt} Z_{i_0}(t_0) - \delta^2 Z_{i_0}(t_0) - pU_{i_0}^{p-1}(t_0) Z_{i_0}(t_0) < 0, \ 0 \leq i_0 \leq I - 1,$$
$$\frac{d}{dt} Z_I(t_0) - \delta^2 Z_I(t_0) - (\frac{2}{h} q U_I^{-q-1}(t_0) + pU_I^{p-1}(t_0)) Z_I(t_0) < 0, \ i_0 = I.$$

But these inequalities contradict (2.1)-(2.3) and this proof is complete. $\square$

The following result gives a property of the operator δ^2.

Lemma 4.2. *Let $U_h \in \mathbb{R}^{I+1}$ be such that $U_h \geq 0$. Then, we have*

$$\delta^2(U_i^p) \geq pU_i^{p-1} \delta^2 U_i, \quad 0 \leq i \leq I.$$

Proof. Let us introduce function $f(s) = s^p$. Using taylor's expansion we get

$$\delta^2 f(U_0) = f'(U_0) \delta^2 U_0 + \frac{(U_1 - U_0)^2}{h^2} f''(\zeta_0),$$
$$\delta^2 f(U_I) = f'(U_I) \delta^2 U_I + \frac{(U_{I-1} - U_I)^2}{h^2} f''(\zeta_I),$$
$$\delta^2 f(U_i) = f'(U_i) \delta^2 U_i + \frac{(U_{i+1} - U_i)^2}{2h^2} f''(\eta_i) + \frac{(U_{i-1} - U_i)^2}{2h^2} f''(\zeta_i), \ 1 \leq i \leq I - 1,$$

where η_i is an intermediate value between U_i and U_{i+1} and ζ_i the one between U_i and U_{i-1}. The result follows taking into account the fact that $U_h \geq 0$. $\square$

Introduce the functionals

$$I(t) = \frac{1}{2} \int_0^1 u_x^2 dx - \frac{1}{p+1} \int_0^1 u^{p+1} dx + \frac{1}{-q+1} u^{-q+1}(1, t) \quad \text{and} \quad J(t) = \int_0^1 u^2 dx$$

and the semidiscrete approximations

$$I_h(t) = \frac{1}{2}\sum_{i=0}^{I-1} \frac{(U_{i+1}-U_i)^2}{h} - \frac{1}{p+1}\sum_{i=0}^{I} hU_i^{p+1} + \frac{1}{-q+1}U_I^{-q+1} \tag{4.3}$$

$$J_h(t) = \sum_{i=0}^{I} hU_i^2.$$

For the functional J using the condition (4.1) and the Jensen's inequality we obtain after integration by parts

$$\begin{aligned}\frac{d}{dt}J(t) =& -2u^{-q+1}(1,t) - 2\int_0^1 u_x^2 dx + 2\int_0^1 u^{p+1}dx \\ \geq& -2u^{-q+1}(1,0) + 2(1-\lambda)\int_0^1 u^{p+1}dx \\ \geq& -2u^{-q+1}(1,0) + 2(1-\lambda)J\frac{p+1}{2}.\end{aligned}$$

In [10] the author showed that if the condition (4.1) is satisfies then there exists $T_b < \infty$ such that $\lim_{t\to T_b} \|u(.,t)\|_{L^1} = \infty$, which implies that $\lim_{t\to T_b} \frac{d}{dt}J(t) = +\infty$ for $p > 1$.

Let the functionals $I_h(t)$ and $J_h(t)$ as defined in (4.3). Multiplying both sides of (2.1) by $h\frac{dU_i}{dt}$ and hdU_i, respectively, and then taking the sum from $i = 0$ to $i = I$, we obtain

$$-\sum_{i=0}^{I} h\Big(\frac{dU_i}{dt}\Big)^2 = \frac{d}{dt}\Big(\frac{1}{2}\sum_{i=0}^{I-1}\frac{(U_{i+1}-U_i)^2}{h} - \frac{1}{p+1}\sum_{i=0}^{I} hU_i^p + \frac{1}{-q+1}U_I^{-q+1}\Big),$$

which implies that the semidiscrete functional $I_h(t)$ is non increasing for $t \in [0, T_b^h)$. Further,

$$\begin{aligned}\frac{d}{dt}\sum_{i=0}^{I} hU_i^2 &= 2\Big(-\sum_{i=0}^{I-1}\frac{(U_{i+1}-U_i)^2}{h} + \sum_{i=0}^{I} hU_i^{p+1} - U_I^{-q+1}\Big) \\ &= -4I_h + \frac{2p-2}{p+1}\sum_{i=0}^{I} hU_i^{p+1} + \frac{2q+2}{-q+1}U_I^{-q+1},\end{aligned}$$

and a straightforward calculation gives

$$\frac{d^2J_h}{dt^2} = -4\frac{dI_h}{dt} + 2(q+1)U_I^{-q}\frac{dU_I}{dt} + 2(p-1)\sum_{i=0}^{I} hU_i^p\frac{dU_i}{dt}.$$

Using the lemma (4.1) we obtain for $p > 1$

$$\frac{d^2 J_h}{dt^2} \geq -4\frac{dI_h}{dt}\,.$$

Using the Hölder inequality and the expressions above, we obtain

$$\Big(\frac{dJ_h}{dt}\Big)^2 \leq 4\Big(\sum_{i=0}^{I} hU_i^2\Big)\Big(\sum_{i=0}^{I} h\Big(\frac{dU_i}{dt}\Big)^2\Big) \leq J_h\frac{d^2 J_h}{dt^2}$$

which implies that $\frac{d^2 J_h}{dt^2} \geq -\lambda\frac{dJ_h}{dt}$ where λ in a non-negative constant and $\frac{dJ_h}{dt} = \sum_{i=0}^{I} hU_i\frac{dU_i}{dt}$.

Using the theorem 3.1 we obtain the following result, see [17].

Theorem 4.1. *Assume that the solution u of (1.1) blows up in a finite time T_b such that $u \in C^{4,1}([0,1] \times [0,T_b))$. We also assume that the initial data at (2.4) satisfies the condition (4.2) and the error of initial data is of order $\circ(1)$. Then for h small enough, the solution U_h of problem (2.1)-(2.4) blows up in finite time T_b^h for $p > 1$, $0 < q < 1$ and we have*

$$\lim_{h\to\infty} T_b^h = T_b.$$

Theorem 4.2. *Assume that the hypotheses in theorem 4.1 remains true. We also assume that the condition*

$$f(s)g^{'}(s) - f^{'}(s)g(s) \geq 0 \quad \textit{for} \quad s \geq 0\,, \tag{4.4}$$

hold. Then, near the blow-up time T_b^h, the solution U_h of problem (2.1)-(2.4) has a following blow-up rate estimate

$$\|U_h(t)\|_\infty \sim (T_b^h - t)^{\frac{-1}{p-1}}, \tag{4.5}$$

in the sense that there exist two positive constants K_1, K_2 such that

$$K_1(T_b^h - t)^{\frac{-1}{p-1}} \leq \|U_h(t)\|_\infty \leq K_2(T_b^h - t)^{\frac{-1}{p-1}} \quad \textit{for} \quad t \in (0, T_b^h).$$

Proof. Introduced the vector $J_h(t)$ defined as follow

$$J_i = \frac{dU_i}{dt} - \varepsilon U_i^p, \quad 0 \leq i \leq I\,, \tag{4.6}$$

with ε is a positive constant. By a straightforward computation we get

$$\frac{d}{dt}J_i - \delta^2 J_i = \frac{d}{dt}\Big(\frac{dU_i}{dt} - \delta^2 U_i\Big) - \varepsilon p U_i^{p-1}\frac{dU_i}{dt} + \varepsilon\delta^2 U_i^p, \quad 0 \leq i \leq I.$$

Using Lemma 4.2 and the equality (4.6), we obtain from the condition (4.4)

$$\frac{d}{dt}J_i - \delta^2 J_i \geq pU_i^{p-1}J_i, \quad 0 \leq i \leq I-1\,,$$
$$\frac{d}{dt}J_I - \delta^2 J_I \geq (pU_I^{p-1} + \frac{2}{h}U_I^{-q-1})J_I.$$

From (4.2),we observe that $J_i(0) \geq 0$ for $0 \leq i \leq I$ if ε is sufficiently small. We deduce from lemma 3.1 that $J_i(t) \geq 0$, $0 \leq i \leq I$, which yields the desired upper bound.

The following result concerns the lower bound for the quenching rate.

Let i_0 be such that $U_{i_0}(t) = \max\limits_{0 \leq i \leq I} U_i(t)$. We can observe that

$$\delta^2 U_{i_0}(t) = \frac{U_{i_{0+1}}(t) - 2U_{i_0}(t) + U_{i_0-1}(t)}{h^2} \leq 0, \quad 1 \leq i_0 \leq I\,,$$
$$\delta^2 U_0(t) = \frac{2U_1(t) - 2U_0(t)}{h^2} \leq 0, \quad i_0 = 0\,,$$
$$\delta^2 U_I(t) = \frac{2U_{I-1}(t) - 2U_I(t)}{h^2} \leq 0, \quad i_0 = I.$$

We can see that $\frac{dU_{i_0}(t)}{dt} \leq U_{i_0}(t)$ for $0 \leq i_0 \leq I$. Integrating this inequality over (t, T_b^h) and we obtain the result desired. □

5. Quenching, Quenching rate

In this section, under some assumptions, we show that the solution U_h of (2.1)-(2.4) quenches in a finite time.

We note that if the initial data is a subsolution then the solution is monotone, non-decreasing function with respect to t and verifies:

(5.1) $$(u_0(x))_{xx} + u_0^p(x) \leq 0 \quad \text{for} \quad x \in [0,1]\,.$$

Lemma 5.1. *Let U_h be a solution of (2.1)-(2.4) and the initial data at (2.4) verifies*

(5.2) $$\delta^2 \varphi_i + \varphi_i^p < 0, \quad 0 \leq i \leq I.$$

Then, $\dfrac{dU_i(t)}{dt} < 0$ *for* $0 \leq i \leq I$, $t \in [0, T_q^h)$.

Proof. Consider the vector $Z_h(t)$ such that $Z_i(t) = \frac{dU_i(t)}{dt}$, $0 \le i \le I$, $t \in [0, T_h^q)$. Let t_0 be the first $t \in (0, T_q^h)$ such that $Z_i(t) < 0$ for $t \in [0, t_0)$, but $Z_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0, \dots, I\}$. Without loss of generality, we suppose that i_0 is the smallest integer checking the inequality above. We observe that:

$$\frac{d}{dt} Z_{i_0}(t_0) = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \ge 0, \quad 0 \le i_0 \le I,$$
$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0-1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0+1}(t_0)}{h^2} < 0, \quad 1 \le i_0 \le I-1,$$
$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_1(t_0) - 2Z_0(t_0)}{h^2} < 0, \quad i_0 = 0,$$
$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_{I-1}(t_0) - 2Z_I(t_0)}{h^2} < 0, \quad i_0 = I.$$

Moreover, by a straightforward computation, we get

$$\frac{d}{dt} Z_{i_0}(t_0) - \delta^2 Z_{i_0}(t_0) - pU_{i_0}^{p-1}(t_0) Z_{i_0}(t_0) > 0, \ 0 \le i_0 \le I-1,$$
$$\frac{d}{dt} Z_I(t_0) - \delta^2 Z_I(t_0) - (\frac{2}{h} q U_I^{-q-1}(t_0) + pU_I^{p-1}(t_0)) Z_I(t_0) > 0, \ i_0 = I.$$

But these inequalities contradict (2.1)-(2.3) and the proof is complete. □

Lemma 5.2. *Let $U_h \in \mathbb{R}^{I+1}$ be such that $U_h > 0$. Then we have*

$$\delta^2(U_i^{-q}) \ge -qU_i^{-q-1}\delta^2 U_i, \quad 0 \le i \le I.$$

Proof. Let us introduce function $f(s) = s^{-q}$. Using Taylor's expansion we get

$$\delta^2 f(U_0) = f'(U_0)\delta^2 U_0 + \frac{(U_1 - U_0)^2}{h^2} f''(\zeta_0),$$
$$\delta^2 f(U_I) = f'(U_I)\delta^2 U_I + \frac{(U_{I-1} - U_I)^2}{h^2} f''(\zeta_I),$$
$$\delta^2 f(U_i) = f'(U_i)\delta^2 U_i + \frac{(U_{i+1} - U_i)^2}{2h^2} f''(\eta_i) + \frac{(U_{i-1} - U_i)^2}{2h^2} f''(\zeta_i), \ 1 \le i \le I-1.$$

where η_i is an intermediate value between U_i and U_{i+1} and ζ_i the one between U_i and U_{i-1}. The result follows taking into account the fact that U_h is nonnegative. □

Define the functional J and its approximation by

$$J(t) = \int_0^1 u(x,t)dx, \quad t \in [0, T_h^q); \quad \text{and } J_h(t) = \sum_{i=0}^{I} hU_i(t), \quad t \in [0, T_q^h).$$

We can easily check that $\lim_{t\to T^q}\dfrac{d}{dt}J[u](t) = -\infty$. For this J we obtained after integration by parts

$$\frac{d}{dt}J[u] = \int_0^1 u_t dx = -u^{-q}(1,t) + \int_0^1 u^p dx.$$

Notice that the condition (5.1) implies $u_t \leq 0$. Let:
$0 < \zeta = \displaystyle\int_0^1 u_0^p(x)dx/u^{-q}(1,0) < 1$, which gives

$$\frac{d}{dt}J[u] \leq -u^{-q}(1,t) + \zeta u^{-q}(1,0) \leq (\zeta - 1)u^{-q}(1,t).$$

Since quenching occur only on the boundary (see [10]), we have $\lim_{t\to T_q} u^{-q}(1,t) = \infty$.

Let us assume that the initial data at (2.4) satisfies (5.2), by a simple computation, we obtain for $t \in [0, T_h^q)$, $p > 0$ and $q > 0$

$$\frac{d^2}{dt^2}J_h(t) = qU_I^{-q-1}(t)\frac{d}{dt}U_I(t) + p\sum_{i=0}^{I} hU_i^{p-1}(t)\frac{d}{dt}U_i(t)\,.$$

From lemma (5.1), we have $\frac{d^2}{dt^2}J_h(t) \leq -c\frac{d}{dt}J_h(t)$ where c is a non-negative constant and $\frac{d}{dt}J_h(t) = \displaystyle\sum_{i=0}^{I} h\frac{dU_i}{dt}$.

Using Theorem 3.1 we obtain the following result , see [17].

Theorem 5.1. *Assume that the solution u of (1.1) quenches in a finite time T_q such that $u \in C^{4,1}([0,1]\times[0,T_q))$. We also assume that the initial data at (2.4) satisfies the condition (5.2) and the error of initial data is of order $\circ(1)$. Then for h small enough, the solution U_h of problem (2.1)-(2.4) quenches in finite time T_q^h for $p > 0$, $q > 0$ and we have $\lim_{h\to\infty} T_q^h = T_q$.*

Theorem 5.2. *Assume that the hypotheses of Theorem 5.1 remains true. We also assume that the condition*

$$f(s)g'(s) - f'(s)g(s) \geq 0 \quad \textit{for} \quad s \geq 0\,, \tag{5.3}$$

holds. Then, near the quenching time T_q^h, the solution U_h of problem (2.1)-(2.4) has a following quenching rate estimate

$$\|U_h(t)\|_{inf} \sim (T_q^h - t)^{\frac{1}{q+1}}, \tag{5.4}$$

in the sense that there exist two positive constants K_1 and K_2, such that

$$K_1(T_q^h - t)^{\frac{1}{q+1}} \leq \|U_h(t)\|_{inf} \leq K_2(T_q^h - t)^{\frac{1}{q+1}} \quad \textit{for} \quad t \in (0, T_q^h)$$

Proof. Introduce the vector $J_h(t)$ defined as follow

$$J_i = \frac{dU_i}{dt}, \quad 0 \leq i \leq I-1, \quad \text{and} \quad J_I = \frac{dU_I}{dt} + \varepsilon U_I^{-q}, \tag{5.5}$$

where ε is a positive constant. By a straightforward computation we get

$$\frac{dJ_i}{dt} - \delta^2 J_i = \frac{d}{dt}(\frac{dU_i}{dt} - \delta^2 U_i), \quad 0 \leq i \leq I-1\,,$$
$$\frac{dJ_I}{dt} - \delta^2 J_I = \frac{d}{dt}(\frac{dU_I}{dt} - \delta^2 U_I) - \varepsilon q U_I^{-q-1}\frac{dU_I}{dt} - \varepsilon \delta^2 U_I^{-q}.$$

Using Lemma 5.2 and the equality (5.5), from condition (5.3) we obtain

$$\frac{dJ_i}{dt} - \delta^2 J_i = pU_i^{p-1}J_i, \quad 0 \leq i \leq I-1\,,$$
$$\frac{dJ_I}{dt} - \delta^2 J_I = (\frac{2q}{h}U_I^{-q-1} + pU_I^{p-1})J_I.$$

From (5.2), we observe that $J_i(0) \leq 0$ for $0 \leq i \leq I$ if ε is sufficiently small. It follows from lemma 3.1 that $J_i(t) \leq 0$ for $0 \leq i \leq I$, $t \in (0, T_q^h)$, which implies that $\frac{dU_I}{dt} + \varepsilon U_I^{-q} \leq 0 \quad$ for $\quad t \in (0, T_q^h)$. Thanks to Lemma 3.4, $U_I(t) = \|U_h(t)\|_{inf}$ and we obtain the desired lower bound. □

The following result concerns the upper bound for the quenching rate.

Let i_0 be such that $U_{i_0}(t) = \min_{0 \leq i \leq I} U_i(t)$. It is not difficult to see that

$$\delta^2 U_{i_0}(t) = \frac{U_{i_0+1}(t) - 2U_{i_0}(t) + U_{i_0-1}(t)}{h^2} \geq 0, \quad 1 \leq i_0 \leq I\,,$$
$$\delta^2 U_0(t) = \frac{2U_1(t) - 2U_0(t)}{h^2} \geq 0, \quad i_0 = 0\,,$$
$$\delta^2 U_I(t) = \frac{2U_{I-1}(t) - 2U_I(t)}{h^2} \geq 0, \quad i_0 = I.$$

We can see that $\dfrac{dU_I(t)}{dt} \geq -\dfrac{2}{h}U_I^{-q}$. Integrating this inequality over (t, T_q^h) and we obtain the result desired.

Remark 5.1. *Let us point out that the quenching rate for the numerical scheme, $(T_q^h - t)^{\frac{1}{q+1}}$, is different from the continuous one, $(T_q - t)^{\frac{1}{2(q+1)}}$, see [10].*

6. Numerical experiments

In the section, we present some numerical approximations to the blow-up and quenching time of problem (1.2), we also discuss to the blow-up and quenching sets. We obtain such numerical approximations by integrating numerically the semidiscrete problem (2.1)-(2.4) using the method presented by Hirota and Ozawa [4]. This method is to transform the ODE into a tractable form by the **arc length transformation technique** [S. Moriguti, C. Okuno, R. Suekane, M. Iri, K. Takeuchi, Ikiteiru Suugaku - Suuri Kougaku no Hatten (in Japanese), Baifukan, Tokyo, 1979.] and to generate a linearly convergent sequence to the blow-up time. The resulting sequence is accelerated by the Aitken Δ^2 method. We use the DOP54 [3] as the adaptive code for the integration of the ODEs. In the following tables, in rows, we present the numerical blow-up and quenching times , the steps, the orders of the approximations and the rates corresponding to meshes of 16, 32, 64, 128, 256 and 512. The order (s) of the method is computed from

$$\frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}.$$

Parameters $InitialStep$, $AbsTol$ and $RelTol$ in DOP54 [3] are set like this $InitialStep = 0$ and $AbsTol = RelTol = 1.d - 15$.

The first set of experiments were performed for the quenching in the case where $u_0(x) = \varepsilon^{-1/q} - \frac{\varepsilon}{2}x^2 + \frac{\varepsilon}{2}$ with $0 < p \leq 1$, $0 < q \leq 1$, $\varepsilon = 1.5$. Let us define the sequence s_l by $s_l = 2^3.2^l$ $(l = 0, \ldots, 10)$.

Table 1. Convergence behaviour of T^n to the quenching time T for $p = 1/2, q = 1$

I	T^n	n	s	p_l
16	0.15911484	2942	...	0.5
32	0.15682314	5609	...	0.5
64	0.15610036	10754	1.66	0.5
128	0.15588244	20734	1.73	0.5
256	0.15581869	40569	1.77	0.5
512	0.15580044	84323	1.80	0.5

Table 2. Convergence behaviour of T^n to the quenching time T for $p = 1/2, q = 1/2$

I	T^n	n	s	p_l
16	0.16100315	3432	...	0.66
32	0.15927478	6701	...	0.66
64	0.15871679	13000	1.63	0.66
128	0.15854545	25160	1.7	0.66
256	0.15849462	49111	1.75	0.66
512	0.15847990	99246	1.79	0.66

Table 3. Convergence behaviour of T^n to the quenching time T for $p = 1, q = 1/2$

I	T^n	n	s	p_l
16	0.14332827	3257	...	0.66
32	0.14167987	6357	...	0.66
64	0.14114245	12332	1.62	0.66
128	0.14097635	23855	1.69	0.66
256	0.14092684	46492	1.75	0.66
512	0.14091245	93327	1.78	0.66

Table 4. Convergence behaviour of T^n to the quenching time T for $p = 1, q = 1$

I	T^n	n	s	p_l
16	0.15309520	2893	...	0.5
32	0.15091834	5526	...	0.5
64	0.15022440	10602	1.65	0.5
128	0.15001371	20439	1.72	0.5
256	0.14995176	39964	1.77	0.5
512	0.14993396	82561	1.80	0.5

Remark 6.1. *The tables 1-4 show the convergence of T^n to the quenching time of the solution of (1.1) when the condition 5.2 is satisfied, since the rate of convergence is near 2, which is just the accuracy of the difference approximation in space. Moreover, the estimated quenching rate converges steadily to that given by (5.4). We also observe relationship between the quenching time and the flow on the boundary and the absorption on the one hand and in the interior of the domain on the other hand. In fact, when the flow on the boundary is constant $q = 1/2$ and that the absorption in the interior of the domain increases by $1/2$ to 1, the quenching time decreases from 0.16 to 0.14 whereas when the absorption in the interior of the domain is constant ($q = 1/2$) and that the flow on the boundary increases from $1/2$ to 1, the quenching time remains substantially the same at 0.16. The absorption in the interior of the domain slows down the quenching.*

Next, we give some plots to illustrate our analysis. In the figures below we have used the case where $I = 64$. We can observe from figures 1-4 that the semidiscrete solution quenches in a finite time at the last node, which is well known in a theoretical point of view ([10]).

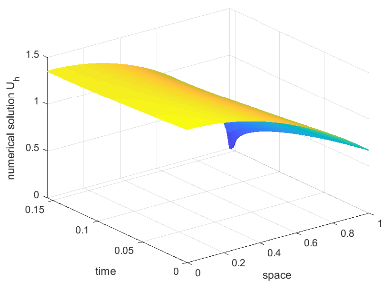

Fig. 1. Evolution of the semidiscrete solution for $p = 0.5, q = 1$

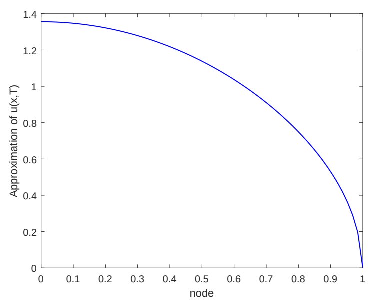

Fig. 2. Profile of the approximation of $u(x, T)$ for $p = 0.5, q = 1$

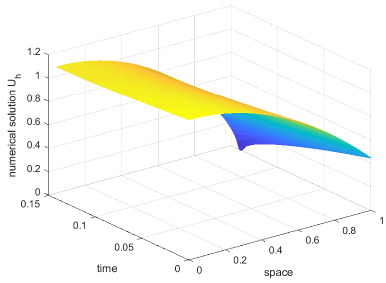

Fig. 3. Evolution of the semidiscrete solution for $p = 1$, $q = 0.5$

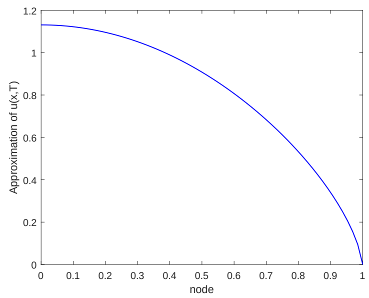

Fig. 4. Profile of the approximation of $u(x, T)$ for $p = 1$, $q = 0.5$

Now we consider blow-up for the two following cases. Firstly, we use $u_0 = \varepsilon^{-1/q} - \frac{\varepsilon}{2}x^2 + \frac{\varepsilon}{2}$, with $p = 1.1$, $q = 0.9$, $\varepsilon = 0.5$, $s_l = 2^{118}.2^l$ $(l = 0, \dots, 10)$ and secondly, $u_0 = \varepsilon^{-1/q} - \frac{\varepsilon}{2}x^2 + \frac{\varepsilon}{2}$, $p = 1.5$, $q = 0.5$, $\varepsilon = 0.5$, $s_l = 2^{52}.2^l$ $(l = 0, \dots, 10)$.

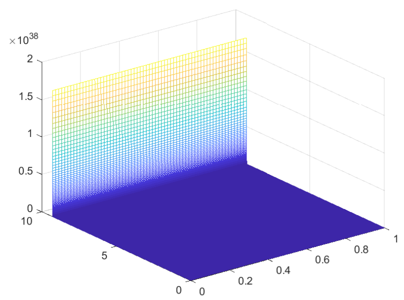

Fig. 5. Evolution of the semidiscrete solution for global blow-up

Table 5. Convergence behaviour of T^n to the blow-up time T for global blow-up

I	T^n	n	s	p_l
16	9.28403723	14086	...	10
32	9.28397671	26522	...	10
64	9.28396158	64409	2.0	10
128	9.28395780	197898	2.0	10
256	9.28395685	737989	2.0	10

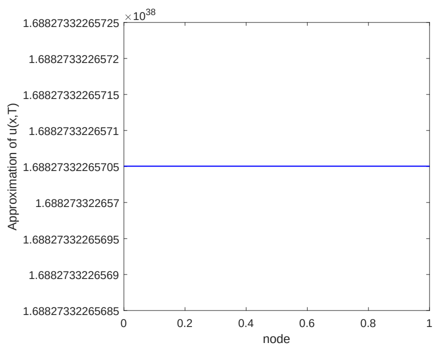

Fig. 6. Profile of the approximation of $u(x, T)$ for global blow-up

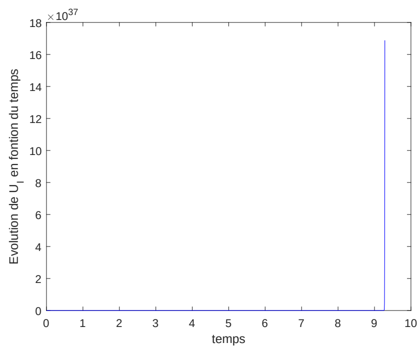

Fig. 7. Evolution of semi-discret solution for global blow-up

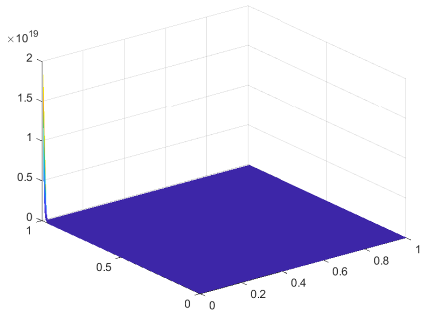

Fig. 8. Evolution of the semidiscrete solution for blow-up inside

Table 6. Convergence behaviour of T^n to the blow-up time T for blow-up inside

I	T^n	n	s	p_l
16	0.99160415	9761	...	2
32	0.99158941	13431	...	2
64	0.99158572	20603	2.0	2
128	0.99158480	35780	2.0	2
256	0.99158457	86308	2.0	2

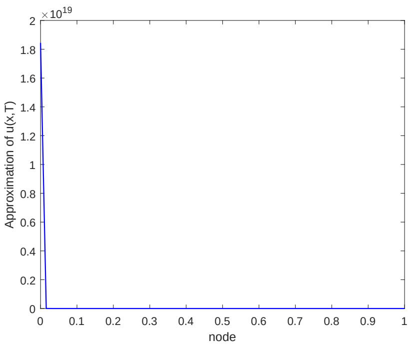

Fig. 9. Profile of the approximation of $u(x, T)$ for blow-up inside

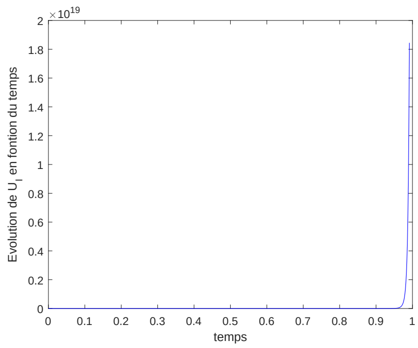

Fig. 10. Evolution of semi-discret solution for blow-up inside

Remark 6.2. *As explained in remark 6.1, we see from tables 5 and 6 that* T^h *converges to the continuous one when condition* (4.2) *is satisfied. From figures 5-9 we can appreciate that blow-up can occur inside the domain, or in the whole interval for different values of* u_0*, which is in agreement with the theoretical results in [10]. We also observe that the estimated blow-up rate converges steadily to that given by (4.5).*

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