

STRONGLY EDGE MULTIPLICATIVE GRAPHS

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ABSTRACT. A graph $G = (V, E)$ with p vertices and q edges is said to be strongly edge multiplicative if the edges of G can be labeled with distinct integers from $\{1, 2, 3, \dots, q\}$ such that labels induced on the vertices obtained by the product of the labels of incident edges are distinct. In this paper, we introduce strongly edge multiplicative graphs and we prove that some families of graphs are strongly edge multiplicative.

1. INTRODUCTION

Graph labellings, where the vertices are assigned values subject to certain conditions, have often been motivated by practical problems, but they are also of interest in their own right. An enormous body of literature has grown around the subject, especially in the last thirty years or so, and even to mention the variety of problems that have been studied would take us too far afield here.

Most interesting graph labelling problems have three ingredients:

- (i) a set of numbers S from which vertex labels are chosen;
- (ii) a rule that assigns a value to each edge;
- (iii) a condition that these values must satisfy.

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Arguably two of the most interesting labelling problems are gracefulnes and harmoniousness. Graceful labellings were introduced under the guise of β -valuations by Rosa [8], and much of their original interest lay in their connection with decompositions of complete graphs, in particular, into trees. (See Bloom [4] for a discussion of this topic.) In a graceful labelling of a graph with q edges, the labels are chosen as distinct values from $\{0, 1, \dots, q\}$ each edge is given the absolute value of the labels on its vertices, and the requirement is that all edge labels be different. Harmonious labellings were introduced by Graham and Sloane [7] and have connections with error-correcting codes. In a harmonious labelling, the vertices have distinct values from $\{1, 2, \dots, q\}$ an edge is given the sum modulo q of the labels on its vertices, and, again, all edge labels must be different. Gallian [6] has written an extensive survey, updated periodically, in which results on many variations of these two types of labelling are compiled.

Before working on labelling problems, readers would be well advised to consult Gallian's work. (Note that it does not consider another important labelling problem, the so-called band-width problem. For a survey of this, see Chung [5].)

In 2001, Beineke and Hegde introduced multiplicative labeling and proved that every graph admits a multiplicative labeling. They defined the concept of strongly multiplicative graphs as follows: A graph with p vertices is said to be strongly multiplicative if its vertices can be labeled $1, 2, \dots, p$ so that values on the edges obtained as the product of the labels of their end vertices, are all distinct. They showed that all graphs in some classes including all trees are strongly multiplicative and considered the question of the maximum number of edges in a strongly multiplicative graph of a given order. For more results on strongly multiplicative graphs, please refer [1–3, 9–12].

In this paper, we consider a labelling that has much the same flavor as graceful and harmonious labellings in its simplicity of definition and its requirement that all of the edge labels be different. However, it uses products rather than sums or differences. The property, which we call “strong edge multiplicativity” (not a catchy name) is this: *Can the edges of a graph be labelled $\{1, 2, \dots, q\}$ in such a way that the resulting products on the vertices are all different?*

After giving formal definitions and some elementary general results in Section 2, we turn in Section 3 to proving that the graphs in certain families are strongly edge multiplicative.

2. RESULTS

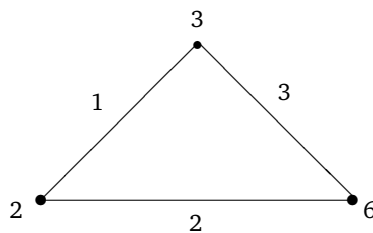
Definition 2.1. A graph $G = (V, E)$ with p vertices and q edges is said to be strongly edge multiplicative if the edges of G can be labeled with distinct integers from $\{1, 2, 3, \dots, q\}$ such that $f(u) = \prod_{v \in N(u)} f(uv)$ for all $u \in V(G)$, are distinct.

Definition 2.2. The friendship graph is the one point union of n copies of cycle C_3 . It is denoted by F_n .

Definition 2.3. An n -sun graph S_n is a cycle C_n with an edge terminating in a vertex of degree one attached to each vertex.

Definition 2.4. The disjoint union of the graphs G and H , denoted $G \cup H$, is the graph obtained by taking the union of G and H on disjoint vertex sets, $V(G)$ and $V(H)$.

Example 1. A spanning subgraph of a strongly edge multiplicative graph need not be strongly edge multiplicative. For example, take a strongly edge multiplicative labeling of C_3 (see Figure 1). Then, a spanning subgraph of C_3 is not a strongly edge multiplicative (see Figure 2).



□

FIGURE 1. Strongly edge multiplicative labeling of C_3

Theorem 2.1. For $n = 1, 2, 3$, the m copies of path mP_n , is not strongly edge multiplicative for any $m \geq 1$.

Proof. Assume $m \geq 1$. If $n = 1$, then $G \cong mK_1$, which is not strongly edge multiplicative. If $n = 2$, then $G \cong mP_2$. Suppose mP_2 is strongly edge multiplicative. Then there exists a mapping $f : E(mP_2) \rightarrow \{1, 2, 3, \dots, m\}$ such

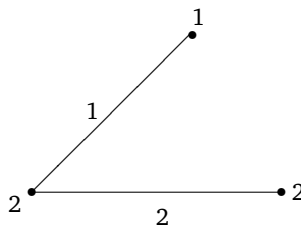


FIGURE 2. Not a Strongly edge multiplicative labeling of spanning subgraph of C_3

that $f(xy) = 1$ for some $xy \in E(mP_2)$. Since xy is an edge, $f(xy) = f(x) = f(y) = 1$. This is contradiction to all the vertex labels are distinct. If $n = 3$, then $G \cong mP_3$. Suppose mP_3 is strongly edge multiplicative. Then there exists a mapping $g : E(mP_3) \rightarrow \{1, 2, \dots, 2m\}$ such that $g(uv) = 1$ for some $uv \in E(mP_3)$. Without loss of generality, we assume that $v \in V(mP_3)$ is adjacent to some $w \in V(mP_3)$. Since $vw \in E(mP_3)$, there exists $n \in \{2, 3, \dots, 2m\}$ such that $g(vw) = n$. Now, we can easily show that $g(v) = g(w) = n$. This is contradiction to all the vertex labels are distinct. Hence the graph $G \cong mP_n$ for $n = 1, 2, 3$ is not strongly edge multiplicative for any $m \geq 1$. $\square$

Theorem 2.2. *For any graph G , the graph $P_2 \cup G$ is not strongly edge multiplicative.*

Proof. Let G be any graph. Suppose $P_2 \cup G$ is strongly edge multiplicative. Then there exists a mapping $f : E(P_2 \cup G) \rightarrow \{1, 2, \dots, q\}$ such that $f(e_i) = x$ for $e_i \in E(P_2)$, $1 \leq x \leq q$. If $v_1, v_2 \in V(P_2)$, then $f(v_1) = f(v_2) = x$. This is contradiction to all vertex labels are distinct. Hence $P_2 \cup G$ is not strongly edge multiplicative. $\square$

Theorem 2.3. *Any graph G with isolated vertex is not a strongly edge multiplicative.*

Proof. It follows from the definition 2.1 of strongly edge multiplicative. $\square$

3. FAMILIES OF STRONGLY EDGE MULTIPLICATIVE GRAPHS

In this section, we consider the question of whether the graphs in certain well known and much studied families are strongly edge multiplicative. We show

that all graphs in some classes, including all cycles, trees, complete graphs, wheels and suns are strongly edge multiplicative.

Theorem 3.1. *The path $P_n (n \geq 4)$ is strongly edge multiplicative.*

Proof. Let us consider P_n for $n \geq 4$. Define $f : E(P_n) \rightarrow \{1, 2, \dots, n-1\}$ as follows:

$$\begin{aligned} f(u_i u_{i+1}) &= i, \quad 1 \leq i \leq n-1 \\ f(u_{n-1} u_n) &= n-1. \end{aligned}$$

The induced vertex labels are:

$$\begin{aligned} f(u_i) &= i, \quad 1 \leq i \leq 2 \\ f(u_n) &= n-1 \\ f(u_i) &= i(i-1), \quad 3 \leq i \leq n-1. \end{aligned}$$

Clearly $f(u_i) = \prod_{u_j \in N(u_i)} f(u_i u_j)$ for all $u_i \in V(P_n)$, are distinct.

Hence P_n is strongly edge multiplicative for $n \geq 4$.

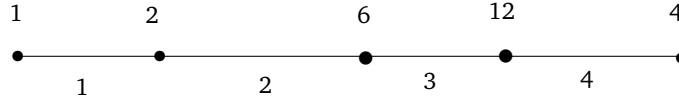


FIGURE 3. Strongly edge multiplicative labeling of P_5

□

Theorem 3.2. *The cycle $C_n (n \geq 3)$ is strongly edge multiplicative.*

Proof. Case (i). Assume n is odd. Define $f : E(C_n) \rightarrow \{1, 2, \dots, n\}$ as follows:

$$\begin{aligned} f(u_{2i-1} u_{2i}) &= i, \quad 1 \leq i \leq \frac{n-1}{2} \\ f(u_n u_1) &= \frac{n+1}{2} \\ f(u_{2i} u_{2i+1}) &= i+1 + \lfloor \frac{n}{2} \rfloor, \quad 1 \leq i < \frac{n}{2}. \end{aligned}$$

Now one can easily check that vertex labels obtained by $f(u_i) = \prod_{u_j \in N(u_i)} f(u_i u_j)$ for all $u_i \in V(C_n)$, are distinct.

Case(ii). Assume n is even. Define $f : E(C_n) \rightarrow \{1, 2, 3, \dots, n\}$ as follows:

$$\begin{aligned} f(u_{2i-1}u_{2i}) &= i, \quad 1 \leq i \leq \frac{n}{2}, \\ f(u_n u_1) &= n \\ f(u_{2i}u_{2i+1}) &= i + \frac{n}{2}, \quad 1 \leq i \leq \frac{n}{2} - 1. \end{aligned}$$

Now one can easily check that vertex labels obtained by $f(u_i) = \prod_{u_j \in N(u_i)} f(u_i u_j)$ for all $u_i \in V(C_n)$, are distinct. $\square$

Theorem 3.3. *The complete graph K_n ($n \geq 3$) is strongly edge multiplicative.*

Proof. First, we assign t internal edges of K_n from $\{1, 2, \dots, t\}$ where $t < q$. Next, we assign $q - t$ external edges of K_n from $\{t + 1, t + 2, \dots, q\}$. Finally, we can easily check that vertex labels obtained by the product of the labels of incident edges are distinct and hence the complete graph K_n is strongly edge multiplicative for $n \geq 3$. $\square$

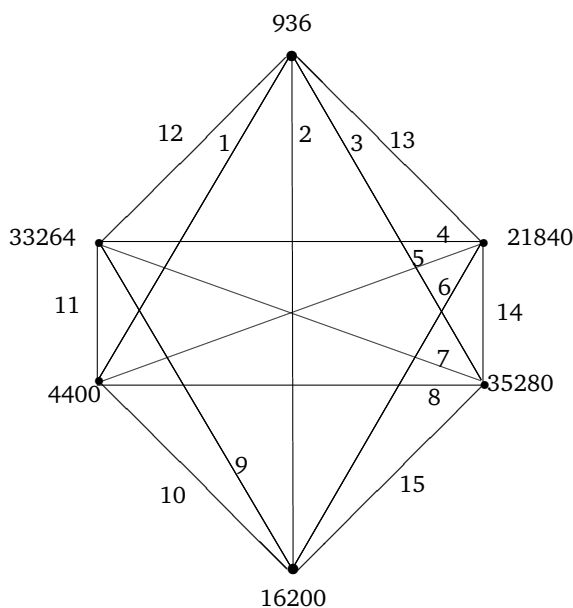


FIGURE 4. Strongly edge multiplicative labeling of K_6 with 9 internal edges and 6 external edges.

Theorem 3.4. *Every tree other than P_n for $n = 1, 2, 3$, has a strongly edge multiplicative labeling.*

Proof. Let T be a tree. If $T \not\cong P_2$ (or) $T \cong P_3$, then by Theorem 2.1, T is not strongly edge multiplicative.

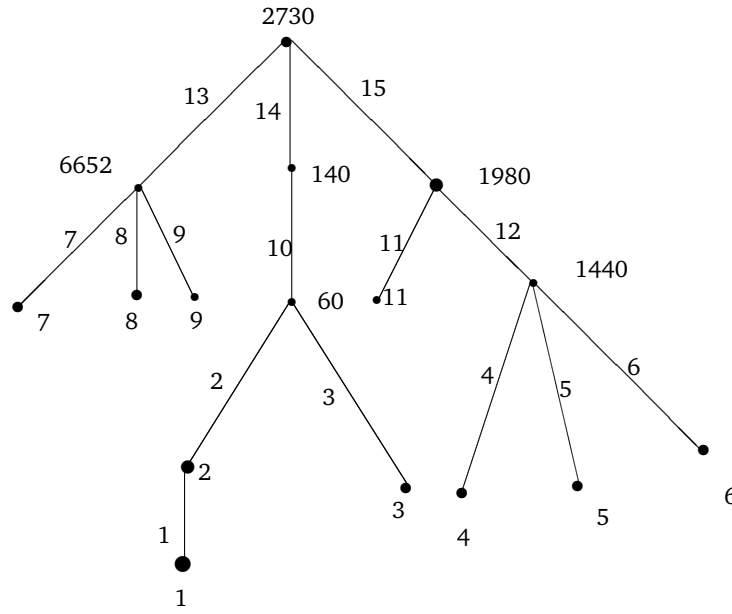


FIGURE 5. Strongly edge multiplicative labeling of a tree

Assign labels $1, 2, \dots, t$ to the pendant edges of T . Hence the pendent vertices receives $1, 2, 3, \dots, t$ and they are distinct. To see that this labeling is strongly edge multiplicative, let u and v be two vertices and assume that the incident edges of u are labeled as l, m with $l < m$ and the incident edges of v are labeled as n, s with $n < s$. Without loss of generality assume also that $l < m < n < s$. From the property of the labeling, it follows that $lm < ns$. Hence all of the vertex labels are distinct. $\square$

Theorem 3.5. Every wheel graph W_{n+1} for $n \geq 5$, is strongly edge multiplicative

Proof. We let $n \geq 5$. Consider the wheel W_{n+1} whose rim is the cycle $u_2u_3, u_3u_4, \dots, u_{n+1}u_2$ and whose center is the vertex u_1 .

Define $f : E(G) \rightarrow \{1, 2, 3, \dots, 2n\}$ as follows:

$$\begin{aligned} f(u_1 u_i) &= i - 1, \quad 2 \leq i \leq n \\ f(u_i u_{i+1}) &= n + i - 1, \quad 2 \leq i \leq n \\ f(u_{n+1} u_2) &= 2n. \end{aligned}$$

Now one can easily verify that vertex labels obtained by $f(u_i) = \prod_{u_j \in N(u_i)} f(u_i u_j)$ for all $u_i \in V(W_{n+1})$, are distinct. Hence every wheel W_{n+1} for $n \geq 5$, is strongly edge multiplicative.

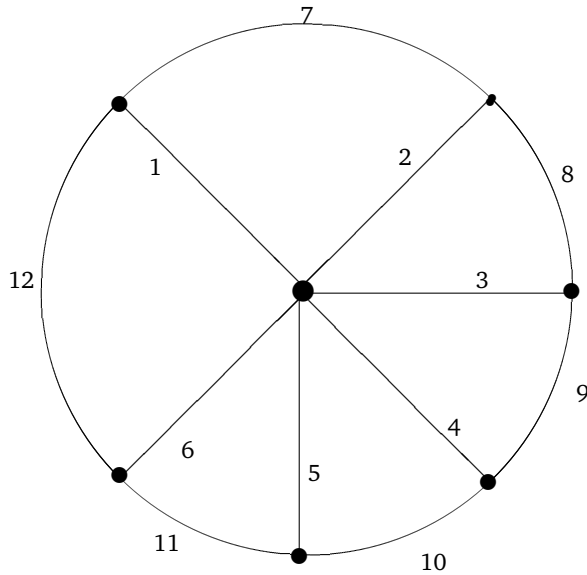


FIGURE 6. Strongly edge multiplicative labeling of W_7

□

Theorem 3.6. For any two positive integers $m, n \geq 2$, the complete bipartite graph $K_{m,n}$ is strongly edge multiplicative.

Proof. Assume $m, n \geq 2$ be any two positive integers.

Let $V(K_{m,n}) = \{u_1, u_2, \dots, u_m\} \cup \{u_{m+1}, u_{m+2}, \dots, u_{m+n}\}$ and $E(K_{m,n}) = \{u_j u_{m+i} : 1 \leq i \leq n, 1 \leq j \leq m\}$. Define $f : E(K_{m,n}) \rightarrow \{1, 2, 3, \dots, mn\}$ as $f(u_j u_{m+i}) =$

$(j-1)n+i$, $1 \leq i \leq n$ and $1 \leq j \leq m$.

Now one can easily check that vertex labels obtained by $f(u_i) = \prod_{u_j \in N(u_i)} f(u_i u_j)$ for all $u_i \in V(K_{m,n})$, are distinct. Hence $K_{m,n}$ is strongly edge multiplicative for any positive integers $m \geq 2$ and $n \geq 2$. $\square$

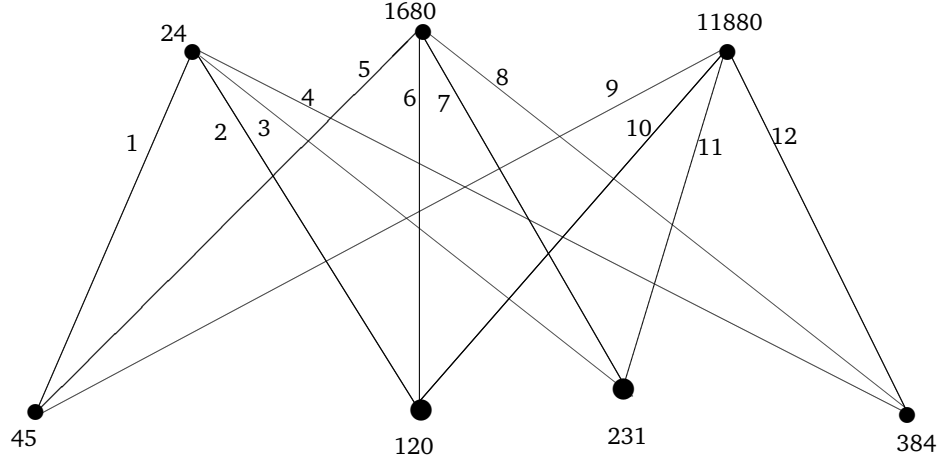


FIGURE 7. Strongly edge multiplicative labeling $K_{3,4}$

Theorem 3.7. *The star graph $K_{1,n}$ is strongly edge multiplicative for all $n \geq 3$.*

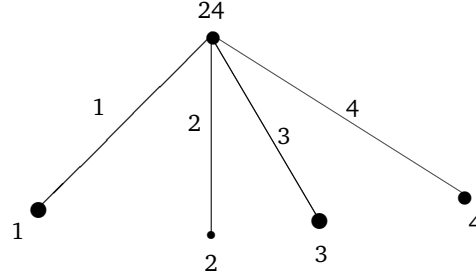
Proof. Consider the star graph $K_{1,n}$ for $n \geq 3$. Denote the center of $K_{1,n}$ by v_1 , the leaves by $v_2, v_3, \dots, v_n$.

Define $f : E(K_{1,n}) \rightarrow \{1, 2, 3, \dots, n\}$ as follows:

$$\begin{aligned} f(v_1 v_i) &= i, \quad 2 \leq i \leq n \\ f(v_1) &= n! \\ f(v_i) &= i-1, \quad 2 \leq i \leq n. \end{aligned}$$

Here, the induced vertex labels obtained by $f(v_i) = \prod_{v_j \in N(v_i)} f(v_i v_j)$ for all $v_i \in V(K_{1,n})$, are distinct. Hence $K_{1,n}$ is strongly edge multiplicative for all $n \geq 3$. $\square$

Theorem 3.8. *For all $n \geq 3$, the n -sun graph S_n is strongly edge multiplicative.*

FIGURE 8. Strongly edge multiplicative labeling of $K_{1,4}$

Proof. Let S_n be the n -sun graph on $2n$ vertices.

Let $V(S_n) = \{u_1, u_2, \dots, u_n\} \cup \{u_{n+1}, u_{n+2}, \dots, u_{2n}\}$ and

$E(S_n) = \{u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{u_n u_1\} \cup \{u_i u_{i+n} : 1 \leq i \leq n\}$.

Define $f : E(S_n) \rightarrow \{1, 2, \dots, 2n\}$ as follows:

$$f(u_i u_{i+1}) = i, \quad 1 \leq i \leq n-1$$

$$f(u_n u_1) = n,$$

$$f(u_i u_{i+n}) = i + n, \quad 1 \leq i \leq n.$$

Now, we can easily verify that vertex labels obtained by $f(u_i) = \prod_{u_j \in N(u_i)} f(u_i u_j)$ for all $u_i \in V(S_n)$, Hence the n -sun graph S_n is strongly edge multiplicative for all $n \geq 3$. □

Theorem 3.9. *The friendship graph F_n is strongly edge multiplicative for any positive integer n*

Proof. Assume n be any positive integer. Let F_n be the friendship graph. Let v_0 be the common vertex and $v_1, v_2, \dots, v_{2n}$ be other vertices of F_n . We note that $|V(F_n)| = 2n + 1$ and $|E(F_n)| = 3n$. We define edge labeling $f : E(F_n) \rightarrow \{1, 2, \dots, 3n\}$ as follows:

$$f(v_0 v_i) = i, \quad 1 \leq i \leq 2n$$

$$f(v_{2i-1} v_{2i}) = 2n + i, \quad 1 \leq i \leq n.$$

The labeling pattern defined above covers all the properties and the graph F_n admits a strongly edge multiplicative labeling. Hence, the friendship graph F_n is strongly edge multiplicative for any positive integer n . □

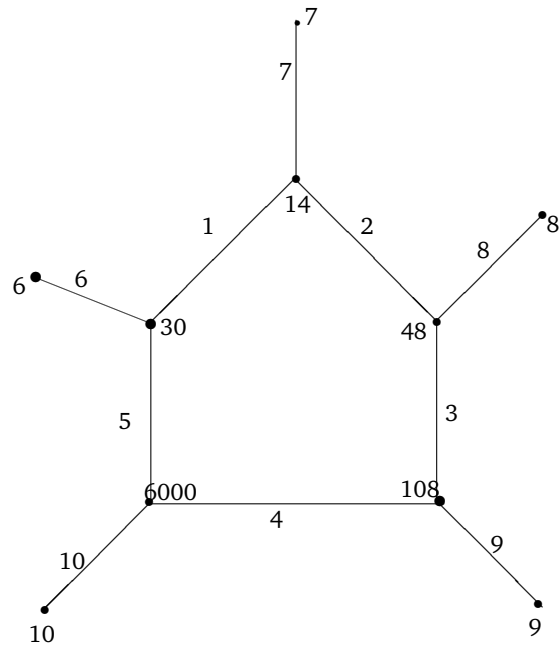


FIGURE 9. Strongly edge multiplicative labeling of S_5

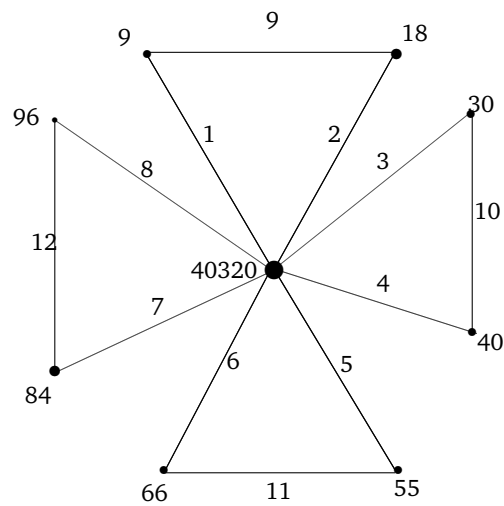


FIGURE 10. Strongly edge multiplicative labeling of friendship graph F_4

4. CONCLUSION

We introduced strongly edge multiplicative labeling and have shown the existence of the strongly edge multiplicative labeling of some well known families of graphs. Finally, we want to conclude the paper with the following two conjecture that we feel that are important.

Conjecture 1. *Every non trivial connected graph other than P_n for $n = 2, 3$, is strongly edge multiplicative.*

Conjecture 2. *Every disconnected graph other than the following graphs is strongly edge multiplicative.*

- (i) mP_n for $n = 2, 3$, and $m \geq 2$.
- (ii) $P_2 \cup G$ for any graph G .

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