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A NEW SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY LINEAR OPERATOR

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ABSTRACT. In this work, we introduce and investigate a new class $k-\widetilde{U}S_s(a,c,\gamma,t)$ of analytic functions in the open unit disc U with negative coefficients. The object of the present paper is to determine coefficient estimates, neighborhoods and partial sums for functions f belonging to this class.

1. INTRODUCTION

Let A denote the class of analytic functions f defined on the unit disk $U = \{z : |z| < 1\}$ with normalization $f(0) = 0$ and $f'(0) = 1$. Such a function has the Taylor series expansion about the origin in the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

denoted by S, the subclass of A consisting of functions that are univalent in U.

For $f \in A$ given by (1.1) and $g(z)$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

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their convolution (or Hadamard product), denoted by $(f * g)$, is defined as

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z) \ (z \in U). \tag{1.2}$$

Note that $f * g \in A$.

A function $f \in A$ is said to be in $k - US(\gamma)$, the class of $k-$uniformly starlike functions of order $\gamma, 0 \leq \gamma < 1$, if satisfies the condition

$$Re \left\{ \frac{zf'(z)}{f(z)} \right\} > k \left| \frac{zf'(z)}{f(z)} - 1 \right| + \gamma, \ (k \geq 0),$$

and a function $f \in A$ is said to be in $k - UC(\gamma)$, the class of $k-$uniformly convex functions of order $\gamma, 0 \leq \gamma < 1$, if satisfies the condition

$$Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > k \left| \frac{zf''(z)}{f'(z)} \right| + \gamma, \ (k \geq 0).$$

Uniformly starlike and uniformly convex functions were first introduced by Goodman [5] and studied by Ronning [7]. It is known that $f \in k - UC(\gamma)$ or $f \in k - US(\gamma)$ if and only if $1 + \frac{zf''(z)}{f'(z)}$ or $\frac{zf'(z)}{f(z)}$, respectively, takes all the values in the conic domain $\mathcal{R}_{k,\gamma}$ which is included in the right half plane given by

$$\mathcal{R}_{k,\gamma} = \left\{ w = u + iv \in C : u > k\sqrt{(u-1)^2 + v^2} + \gamma, \ k \geq 0 \text{ and } \gamma \in [0, 1) \right\}.$$

Denote by $\mathcal{P}(P_{k,\gamma}), (k \geq 0, 0 \leq \gamma < 1)$ the family of functions p, such that $p \in \mathcal{P}$, where $\mathcal{P}$ denotes well-known class of Caratheodory functions. The function $P_{k,\gamma}$ maps the unit disk conformally onto the domain $\mathcal{R}_{k,\gamma}$ such that $1 \in \mathcal{R}_{k,\gamma}$ and $\partial\mathcal{R}_{k,\gamma}$ is a curve defined by the equality

$$\partial\mathcal{R}_{k,\gamma} = \left\{ w = u + iv \in C : u^2 = \left(k\sqrt{(u-1)^2 + v^2} + \gamma \right)^2, \ k \geq 0 \text{ and } \gamma \in [0, 1) \right\}. \tag{1.3}$$

From elementary computations we see that (1.3) represents conic sections symmetric about the real axis. Thus $\mathcal{R}_{k,\gamma}$ is an elliptic domain for $k > 1$, a parabolic domain for $k = 1$, a hyperbolic domain for $0 < k < 1$ and the right half plane $u > \gamma$, for $k = 0$.

In [9], Sakaguchi defined the class S_s of starlike functions with respect to symmetric points as follows:

Let $f \in A$. Then f is said to be starlike with respect to symmetric points in U if and only if

$$Re\left\{\frac{2zf'(z)}{f(z)-f(-z)}\right\} > 0,\ (z \in U).$$

Recently, Owa et al. [6] defined the class $S_s(\alpha, t)$ as follows:

$$Re\left\{\frac{(1-t)zf'(z)}{f(z)-f(tz)}\right\} > \alpha,\ (z \in U),$$

where $0 \le \alpha < 1, |t| \le 1, t \ne 1$. Note that $S_s(0,-1) = S_s$ and $S_s(\alpha,-1) = S_s(\alpha)$ is called Sakaguchi function of order α.

The function $\Psi(a,c;z)$ be the incomplete beta function defined by

$$\Psi(a,c;z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} z^n, c \ne 0,-1,-2,\cdots$$

where $(\lambda)_n$ is the Pochhammer symbol defined in terms of the Gamma function by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if } n = 0; \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & \text{if } n \in N \end{cases}.$$

Further, for $f \in A$,

$$L(a,c)f(z) = \Psi(a,c;z) * f(z) = z + \sum_{n=2}^{\infty} \phi(a,c,n) a_n z^n$$

where $\phi(a,c,n) = \dfrac{(a)_{n-1}}{(c)_{n-1}}$.

where $L(a,c)$ is called Carlson-Shaffer operator [3] and the operator $*$ stands for the Hadamard product (or convolution product) of two power series is given by (1.2).

We notice that $L(a,a)f(z) = f(z),\ L(2,1)f(z) = zf'(z)$.

Now, by making use of the linear operator $L(a,c)$, we define a new subclass of functions belonging to the class A.

Definition 1.1. *A function $f \in A$ is said to be in the class $k - US_s(a,c,\gamma,t)$ if for all $z \in U$*

$$Re\left\{\frac{(1-t)z\left(L(a,c)f(z)\right)'}{L(a,c)f(z)-L(a,c)f(tz)}\right\} \ge k\left|\frac{(1-t)z\left(L(a,c)f(z)\right)'}{L(a,c)f(z)-L(a,c)f(tz)} - 1\right| + \gamma,$$

for $k \ge 0, |t| \le 1, t \ne 1, 0 \le \gamma < 1$.

Furthermore, we say that a function $f \in k - US_s(a,c,\gamma,t)$ is in the subclass $k - \widetilde{U}S_s(a,c,\gamma,t)$ if $f(z)$ is of the following form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \; a_n \geq 0, n \in \mathbb{N}, \; z \in U. \tag{1.4}$$

The aim of the present paper is to study the coefficient bounds, partial sums and certain neighborhood results of the class $k - \widetilde{U}S_s(a,c,\gamma,t)$.

Firstly, we shall need the following lemmas.

Lemma 1.1. *Let $w = u + iv$. Then*

$$Re\ w \geq \alpha \text{ if and only if } |w-(1+\alpha)| \leq |w+(1-\alpha)|.$$

Lemma 1.2. *Let $w = u + iv$ and α, γ be real numbers. Then*

$$Re\ w > \alpha|w-1| + \gamma \text{ if and only if } Re\{w(1+\alpha e^{i\theta}) - \alpha e^{i\theta}\} > \gamma\,.$$

2. Coefficient bounds of the function class $k - \widetilde{U}S_s(a,c,\gamma,t)$

Theorem 2.1. *The function f defined by* (1.4) *is in the class $k - \widetilde{U}S_s(a,c,\gamma,t)$ if and only if*

$$\sum_{n=2}^{\infty} \phi(a,c,n)|n(k+1) - u_n(k+\gamma)|a_n \leq 1-\gamma, \tag{2.1}$$

where $k \geq 0, |t| \leq 1, t \neq 1, 0 \leq \gamma < 1$ and $u_n = 1 + t + \cdots + t^{n-1}$.
The result is sharp for the function $f(z)$ given by

$$f(z) = z - \frac{1-\gamma}{\phi(a,c,n)|n(k+1) - u_n(k+\gamma)|} z^n$$

Proof. By Definition 1.1, we get

$$Re\left\{\frac{(1-t)z\,(L(a,c)f(z))'}{L(a,c)f(z) - L(a,c)f(tz)}\right\} \geq k\left|\frac{(1-t)z\,(L(a,c)f(z))'}{L(a,c)f(z) - L(a,c)f(tz)} - 1\right| + \gamma.$$

Then by Lemma 1.2, we have

$$Re\left\{\frac{(1-t)z\,(L(a,c)f(z))'}{L(a,c)f(z) - L(a,c)f(tz)}(1+ke^{i\theta}) - ke^{i\theta}\right\} \geq \gamma, \; -\pi < \theta \leq \pi$$

or equivalently

(2.2)
$$Re\left\{\frac{(1-t)z\,(L(a,c)f(z))'\,(1+ke^{i\theta})}{L(a,c)f(z)-L(a,c)f(tz)}-\frac{ke^{i\theta}\,[L(a,c)f(z)-L(a,c)f(tz)]}{L(a,c)f(z)-L(a,c)f(tz)}\right\}\geq\gamma.$$

Let $F(z)=(1-t)z\,(L(a,c)f(z))'\,(1+ke^{i\theta})-ke^{i\theta}\,[L(a,c)f(z)-L(a,c)f(tz)]$ and $E(z)=L(a,c)f(z)-L(a,c)f(tz)$.

By Lemma 1.1, (2.2) is equivalent to

$$|F(z)+(1-\gamma)E(z)|\geq|F(z)-(1+\gamma)E(z)|,\ \text{for } 0\leq\gamma<1.$$

But:

$$\begin{aligned}
&|F(z)+(1-\gamma)E(z)|\\
&=\Big|(1-t)\big\{(2-\gamma)z-\sum_{n=2}^{\infty}\phi(a,c,n)(n+u_n(1-\gamma))a_nz^n-ke^{i\theta}\sum_{n=2}^{\infty}\phi(a,c,n)(n-u_n)a_nz^n\big\}\Big|\\
&\geq|1-t|\Big\{(2-\gamma)|z|-\sum_{n=2}^{\infty}\phi(a,c,n)|n+u_n(1-\gamma)|a_n|z^n|-k\sum_{n=2}^{\infty}\phi(a,c,n)|n-u_n|a_n|z^n|\Big\}.
\end{aligned}$$

Also:

$$\begin{aligned}
&|F(z)-(1+\gamma)E(z)|\\
&=\Big|(1-t)\big\{-\gamma z-\sum_{n=2}^{\infty}\phi(a,c,n)(n-u_n(1+\gamma))a_nz^n-ke^{i\theta}\sum_{n=2}^{\infty}\phi(a,c,n)(n-u_n)a_nz^n\big\}\Big|\\
&\leq|1-t|\Big\{\gamma|z|+\sum_{n=2}^{\infty}\phi(a,c,n)|n-u_n(1+\gamma)|a_n|z^n|+k\sum_{n=2}^{\infty}\phi(a,c,n)|n-u_n|a_n|z^n|\Big\}.
\end{aligned}$$

So, we have:

$$\begin{aligned}
&|F(z)+(1-\gamma)E(z)|-|F(z)-(1+\gamma)E(z)|\\
&\geq|1-t|\Big\{2(1-\gamma)|z|-\sum_{n=2}^{\infty}\phi(a,c,n)\big[|n+u_n(1-\gamma)|+|n-u_n(1+\gamma)|+2k|n-u_n|\ \big]a_n|z^n|\Big\}\\
&\geq 2(1-\gamma)|z|-\sum_{n=2}^{\infty}2\phi(a,c,n)\big|n(k+1)-u_n(k+\gamma)\big|a_n|z^n|\geq 0
\end{aligned}$$

or

$$\sum_{n=2}^{\infty}\phi(a,c,n)\big|n(k+1)-u_n(k+\gamma)\big|a_n\leq 1-\gamma.$$

Conversely, suppose that (2.1) holds. Then we must show

$$Re\left\{\frac{(1-t)z\,(L(a,c)f(z))'\,(1+ke^{i\theta})-ke^{i\theta}\,[L(a,c)f(z)-L(a,c)f(tz)]}{L(a,c)f(z)-L(a,c)f(tz)}\right\}\geq\gamma.$$

Upon choosing the values of z on the positive real axis where $0 \leq z = r < 1$, the above inequality reduces to

$$Re\left\{\frac{(1-\gamma)-\sum\limits_{n=2}^{\infty}\phi(a,c,n)[n(1+ke^{i\theta})-u_n(\gamma+ke^{i\theta})]a_nz^{n-1}}{1-\sum\limits_{n=2}^{\infty}\phi(a,c,n)u_na_nz^{n-1}}\right\}\geq 0.$$

Since $Re(-e^{i\theta}) \geq -|e^{i\theta}| = -1$, the above inequality reduces to

$$Re\left\{\frac{(1-\gamma)-\sum\limits_{n=2}^{\infty}\phi(a,c,n)[n(1+k)-u_n(\gamma+k]a_nr^{n-1}}{1-\sum\limits_{n=2}^{\infty}\phi(a,c,n)u_na_nr^{n-1}}\right\}\geq 0.$$

Letting $r \to 1^-$, we have desired conclusion. □

Corollary 2.1. *If $f(z) \in k-\widetilde{U}S_s(a,c,\gamma,t)$ then*

$$a_n \leq \frac{1-\gamma}{\phi(a,c,n)|n(k+1)-u_n(k+\gamma)|}$$

where $k \geq 0, |t| \leq 1, t \neq 1, 0 \leq \gamma < 1$ and $u_n = 1+t+\cdots+t^{n-1}$.

3. Neighborhood of the function class $k-\widetilde{U}S_s(a,c,\gamma,t)$

Following the earlier investigations (based upon the familiar concept of neighborhoods of analytic functions) by Goodman [4], Ruscheweyh [8], Altintas et al. [1,2].

Definition 3.1. *Let $k \geq 0, |t| \leq 1, t \neq 1, 0 \leq \gamma < 1, \alpha \geq 0$ and $u_n = 1+t+\cdots+t^{n-1}$. We define the $\alpha-$neighborhood of a function $f \in A$ and denote by $N_\alpha(f)$ consisting of all functions $g(z) = z-\sum\limits_{n=2}^{\infty} b_nz^n \in S(b_n \geq 0, n \in \mathbb{N})$ satisfying*

$$\sum_{n=2}^{\infty}\frac{\phi(a,c,n)|n(k+1)-u_n(k+\gamma)|}{1-\gamma}|a_n-b_n| \leq 1-\alpha.$$

Theorem 3.1. *Let $f(z) \in k - \widetilde{U}S_s(a, c, \gamma, t)$ and for all real θ we have $\gamma(e^{i\theta} - 1) - 2e^{i\theta} \neq 0$. For any complex number ϵ with $|\epsilon| < \alpha(\alpha \geq 0)$, if f satisfies the following condition:*

$$\frac{f(z) + \epsilon z}{1 + \epsilon} \in k - \widetilde{U}S_s(a, c, \gamma, t)$$

then $N_\alpha(f) \subset k - \widetilde{U}S_s(a, c, \gamma, t)$.

Proof. It is obvious that $f \in k - \widetilde{U}S_s(a, c, \gamma, t)$ if and only if

$$\left| \frac{(1-t)z\,(L(a,c)f(z))'\,(1+ke^{i\theta}) - (ke^{i\theta}+1+\gamma)\,(L(a,c)f(z) - L(a,c)f(tz))}{(1-t)z\,(L(a,c)f(z))'\,(1+ke^{i\theta}) + (1-ke^{i\theta}-\gamma)\,(L(a,c)f(z) - L(a,c)f(tz))} \right| < 1\,,$$

when $-\pi \leq \theta \leq \pi$. For any complex number s with $|s| = 1$, we have

$$\frac{(1-t)z\,(L(a,c)f(z))'\,(1+ke^{i\theta}) - (ke^{i\theta}+1+\gamma)\,(L(a,c)f(z) - L(a,c)f(tz))}{(1-t)z\,(L(a,c)f(z))'\,(1+ke^{i\theta}) + (1-ke^{i\theta}-\gamma)\,(L(a,c)f(z) - L(a,c)f(tz))} \neq s.$$

In other words, we must have

$$\begin{aligned}(1-s)(1-t)z\,(L(a,c)f(z))'\,(1+ke^{i\theta}) - (ke^{i\theta}+1+\gamma+s(-1+ke^{i\theta}+\gamma))\\ (L(a,c)f(z) - L(a,c)f(tz)) \neq 0.\end{aligned}$$

which is equivalent to

$$z - \sum_{n=2}^{\infty} \frac{\phi(a,c,n)\,\big((n-u_n)(1+ke^{i\theta}-ske^{i\theta}) - s(n+u_n) - u_n\gamma(1-s)\big)}{\gamma(s-1)-2s} z^n \neq 0.$$

However, $f \in k - \widetilde{U}S_s(a, c, \gamma, t)$ if and only $\dfrac{(f * h)}{z} \neq 0, z \in U - \{0\}$, where $h(z) = z - \sum\limits_{n=2}^{\infty} c_n z^n$ and

$$c_n = \frac{\phi(a,c,n)\,\big((n-u_n)(1+ke^{i\theta}-ske^{i\theta}) - s(n+u_n) - u_n\gamma(1-s)\big)}{\gamma(s-1)-2s}\,.$$

We note that $|c_n| \leq \dfrac{\phi(a,c,n)\,|n(1+k) - u_n(k+\gamma)|}{1-\gamma}$ since $\dfrac{f(z)+\epsilon z}{1+\epsilon}$ belongs in the class $k - \widetilde{U}S_s(a, c, \gamma, t)$, therefore $z^{-1}\left(\frac{f(z)+\epsilon z}{1+\epsilon} * h(z)\right) \neq 0$, which is equivalent to

$$\frac{(f*h)(z)}{(1+\epsilon)z} + \frac{\epsilon}{1+\epsilon} \neq 0. \tag{3.1}$$

Now suppose that $\left|\frac{(f*h)(z)}{z}\right| < \alpha$. Then by (3.1), we must have

$$\left|\frac{(f*h)(z)}{(1+\epsilon)z} + \frac{\epsilon}{1+\epsilon}\right| \geq \frac{|\epsilon|}{|1+\epsilon|} - \frac{1}{|1+\epsilon|}\left|\frac{(f*h)(z)}{z}\right| > \frac{|\epsilon|-\alpha}{|1+\epsilon|} \geq 0,$$

this is a contradiction by $|\epsilon| < \alpha$ and however, we have $\left|\frac{(f*h)(z)}{z}\right| \geq \alpha$.

If $g(z) = z - \sum\limits_{n=2}^{\infty} b_n z^n \in N_\alpha(f)$, then

$$\begin{aligned}\alpha - \left|\frac{(g*h)(z)}{z}\right| \leq \left|\frac{((f-g)*h)(z)}{z}\right| &\leq \sum_{n=2}^{\infty} |a_n - b_n||c_n||z^n| \\ &< \sum_{n=2}^{\infty} \frac{\phi(a,c,n)|n(1+k) - u_n(k+\gamma)|}{1-\gamma}|a_n - b_n| \leq \alpha.\end{aligned}$$

□

4. Partial sums of the function class $k - \widetilde{U}S_s(a,c,\gamma,t)$

In this section, applying methods used by Silverman [10] and Silvia [11], we investigate the ratio of a function of the form (1.4) to its sequence of partial sums $f_m(z) = z + \sum\limits_{n=2}^{m} a_n z^n$.

Theorem 4.1. *If f of the form* (1.1) *satisfies the condition* (2.1) *then*

$$Re\left\{\frac{f(z)}{f_m(z)}\right\} \geq 1 - \frac{1}{\delta_{m+1}} \tag{4.1}$$

and

$$\delta_n = \begin{cases} 1, & \text{if } n = 2,3,\cdots,m; \\ \delta_{m+1}, & \text{if } n = m+1, m+2, \cdots. \end{cases}$$

where

$$\delta_n = \frac{\phi(a,c,n)|n(1+k) - u_n(k+\gamma)|}{1-\gamma}. \tag{4.2}$$

The result in (4.1) *is sharp for every m, with the extremal function*

$$f(z) = z + \frac{z^{m+1}}{\delta_{m+1}}. \tag{4.3}$$

Proof. Define the function w, we may write

$$\frac{1+w(z)}{1-w(z)} = \delta_{m+1}\left\{\frac{f(z)}{f_m(z)} - \left(1-\frac{1}{\delta_{m+1}}\right)\right\} = \left\{\frac{1+\sum_{n=2}^{m} a_n z^{n-1} + \delta_{m+1}\sum_{n=m+1}^{\infty} a_n z^{n-1}}{1+\sum_{n=2}^{m} a_n z^{n-1}}\right\}. \tag{4.4}$$

Then, from (4.4), we can obtain

$$w(z) = \frac{\delta_{m+1}\sum_{n=m+1}^{\infty} a_n z^{n-1}}{2+2\sum_{n=2}^{m} a_n z^{n-1} + \delta_{m+1}\sum_{n=m+1}^{\infty} a_n z^{n-1}},$$

and

$$|w(z)| \le \frac{\delta_{m+1}\sum_{n=m+1}^{\infty} a_n}{2-2\sum_{n=2}^{m} a_n - \delta_{m+1}\sum_{n=m+1}^{\infty} a_n}.$$

Now $|w(z)| \le 1$ if $2\delta_{m+1}\sum_{n=m+1}^{\infty} a_n \le 2-2\sum_{n=2}^{m} a_n$,
which is equivalent to

$$\sum_{n=2}^{m} a_n + \delta_{m+1}\sum_{n=m+1}^{\infty} a_n \le 1. \tag{4.5}$$

It is suffices to show that the left hand side of (4.5) is bounded above by $\sum_{n=2}^{\infty}\delta_n a_n$, which is equivalent to

$$\sum_{n=2}^{m}(\delta_n - 1)a_n + \sum_{n=m+1}^{\infty}(\delta_n - \delta_{m+1})a_n \ge 0.$$

To see that the function given by (4.3) gives the sharp result, we observe that for $z = re^{i\pi/n}$, $\frac{f(z)}{f_m(z)} = 1 + \frac{z^m}{\delta_{m+1}}$. Taking $z \to 1^-$, we have

$$\frac{f(z)}{f_m(z)} = 1 - \frac{1}{\delta_{m+1}}.$$

This completes the proof of Theorem 4.1. □

We next determine bounds for $\frac{f_m(z)}{f(z)}$.

Theorem 4.2. *If f of the form* (1.1) *satisfies the condition* (2.1) *then*

$$Re\left\{\frac{f_m(z)}{f(z)}\right\} \geq \frac{\delta_{m+1}}{1+\delta_{m+1}}.$$

The result is sharp with the function given by (4.3).

Proof. We may write

$$\begin{aligned}\frac{1+w(z)}{1-w(z)} =&(1+\delta_{m+1})\left\{\frac{f_m(z)}{f(z)} - \frac{\delta_{m+1}}{1+\delta_{m+1}}\right\}\\ =&\left\{\frac{1+\sum\limits_{n=2}^{m} a_n z^{n-1} - \delta_{m+1}\sum\limits_{n=m+1}^{\infty} a_n z^{n-1}}{1+\sum\limits_{n=2}^{\infty} a_n z^{n-1}}\right\},\end{aligned}$$

where

$$w(z) = \frac{(1+\delta_{m+1})\sum\limits_{n=m+1}^{\infty} a_n z^{n-1}}{-\left(2+2\sum\limits_{n=2}^{m} a_n z^{n-1} - (1-\delta_{m+1})\sum\limits_{n=m+1}^{\infty} a_n z^{n-1}\right)}$$

and

$$|w(z)| \leq \frac{(1+\delta_{m+1})\sum\limits_{n=m+1}^{\infty} a_n}{2-2\sum\limits_{n=2}^{m} a_n + (1-\delta_{m+1})\sum\limits_{n=m+1}^{\infty} a_n} \leq 1.$$

This last inequality is equivalent to

$$\sum_{n=2}^{m} a_n + \delta_{m+1}\sum_{n=m+1}^{\infty} a_n \leq 1. \tag{4.6}$$

It is suffices to show that the left hand side of (4.6) is bounded above by $\sum\limits_{n=2}^{\infty}\delta_n a_n$, which is equivalent to

$$\sum_{n=2}^{m}(\delta_n - 1)a_n + \sum_{n=m+1}^{\infty}(\delta_n - \delta_{m+1})a_n \geq 0.$$

This completes the proof of Theorem 4.2. □

We next turn to ratios involving derivatives.

Theorem 4.3. *If f of the form* (1.1) *satisfies the condition* (2.1) *then*

$$Re\left\{\frac{f'(z)}{f'_m(z)}\right\} \geq 1 - \frac{m+1}{\delta_{m+1}}, \tag{4.7}$$

$$Re\left\{\frac{f'_m(z)}{f'(z)}\right\} \geq \frac{\delta_{m+1}}{1+m+\delta_{m+1}} \tag{4.8}$$

where

$$\delta_n \geq \begin{cases} 1, & \text{if } n = 1, 2, 3, \cdots, m; \\ n\frac{\delta_{m+1}}{m+1}, & \text{if } n = m+1, m+2, \cdots. \end{cases}$$

and δ_n is defined by (4.2). *The estimates in* (4.7) *and* (4.8) *are sharp with the extremal function given by* (4.3).

Proof. Firstly, we will give proof of (4.7). We write

$$\begin{aligned}\frac{1+w(z)}{1-w(z)} =& \delta_{m+1}\left\{\frac{f'(z)}{f'_m(z)} - \left(1 - \frac{m+1}{\delta_{m+1}}\right)\right\} \\ =& \left\{\frac{1 + \sum\limits_{n=2}^{m} na_n z^{n-1} + \frac{\delta_{m+1}}{m+1}\sum\limits_{n=m+1}^{\infty} na_n z^{n-1}}{1 + \sum\limits_{n=2}^{m} a_n z^{n-1}}\right\},\end{aligned}$$

where

$$w(z) = \frac{\frac{\delta_{m+1}}{m+1}\sum\limits_{n=m+1}^{\infty} na_n z^{n-1}}{2 + 2\sum\limits_{n=2}^{m} na_n z^{n-1} + \frac{\delta_{m+1}}{m+1}\sum\limits_{n=m+1}^{\infty} na_n z^{n-1}}$$

and

$$|w(z)| \leq \frac{\frac{\delta_{m+1}}{m+1}\sum\limits_{n=m+1}^{\infty} na_n}{2 - 2\sum\limits_{n=2}^{m} na_n + \frac{\delta_{m+1}}{m+1}\sum\limits_{n=m+1}^{\infty} na_n}.$$

Now $|w(z)| \leq 1$ if and only if

$$\sum_{n=2}^{m} na_n + \frac{\delta_{m+1}}{m+1}\sum_{n=m+1}^{\infty} na_n \leq 1, \tag{4.9}$$

since the left hand side of (4.9) is bounded above by $\sum\limits_{n=2}^{\infty} \delta_n a_n$.
The proof of (4.8) follows the pattern of that in Theorem 4.1.
This completes the proof of Theorem 4.3. $\square$

REFERENCES

[1] S.O. ALTINTA, S. OWA: *Neighborhoods of certain analytic functions with negative coefficients*, Int. J. Math. and Math. Sci., **19** (1996) , 797–800.

[2] S.O. ALTINTA, E. OZKAN, H.M.SRIVASTAVA: *Neighborhoods of a class of analytic functions with negative coefficients*, Appl. Math. Let., **13** (2000), 63–67.

[3] B.C. CARLSON, S.B.SHAFFER: *Starlike and prestarlike hypergrometric functions*, SIAM J. Math. Anal., **15** (2002), 737-745.

[4] A. W. GOODMAN : *Univalent functions and nonanalytic curves*, Proc. Amer. Math. Soc., **8** (1957), 598–601 .

[5] A. W. GOODMAN : *On uniformly starlike functions*, J. Math. Anal. Appl., **155** (1991), 364–370.

[6] S. OWA, T. SEKINE, R. YAMAKAWA: *On Sakaguchi type functions*, Appl. Math. Comput., **187** (2007), 356–361 .

[7] F. RONNING: *Uniformly convex functions and a corresponding class of starlike functions*, Proc. Amer. Math. Soc., **118** (1993) , 189-196.

[8] S. RUSCHEWEYH: *Neighborhoods of univalent functions*, Proc. Amer. Math. Soc., **81** (4) (1981), 521–527.

[9] K. SAKAGUCHI: *On a certain univalent mapping*, J. Math. Soc. Japan, **11** (1959), 72–75.

[10] H. SILVERMAN: *Partial sums of starlike and convex functions*, J. Anal. Appl., **209** (1997), 221–227.

[11] E.M. SILVIA: *Partial sums of convex functions of order R*, Houston J. Math., **11** (3) (1985), 397–404.

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