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ADOMIAN METHOD FOR SOLVING EMDEN-FOWLER EQUATION OF HIGHER ORDER

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ABSTRACT. Adomian Decomposition Method (ADM) is prsented in this article to solve Emden-Fowler equation of higher order. We tested this method with several numerical examples that showed the reliability of the method in the finding of good approximate solutions.

1. INTRODUCTION

Many researchers were interested in studying different types of Emden-Fowler equation, where in the last five years a lot of researches have been presented to solve Emden equation using the Adomian method as in [6, 8, 9]. Adomian method is considered as one of the most effective methods in finding convergence solutions as well as the complete solution. As it started by George Adomian in 1980s [1–4], many suggested amendments were presented on this method as in [5, 7]. Our target in this work is to find a solution of the equation under study. We suggested an unused differential operator that can find solution of the equation subject of our study.

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2. DISCUSSION THE METHOD

We suggested a new differential operator as below:

$$L(.) = x^{-n}\frac{d}{dx}x^{n-m}\frac{d^k}{dx^k}x^m(.) = g(x,y), \tag{2.1}$$

where $k \geq 1, n \geq 1$ and $m \geq 0$.

If we put $k = 1$ in eq.(2.1), we obtain the frist type of Emden-Fowler equation.

$$y'' + \frac{n+m}{x}y' + \frac{m(n-1)}{x^2}y = g(x,y),$$

If we put $k = 2$ in eq.(2.1), we obtain the second type of Emden-Fowler equation.

$$y''' + \frac{n+2m}{x}y'' + \frac{m(2(n-1)+(m-1))}{x^2}y' + \frac{m(m-1)(n-2)}{x^3}y = \\ = g(x,y)\,.$$

If we put $k = 3$ in eq.(2.1), we obtain the third type of Emden-Fowler equation.

$$\begin{aligned} y^{(iv)} &+ \frac{n+3m}{x}y''' + \frac{m(3(n-1)+3(m-1))}{x^2}y'' + \frac{m(m-1)(3(n-2)+(m-2))}{x^3}y' \\ &+ \frac{m(m-1)(m-2)(n-3)}{x^4}y = g(x,y)\,. \end{aligned}$$

If we put $k = 4$ in eq.(2.1), we obtain the forth type of Emden-Fowler equation.

$$\begin{aligned} y^{(v)} &+ \frac{n+4m}{x}y^{(iv)} + \frac{m(4(n-1)+6(m-1))}{x^2}y''' + \frac{m(m-1)(6(n-2)+4(m-2))}{x^3}y'' \\ &+ \frac{m(m-1)(m-2)(4(n-3)+(m-3))}{x^4}y' \\ &+ \frac{m(m-1)(m-2)(m-3)(n-4)}{x^5}y = g(x,y)\,. \end{aligned}$$

If we continue with this procedure we have:

$$\begin{aligned} y^{(k+1)} &+ \sum_{r=0}^{k}\frac{\Gamma(m+1)}{\Gamma(m-r+1)}\binom{k}{r}(n-r)x^{-r-1}y^{(k-r)} + \\ &+ \sum_{r=1}^{k}\frac{\Gamma(m+1)}{\Gamma(m-r+2)}\binom{k}{r}(m+1-r)x^{-r}y^{(k+1-r)} = g(x,y) \end{aligned} \tag{2.2}$$

3. Adomian Method

Assume the differential equation as eq.(2.2) with

$$y(0)=A, y'(0)=B, y''(0)=C, ...y^{(k)}(0)=D,$$

where $g(x,y)$ is a known function and A,B,C,D are constants. We define an operator form eq.(2.2) written as

$$Ly=g(x,y)\,, \tag{3.1}$$

where

$$L(.)=x^{-n}\frac{d}{dx}x^{n-m}\frac{d^k}{dx^k}x^m(.)\,,$$

and

$$L^{-1}(.)=x^{-m}\underbrace{\int_0^x\int_0^x\int_0^x...\int_0^x}_{(k)}x^{m-n}\int_0^x x^n(.)\,\underbrace{dxdxdx...dxdx}_{(k+1)},$$

By applying L^{-1} on (3.1), we have

$$y(x)=\delta(x)+L^{-1}(g(x,y)),$$

such that

$$L(\delta(x))=0\,.$$

The method by Adomian is given the solution $y(x)$ and the function $g(x,y)$ by infinite series

$$y(x)=\sum_{n=0}^{\infty}y_n(x)\,, \tag{3.2}$$

and

$$g(x,y)=\sum_{n=0}^{\infty}A_n\,, \tag{3.3}$$

where the elements $y_n(x)$ of the solution $y(x)$ will be determined repeatable. Specific algorithms were seen in [2,4] to formulate Adomian polynomials. The following algorithm:

$$
\begin{aligned}
A_0 &= G(y_0)\,, \\
A_1 &= y_1 G'(y_0)\,, \\
A_2 &= y_2 G'(y_0) + \frac{1}{2!} y_1^2 G'' y_0)\,, \\
A_3 &= y_3 G'(y_0) + y_1 y_2 G''(y_0) + \frac{1}{3!} y_1^3 G''' y_0\,, \\
&\cdots
\end{aligned}
\tag{3.4}
$$

can be used to build Adomian polynomials, when $G(y)$ is any function. From (3.2),(3.3) and (3.4) we have:

$$
\sum_{n=0}^{\infty} y_n(x) = \delta(x) + L^{-1} \sum_{n=0}^{\infty} A_n. \tag{3.5}
$$

The component $y(x)$ can be given by using Adomian decomposition method as follows

$$
\begin{aligned}
y_0 &= \delta(x), \\
y_{(n+1)} &= L^{-1} A_n,\ n \geq 0,
\end{aligned}
\tag{3.6}
$$

thus

$$
\begin{aligned}
y_0 &= \delta(x) \\
y_1 &= L^{-1} A_0\,, \\
y_2 &= L^{-1} A_1\,, \\
y_3 &= L^{-1} A_2\,, \\
&\cdots
\end{aligned}
\tag{3.7}
$$

Using the equations (3.4) and (3.7) we can determine the components y_n and therefore we can immediately obtain series solutions of $y(x)$ in (3.5). In addition, and for numerical reasons, we can use the n-term approximate

$$
\zeta_n = \sum_{k=0}^{n-1} y_k,
$$

in order to approximate the exact solution.

4. Experiment of the Method

This part of the article allocated to test the method, where we gave three examples of different orders.

Example 1. *If we put $m = \frac{1}{2}, n = 2, k = 1,$ in equation (2.2) we obtain*

$$y'' + \frac{2.5}{x}y' + \frac{0.5}{x^2}y = \frac{1 + 45\,x^2 + 135\,x^4 + 91\,x^6}{2\,x^2} + (1+x^2)^{15}y^5 - y^{10}, \tag{4.1}$$

$$y(0) = 1, y'(0) = 0,$$

$y(x) = \left(1+x^2\right)^3$, *is the solution of eq.(4.1).*
We can write eq.(4.1) as

$$Ly = \frac{1 + 45\,x^2 + 135\,x^4 + 91\,x^6}{2\,x^2} + (1+x^2)^{15}y^5 - y^{10}, \tag{4.2}$$

interms

$$L(.) = x^{-2}\frac{d}{dx}x^{1.5}\frac{d}{dx}x^{0.5}(.),$$

as well

$$L^{-1}(.) = x^{-0.5}\int_0^x x^{-1.5}\int_0^x x^2(.)dxdx\,.$$

Applying L^{-1} on (4.2) we find

$$y = L^{-1}(\frac{1 + 45\,x^2 + 135\,x^4 + 91\,x^6}{2\,x^2}) + L^{-1}((1+x^2)^{15}y^5 - y^{10}),$$

therefore

$$y = \left(1+x^2\right)^3 + L^{-1}((1+x^2)^{15}y^5 - y^{10})\,. \tag{4.3}$$

Replace $y_n(x)$ for $y(x)$ into (4.3) gives

$$\begin{aligned}
\sum_{n=0}^{\infty} y_n(x) &= \left(1+x^2\right)^3 + L^{-1}((1+x^2)^{15}A_n - L^{-1}(A_n) \\
y_0 &= \left(1+x^2\right)^3 \\
y_{n+1} &= L^{-1}(((1+x^2)^{15}A_n) - L^{-1}(A_n), \;\; n \geq 0, && (4.4)\\
A_0 &= (1+x^2)^{15}y_0^5 - y_0^{10}. && (4.5)
\end{aligned}$$

From (4.4) and (4.5) we get

$$\begin{aligned}
y_0 &= \left(1+x^2\right)^3 \\
y_1 &= 0\,.
\end{aligned}$$

Then the series solution by (ADM) is

$$y(x) = y_0 + y_1 = \left(1 + x^2\right)^3.$$

As we can see in the above illustrated example, we got the exact solution using ADM. Therefore, we canclude that the ADM is a reliable in finding the exact solution.

Example 2. *If we put $m = 4, n = 5, k = 3,$ in equation (2.2) we obtain*

$$\begin{aligned} y^{(4)} &+ \frac{17}{x}y''' + \frac{84}{x^2}y'' + \frac{132}{x^3}y' + \frac{48}{x^4}y = \\ &= \frac{3\,e^{x^3}\left(16 + 334\,x^3 + 618\,x^6 + 261\,x^9 + 27\,x^{12} + \frac{1}{3}x^3 e^{x^{-3}}\right)}{x^4} - \ln y\,, \qquad (4.6) \\ & y(0) = 1, y'(0) = y''(0) = 0, y'''(0) = 6\,, \end{aligned}$$

$y(x) = e^{x^3}$ *is the exact solution of eq.(4.6). We can write equation (4.6) as*

$$Ly = \frac{3\,e^{x^3}\left(16 + 334\,x^3 + 618\,x^6 + 261\,x^9 + 27\,x^{12} + \frac{1}{3}x^3 e^{x^{-3}}\right)}{x^4} - \ln y, \qquad (4.7)$$

in terms

$$L(.) = x^{-5}\frac{d}{dx}x^1\frac{d^3}{dx^3}x^4(.)\,,$$

as well

$$L^{-1}(.) = x^{-4}\int_0^x\int_0^x\int_0^x x^{-1}\int_0^x x^5(.)dxdxdxdx\,.$$

Applying L^{-1} on (4.7) we find

$$y = L^{-1}\left(\frac{3\,e^{x^3}\left(16 + 334\,x^3 + 618\,x^6 + 261\,x^9 + 27\,x^{12} + \frac{1}{3}x^3 e^{x^{-3}}\right)}{x^4}\right) - L^{-1}\ln y,$$

therefore

$$y = e^{x^3} + 0.000112233\,x^7 - L^{-1}\ln y. \qquad (4.8)$$

Replace $y_n(x)$ for $y(x)$ into (4.8) gives

$$\begin{aligned} \sum_{n=0}^{\infty} y_n(x) &= e^{x^3} + 0.000112233\,x^7 - L^{-1}A_n, \\ y_0 &= e^{x^3} + 0.000112233\,x^7, \\ y_{n+1} &= -L^{-1}A_n,\ n \geq 0, \end{aligned}$$

we get the series of $\ln y$, *by Adomian polynomials*

$$\begin{aligned} A_0 &= \ln y_0, \\ A_1 &= \frac{y_1}{y_0}, \\ A_2 &= \frac{y_2}{y_0} - \frac{y_1^2}{2y_0^2}. \end{aligned}$$

Then

$$\begin{aligned} y_0 &= e^{x^3} + 0.000112233\, x^7, \\ y_1 &= -0.000112233\, x^7 - 3.1624 \times 10^{-9}\, x^{11} + 1.43272 \times 10^{-9}\, x^{14} - \\ &\quad -3.70114 \times 10^{-10}\, x^{17} + 3.4081 \times 10^{-14}\, x^{18}, \\ y_2 &= 3.1624 \times 10^{-9}\, x^{11} - 1.43272 \times 10^{-9}\, x^{14} + 3.19958 \times 10^{-14}\, x^{15} + \\ &\quad +3.70114 \times 10^{-10}\, x^{17} - 9.30274 \times 10^{-14}\, x^{18}, \\ y(x) &= y_0 + y_1 + y_2 = e^{x^3} + 3.19958 \times 10^{-14}\, x^{15} - 5.89464 \times 10^{-14}\, x^{18}. \end{aligned}$$

In above example some terms appear in the first elements of y_n with opposite signs, like the term $0.000112233\, x^7$ appear in y_0 and y_1 with opposite sings. So if we continue finding values of y, we will get the right solution.

Example 3. *If we put* $m = 6, n = 1, k = 6,$ *in equation (2.2) we obtain*

$$\begin{aligned} y^{(7)} &+ \frac{38}{x}y^{(6)} + \frac{486}{x^2}y^{(5)} + \frac{2400}{x^3}y^{(4)} + \frac{3000}{x^4}y^{(3)} - \frac{6480}{x^5}y^{(2)} - \frac{12240}{x^6}y' - \frac{2880}{x^7}y \\ &= 3706560 + x^{14} - y^2 \end{aligned} \tag{4.9}$$

$$y(0) = y'(0) = y''(0) = y'''(0) = y^{(4)}(0) = y^{(5)}(0) = y^{(6)}(0) = 0,$$

$y(x) = x^7$ *is the solution of eq.(4.9).*

We can write equation (4.9) as

$$Ly = 3706560 + x^{14} - y^2, \tag{4.10}$$

in terms

$$L(.) = x^{-2}\frac{d}{dx}x^{-4}\frac{d^6}{dx^6}x^6(.),$$

as well

$$L^{-1}(.) = x^{-6}\int_0^x \int_0^x \int_0^x \int_0^x \int_0^x \int_0^x x^4 \int_0^x x^2(.)dxdxdxdxdxdxdx.$$

Applying L^{-1} on (4.10) we find

$$y = L^{-1}(3706560 + x^{14}) - L^{-1}y^2,$$

then

$$y(x) = x^7 + 2.76002 \times 10^{-10}\, x^{21} - L^{-1}y^2\,. \tag{4.11}$$

Replace $y_n(x)$ for $y(x)$ into (4.11) gives

$$\begin{aligned}
\sum_{n=0}^{\infty} y_n(x) &= x^7 + 2.76002 \times 10^{-10}\, x^{21} - L^{-1}A_n,\\
y_0 &= x^7 + 2.76002 \times 10^{-10}\, x^{21},\\
y_{n+1} &= -L^{-1}A_n, \quad n \geq 0\,.
\end{aligned}$$

We get the series of y^2, by Adomian polynomials

$$\begin{aligned}
A_0 &= y_0^2,\\
A_1 &= 2y_0y_1\,.
\end{aligned}$$

Then

$$\begin{aligned}
y_0 &= x^7 + 2.76002 \times 10^{-10}\, x^{21},\\
y_1 &= -2.76002 \times 10^{-10}\, x^{21} - 5.50027 \times 10^{-21}\, x^{35} - 8.11028 \times 10^{-32}\, x^{49},\\
y_2 &= 5.50027 \times 10^{-21}\, x^{35} + 1.73918 \times 10^{-31}\, x^{49} + 6.2807 \times 10^{-43}\, x^{63} +\\
&\quad +2.25666 \times 10^{-54}\, x^{77},\\
y(x) &= y_0 + y_1 + y_2 = x^7 + 9.28147 \times 10^{-32}\, x^{49} + 6.2807 \times 10^{-43}\, x^{63} +\\
&\quad +2.25666 \times 10^{-54}\, x^{77}.
\end{aligned}$$

In the following, Table 1 and Figure 1 explain the convergence between (ADM) and the right solution

TABLE 1. Comparison of numerical errors between the right solution $y = x^7$ and the ADM solution $y = \sum_{n=0}^{2} y_n(x)$.

x	Right	ADM	Absolute error
0.0	0.000	0.000000000	0.0000000
0.1	1×10^{-7}	1×10^{-7}	0.0000000
0.2	1.28×10^{-5}	1.28×10^{-5}	0.0000000
0.3	2.187×10^{-4}	2.187×10^{-4}	0.0000000
0.4	1.6384×10^{-3}	1.6384×10^{-3}	0.0000000
0.5	7.8125×10^{-3}	7.8125×10^{-3}	0.0000000
0.6	0.0279936	0.0279936	0.0000000
0.7	0.823543	0.823543	0.0000000
0.8	0.209715	0.209715	0.0000000
0.9	0.478297	0.47829	0.0000000
1.0	1.000000	1.0000000	0.0000000

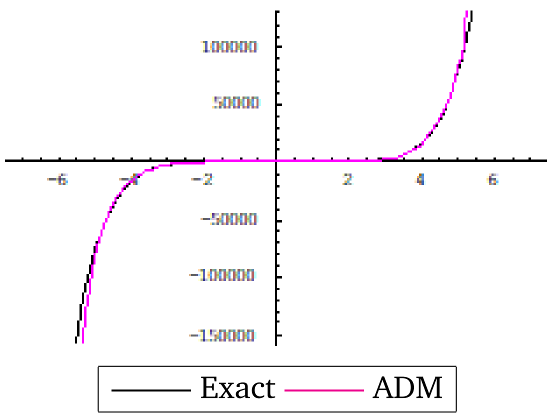

FIGURE 1. The right solution $y = x^7$ and the ADM solution $y = \sum_{n=0}^{2} y_n(x)$.

5. CONCLUSION

In this article, we offered a new differential operator for solving different types of Emden-Folwer equation of higher order. The examples presented in this article illustrated the quality of the method for finding the solutions. In Examples 2 and 3 the results were very close to exact solution. In Examples 1 we got the exact solution. This indicates that the presented method is very efficient to solve equations considered.

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