

NUMERICAL APPROXIMATION OF THE SPECTRUM FOR A FLEXIBLE EULER-BERNOULLI BEAMS WITH A FORCE CONTROL IN ROTATION AND VELOCITY ROTATION

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ABSTRACT. In this paper, we use asymptotic techniques and the finite differences method to study the spectrum of the differential operator arising in the exponential stabilization of the following differential equation $y_{tt} + y_{xxxx} = 0$, $0 < x < 1$, $t \geq 0$ with boundary conditions $y(0, t) = y_x(0, t) = 0$, $t \geq 0$, $y_{xxx}(1, t) = 0$, $t \geq 0$, $y_{xx}(1, t) = -\alpha y_{xt}(1, t) - \beta y_x(1, t)$, $t \geq 0$.

1. INTRODUCTION

We consider the following evolutive system

$$(1.1) \quad y_{tt} + y_{xxxx} = 0, \quad 0 < x < 1, \quad t \geq 0,$$

$$(1.2) \quad y(0, t) = y_x(0, t) = 0, \quad t \geq 0,$$

$$(1.3) \quad y_{xxx}(1, t) = 0, \quad t \geq 0,$$

$$(1.4) \quad y_{xx}(1, t) = -\alpha y_{xt}(1, t) - \beta y_x(1, t), \quad t \geq 0,$$

where y is a scalar function of variable x and t ; α and β are two given positive constants. This simplified model may represent a cable that is clamped at one end and submitted to a force control in rotation and velocity rotation. We

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assumed that the cable is completely flexible with constant length. The system (1.1)–(1.4) has been studied when $\beta = 0$ in [7], where it has studied the exponential stabilization. Then they get by explicit formulae the location of the spectrum. When control in velocity is added to the existing feedback, only the behavior of the eigenvalues of large modulus can be analytically obtained. One use in this case asymptotic techniques. The main goal of this work is to use the finite differences method to elaborate a program that gives the complete eigenvalues location of the system defined by (1.1)–(1.4), as a function of positive feedback parameter α and β . the paper is organized as follows. In the next section we recall the formulation of the system (1.1)–(1.4) in the context of the C_0 semigroup of contractions. We then formulate the eigenvalue problem and give the exact spectrum when $\beta = 0$. When $\beta > 0$ we show asymptotic techniques are used to estimate eigenvalues of large modulus .

In the third section, we develop a numerical scheme based on the finite differences method for the eigenvalues problem. Finally, in the last section, we give some numerical experiments from our scheme.

2. FORMULATION OF THE SYSTEM IN THE C_0 SEMIGROUP OF CONTRACTIONS THEORY

2.1. Energy space and energy norm. We consider the system (1.1)–(1.4) and we let $v = u_t$, $W = (u, v)^T$. We introduce the following spaces:

$$V = \{u \in H^2(0, 1); u(0) = u_x(0) = 0\},$$

$$H = \{(u, v)^T; u \in V, v \in L^2(0, 1)\} = V \times L^2(0, 1).$$

The space H is called energy space of the system. We define the inner-product

$$\langle (u_1, v_1), (u_2, v_2) \rangle = \int_0^1 (u_1'' u_2'' + v_1 v_2) dx + \beta u_1' u_2'.$$

The energy norm induced by the inner product is hence defined by

$$\|(u, v)\|_H^2 = \int_0^1 [(u'')^2 + v^2] dx + \beta (u')^2.$$

Finally letting

$$A = \begin{pmatrix} 0 & 1 \\ \frac{d^4}{dx^4} & 0 \end{pmatrix},$$

the system (1.1)–(1.4) can be written as formally as

$$\frac{dW(t)}{dt} = AW(t), \quad W(0) = W_0 \in H,$$

with $D(A) = \{(u, v)^T \in (H^4(0, 1) \cap V) \times V / u_{xxx}(1) = 0, u_{xx}(1) = -\alpha v_x(1) - \beta u_x(1)\}$.

2.2. Spectrum of the operator A . Let λ denotes an eigenvalue of A and $W = (u, v)^T \in D(A)$ a corresponding eigenvector. We have

$$AW = \lambda W,$$

or

$$\begin{aligned} v &= \lambda u \\ u_{xxxx} &= \lambda v \\ u(0) &= 0 \\ u_x(0) &= 0 \\ u_{xx}(1) &= -\alpha v_x(1) - \beta u_x(1) \\ u_{xxx}(1) &= 0. \end{aligned}$$

Eliminating v from the above equations, we get

$$\begin{aligned} u_{xxxx} + \lambda^2 u &= 0, \\ u(0) &= 0, \\ u_x(0) &= 0, \\ u_{xx}(1) &= -(\alpha\lambda + \beta)u_x(1), \\ u_{xxx}(1) &= 0. \end{aligned}$$

Letting $\lambda = \tau^2$, we get

$$\begin{aligned} (2.1) \quad u_{xxxx} + \tau^4 u &= 0, \\ u_{xxx}(1) &= 0, \\ u_{xx}(1) &= -(\alpha\tau^2 + \beta)u_x(1), \\ u_x(0) &= 0, \\ u(0) &= 0. \end{aligned}$$

The Shkalikov's characteristic polynomial associated with equation (2.1) is $\omega^4 + 1 = (\omega^2 - \sqrt{2}\omega + 1)(\omega^2 + \sqrt{2}\omega + 1) = 0$ and its zeros are $\omega_1 = \frac{1+i}{\sqrt{2}}$, $\omega_2 = \frac{-1+i}{\sqrt{2}}$, $\omega_3 = \frac{-1-i}{\sqrt{2}}$, $\omega_4 = \frac{1-i}{\sqrt{2}}$. The solutions of (2.1) take the form:

$$u(x) = C_1 e^{\tau\omega_1 x} + C_2 e^{\tau\omega_2 x} + C_3 e^{\tau\omega_3 x} + C_4 e^{\tau\omega_4 x}.$$

Let us set:

$$\begin{aligned} F_i &= \tau^3 \omega_i^3 e^{\tau\omega_i}, \quad i = 1, \dots, 4 \\ G_i &= (\alpha\tau^2 + \tau\omega_i + \beta) \tau\omega_i e^{\tau\omega_i} \quad i = 1, \dots, 4. \end{aligned}$$

Then we get the following matrix equation:

$$\begin{bmatrix} F_1 & F_2 & F_3 & F_4 \\ G_1 & G_2 & G_3 & G_4 \\ \tau\omega_1 & \tau\omega_2 & \tau\omega_3 & \tau\omega_4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

A necessary and sufficient condition for this matrix equation to have nontrivial solutions for C_1, C_2, C_3 , and C_4 is that the following characteristic determinant

$$\Delta(\tau) = \begin{vmatrix} F_1 & F_2 & F_3 & F_4 \\ G_1 & G_2 & G_3 & G_4 \\ \tau\omega_1 & \tau\omega_2 & \tau\omega_3 & \tau\omega_4 \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

vanishes. By developing $\Delta(\tau)$ with respect to the last row, we get

$$\begin{aligned} \Delta(\tau) &= \tau \left\{ (\omega_3 - \omega_4)(F_1 G_2 - F_2 G_1) + (\omega_2 - \omega_3)(F_1 G_4 - F_4 G_1) \right. \\ &\quad + (\omega_3 - \omega_1)(F_2 G_4 - F_4 G_2) + (\omega_4 - \omega_1)(F_3 G_2 - F_2 G_3) \\ &\quad \left. + (\omega_2 - \omega_4)(F_3 G_1 - F_1 G_3) + (\omega_1 - \omega_2)(F_3 G_4 - F_4 G_3) \right\}. \end{aligned}$$

Next, we set

$$\begin{aligned} W_{ij} &= (\omega_i - \omega_j) \quad 1 \leq i, j \leq 4 \\ T_{ij} &= F_i G_j - F_j G_i \quad 1 \leq i, j \leq 4 \end{aligned}$$

so that

$$\Delta(\tau) = \tau (W_{34}T_{12} + W_{23}T_{14} + W_{31}T_{24} + W_{41}T_{32} + W_{24}T_{31} + W_{12}T_{34}) ,$$

where

$$\begin{aligned} W_{34} &= -\sqrt{2}, \quad W_{23} = i\sqrt{2}, \quad W_{31} = (-1 - i)\sqrt{2} \\ W_{41} &= i\sqrt{2}, \quad W_{24} = (-1 - i)\sqrt{2}, \quad W_{12} = \sqrt{2} \\ T_{12} &= \tau^3 \left[\sqrt{2}\tau^2 - 2i(\alpha\tau^3 + \beta\tau) \right] e^{i\tau\sqrt{2}} \\ T_{14} &= \tau^3 \left[i\sqrt{2}\tau^2 + 2i(\alpha\tau^3 + \beta\tau) \right] e^{\tau\sqrt{2}} \\ T_{24} &= \sqrt{2}(1 - i)\tau^5 \\ T_{32} &= \tau^3 \left[-i\sqrt{2}\tau^2 + 2i(\alpha\tau^3 + \beta\tau) \right] e^{-\tau\sqrt{2}} \\ T_{31} &= \sqrt{2}(1 + i)\tau^5 \\ T_{34} &= \tau^3 \left[-\sqrt{2}\tau^2 - 2i(\alpha\tau^3 + \beta\tau) \right] e^{-i\tau\sqrt{2}}. \end{aligned}$$

After simplification, we get

$$\begin{aligned} \Delta(\tau) &= \tau^7 \left\{ \left[-2\tau^{-1} + 2\sqrt{2}i\alpha + 2\sqrt{2}i\beta\tau^{-2} \right] e^{i\tau\sqrt{2}} \right. \\ &\quad - \left[2\tau^{-1} + 2\sqrt{2}i\alpha + 2\sqrt{2}i\beta\tau^{-2} \right] e^{-i\tau\sqrt{2}} \\ &\quad - \left[2\tau^{-1} + 2\sqrt{2}\alpha + 2\sqrt{2}\beta\tau^{-2} \right] e^{\tau\sqrt{2}} \\ &\quad \left. + \left[-2\tau^{-1} + 2\sqrt{2}\alpha + 2\sqrt{2}\beta\tau^{-2} \right] e^{-\tau\sqrt{2}} - 8\tau^{-1} \right\}. \end{aligned}$$

We observe that for $|\tau|$ sufficiently large, the dominant term of each expression in bracket is nonzero. In the view of Shkalikov's theory, the boundary conditions are said to be *regular*.

We also mention that for $\beta = 0$, the previous characteristic determinant is the same as that found in [7] (p.37), where the author proves that the eigenvalues with sufficiently large modulus are asymptotically simple and isolated.

We claim that the above result is also true when $\beta \neq 0$. The properties just mentioned are very essential in order to use the fundamental Theorem 3.1 of Shkalikov's theory [8] (p.1328) in the perspective of finding a Riesz basis property for the operator A in the energy space H .

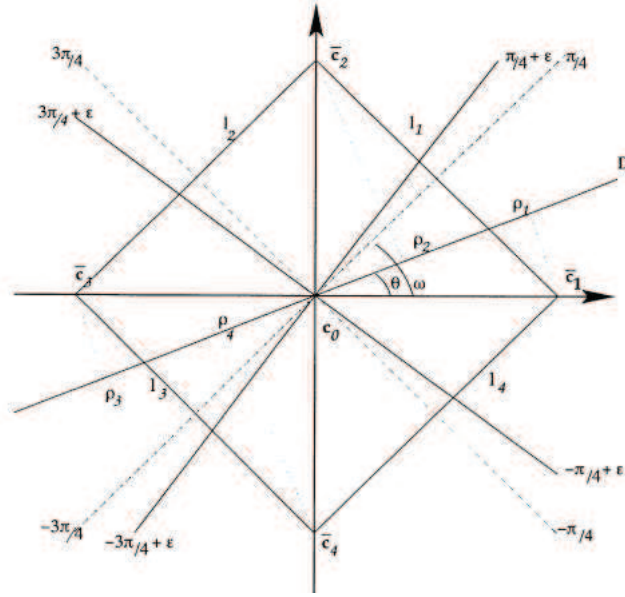
We set

$$\begin{aligned}
 P(\tau) &= (-\tau + \sqrt{2}i\alpha\tau^2 + 2i\beta)e^{i\tau\sqrt{2}} - (\tau + \sqrt{2}i\alpha\tau^2 + \sqrt{2}i\beta)e^{-i\tau\sqrt{2}} \\
 &\quad - (\tau + \sqrt{2}\alpha\tau^2 + \sqrt{2}\beta)e^{\tau\sqrt{2}} + (-\tau + \sqrt{2}\alpha\tau^2 + \sqrt{2}\beta)e^{-\tau\sqrt{2}} - 4\tau \\
 Q(\tau) &= (\tau + \sqrt{2}\alpha\tau^2 + \sqrt{2}\beta)e^{\tau\sqrt{2}} + (\tau + \sqrt{2}i\alpha\tau^2 + \sqrt{2}i\beta)e^{-i\tau\sqrt{2}}
 \end{aligned}$$

and note that

$$\Delta(\tau) = 2\tau^5 P(\tau).$$

Our next task is to show that the zeros of Q are asymptotically those of P , in a sector of the complex plan to be determined. For the approach used, the reader is referred to [5], [7]. We consider the convex polygon C whose center is O and which has vertices only at the following complex plane points: $\sqrt{2}$, $i\sqrt{2}$, $-\sqrt{2}$, $-i\sqrt{2}$ denoted by $\bar{C}_1$, $\bar{C}_2$, $\bar{C}_3$, $\bar{C}_4$ and plotted in the complex plane according to the following underlined figure:



Let the sides of the polygon C be taken in counterclockwise succession with any one be designated by l_i , that is to say the segment $[\bar{C}_i, \bar{C}_{i+1}]$ and θ_i be the inclination angle of the outer normal to the side l_i . We denote by D_i the line defined by

$$D_i = \{z \in \mathbb{C} / \text{Arg}(z) = \theta, \theta_{i-1} + \varepsilon \leq \theta \leq \theta_i + \varepsilon, i = 1, \dots, 4\},$$

where ε is sufficiently small but otherwise arbitrary positive real number. Let S_i be the sector described by the following relation:

$$\{z \in \mathbb{C} / |z| \text{ large enough, } \theta_{i-1} + \varepsilon \leq \text{Arg}(z) \leq \theta_i + \varepsilon\}.$$

Such a sector is thus associated with each side of the polygon and the set of these sectors completely fills out the complex plane, so that we have:

$$\bigcup_{i=1}^4 S_i = \mathbb{C}.$$

A straightforward calculation shows that the following holds for any $\tau \in \mathbb{C}$

$$P(-\tau) = -P(\tau), \quad P(\bar{\tau}) = \overline{P(\tau)}.$$

It follows that the set of zeros of $P(\tau)$ is symmetric with respect to the origin of the complex plane and also symmetric with respect to the real axis. This allows us to study $P(\tau)$, for instance, in the following typical sector:

$$S_1 = \left\{ z \in \mathbb{C} / |z| \text{ large enough, } -\frac{\pi}{4} + \varepsilon \leq \text{Arg}(z) \leq \frac{\pi}{4} + \varepsilon \right\}.$$

When z lies on the line D_1 , it is readily computed that

$$|z^{\alpha_j} e^{C_j z}| = e^{|z|(x_j \cos(\theta) + y_j \sin(\theta) + \alpha_j \frac{\log|z|}{|z|})},$$

where $\overline{C_j} = x_j + iy_j$. It follows that the relative magnitude of the terms $|z^{\alpha_j} e^{C_j z}|$ are ultimately determined by the relative magnitude of the associated quantities $x_j \cos(\theta) + y_j \sin(\theta)$; these, however, are represented geometrically by the orthogonal projection vectors $\overline{C_j} = x_j + iy_j$ upon the line D_1 . These projections are as follows:

$$\rho_1 = \sqrt{2} \cos(\theta), \quad \rho_2 = \sqrt{2} \sin(\theta), \quad \rho_3 = -\sqrt{2} \cos(\theta), \quad \rho_4 = -\sqrt{2} \sin(\theta),$$

and from this interpretation, it is easily seen that by virtue of the restriction upon the range of θ , $C_j \notin l_1$ here ($j \neq 1, 2$) and

$$x_j \cos(\theta) + y_j \sin(\theta) < x_1 \cos(\theta) + y_1 \sin(\theta).$$

Thus, for $z \in S_1$, $j \neq 1, 2$, we get:

$$z^{\alpha_j} e^{C_j z} = z^{\alpha_j} e^{C_j z} \varepsilon_j(z),$$

where

$$\varepsilon_j(z) = |z|^{\alpha_j - \alpha_1} e^{(x_j \cos(\theta) + y_j \sin(\theta)) - (x_1 \cos(\theta) + y_1 \sin(\theta))}$$

and where $\varepsilon_j(z) \rightarrow 0$ when $|z| \rightarrow +\infty$. We recall here the fundamental result of Langer [6], which allows us to study the asymptotical behavior of the zeros of the exponential sums.

2.3. Asymptotic behavior of the eigenvalues.

Theorem 2.1. *Let Q be a function defined in $\mathbb{C}$ as*

$$Q(z) = \sum_{j=0}^n z^{v_j} (a_j + \varepsilon_j(z)) e^{C_j z},$$

where $a_j \in \mathbb{C}$, $a_0, a_n \neq 0$, $v_j \in \mathbb{R}$, $C_j \in \mathbb{C}$ and the functions ε_j are analytic in all sector S_i (for a fixed i) and which approach zero uniformly when $|z|$ goes to $+\infty$, such a function will be called an ε -function. Then in this sector S_i , the zeros of Q are asymptotically represented by those of the relative sum $\sum_{j \in K_i} a_j z^{v_j} e^{C_j z}$, where K_i is the subset of $\{1, \dots, n\}$ which does not contain indexes j such that $C_j \in l_i$.

According to this theorem, the zeros of P in S_1 are asymptotically those of Q which are themselves asymptotically represented by those of

$$e^{\tau\sqrt{2}} + ie^{-i\tau\sqrt{2}} = 0.$$

Now, the zeros of the above equation take the form $\omega\pi(n - \frac{1}{4})$, where $n \in \mathbb{N}$ and $\omega = \frac{1+i}{\sqrt{2}}$ is a zero of the Shkalikov's characteristic polynomial (2.1). This allows us to search for an asymptotic expansion of zeros of P in the form

$$\tau_n = \omega\pi\left(n - \frac{1}{4}\right) + \frac{x}{n} + \frac{y}{n^2} + O\left(\frac{1}{n^3}\right),$$

where x and y are complex constants to be determined later. We insert the above expression of τ_n in the one of $Q(\tau)$. After a direct computation we get:

$$\begin{aligned} Q(\tau_n) = & (-1)^n K(1-i) \left\{ n \left[\omega\pi(1+i) + \sqrt{2}\alpha\omega^2\pi^2 X(1+i) \right] - \frac{1}{4}(1+i)\omega\pi X \right. \\ & + 2\omega\pi - \frac{\sqrt{2}}{2}\alpha\omega^2\pi^2 X(1+i) + \sqrt{2}\alpha\omega^2\pi^2 Y(1+i) \\ & \left. + \sqrt{2}\alpha\omega^2\pi^2 X^2 + \frac{A}{n} + O\left(\frac{1}{n^2}\right) \right\} \end{aligned}$$

where A is a complex which depends on x and y . By choosing the complex number in such a way that the coefficients of n^{2-i} , $i = 1, 2$ in $Q(\tau_n)$ vanish, we

get

$$x = \frac{-1+i}{4\alpha\pi}\sqrt{2}, \quad y = \frac{3\sqrt{2}(1+\pi) + i\sqrt{2}(2+3\pi)}{16\pi^2},$$

upon substitution of these values in the expression of τ_n , we get:

$$\tau_n = \left(n - \frac{1}{4}\right)\omega\pi + \frac{1}{4\alpha\pi n}\sqrt{2}(-1+i) + O\left(\frac{1}{n^2}\right), \quad n \in \mathbb{N}.$$

Taking into account that $\lambda_n = \tau_n^2$, we finally get

$$\lambda_n = -\frac{1}{\alpha} + O\left(\frac{1}{n}\right) + i \left[\left(n - \frac{1}{4}\right)^2 \pi^2 + O\left(\frac{1}{n^2}\right) \right], \quad n \in \mathbb{N}.$$

We have then proven the following assertion:

$$\operatorname{Re}(\lambda_n) \longrightarrow -\frac{1}{\alpha} \text{ as } n \longrightarrow +\infty.$$

3. FINITE DIFFERENCES METHOD

In this section we develop a numerical scheme based on the finite differences method for the eigenvalue problem associated with the evolutive system defined by (1.1)–(1.4). In practice the spectral problem for such system is not simple and cannot be solved by formula. Even when there is a formula, it might be so complicated that we would prefer to visualize the eigenvalues by looking at a graph. The finite differences method is an important technique of computation using quite simple equations and consists of replacing each derivative by a difference quotient. Consider for instance, a function $u(x)$ of one variable. Choose a mesh size $h = \Delta x$. We approximate the value $u(jh)$ for $x = jh$ by a number u_j indexed by an integer $j : u_j \sim u(jh)$. Using Taylor expansions

$$\begin{aligned} y(x_i - 4h) &= y(x_i) - 4hy'(x_i) + \frac{16}{2}h^2y''(x_i) - \frac{64}{3!}h^3y'''(x_i) + \frac{256}{4!}h^4y^{(4)}(x_i) \\ &\quad - \frac{1024}{5!}h^5y^{(5)}(x_i) + O(h^6) \end{aligned}$$

$$\begin{aligned} y(x_i - 3h) &= y(x_i) - 3hy'(x_i) + \frac{9}{2}h^2y''(x_i) - \frac{27}{3!}h^3y'''(x_i) + \frac{81}{4!}h^4y^{(4)}(x_i) \\ &\quad - \frac{243}{5!}h^5y^{(5)}(x_i) + O(h^6) \end{aligned}$$

$$\begin{aligned} y(x_i - 2h) &= y(x_i) - 2hy'(x_i) + 2h^2y''(x_i) - \frac{8}{6}h^3y^3(x_i) + \frac{16}{24}h^4y^4 \\ &\quad - \frac{32}{5!}h^5y^5(x_i) + O(h^6) \end{aligned}$$

$$\begin{aligned} y(x_i - h) &= y(x_i) - hy'(x_i) + \frac{1}{2}h^2y''(x_i) - \frac{1}{6}h^3y^3(x_i) + \frac{1}{24}h^4y^4 \\ &\quad - \frac{1}{5!}h^5y^5(x_i) + O(h^6) \end{aligned}$$

$$\begin{aligned} y(x_i + h) &= y(x_i) + hy'(x_i) + \frac{1}{2}h^2y''(x_i) + \frac{1}{6}h^3y^3(x_i) + \frac{1}{24}h^4y^4(x_i) \\ &\quad + \frac{1}{5!}h^5y^5(x_i) + O(h^6) \end{aligned}$$

$$\begin{aligned} y(x_i + 2h) &= y(x_i) + 2hy'(x_i) + 2h^2y''(x_i) + \frac{8}{6}h^3y^3(x_i) + \frac{16}{24}h^4y^4(x_i) \\ &\quad + \frac{32}{5!}h^5y^5(x_i) + O(h^6) \end{aligned}$$

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$$\begin{aligned} y(x_i + 4h) &= y(x_i) + 4hy'(x_i) + \frac{16}{2}h^2y''(x_i) + \frac{64}{3!}h^3y^3(x_i) + \frac{256}{4!}h^4y^4(x_i) \\ &\quad + \frac{1024}{5!}h^5y^5(x_i) + O(h^6) \end{aligned}$$

Also, using Taylor expansions we get

$$y^{(4)}(x_i) = \frac{y(x_i - 2h) - 4y(x_i - h) + 6y(x_i) - 4y(x_i + h) + y(x_i + 2h)}{h^4} + O(h^2),$$

$$\begin{aligned} y^{(3)}(x_i) &= \frac{5y(x_i) - 18y(x_i - h) + 24y(x_i - 2h) - 14y(x_i - 3h) + 3y(x_i - 4h)}{2h^3} \\ &\quad + O(h^2), \end{aligned}$$

$$\begin{aligned} y^{(3)}(x_i) &= \frac{-5y(x_i) + 18y(x_i + h) - 24y(x_i + 2h) + 14y(x_i + 3h) - 3y(x_i + 4h)}{2h^3} \\ &\quad + O(h^2), \end{aligned}$$

$$y^{(2)}(x_i) = \frac{2y(x_i) - 5y(x_i - h) + 4y(x_i - 2h) - y(x_i - 3h)}{h^2} + O(h^2).$$

We recall that the spectral problem to be solved is the following one

$$\begin{aligned} y_{xxxx} + \lambda^2 y &= 0 \\ y_{xxx}(1) &= 0 \\ y_{xx}(1) &= -(\alpha\lambda + \beta)y_x(1) \\ y_x(0) &= 0 \\ y(0) &= 0. \end{aligned}$$

With the above approximations in mind our spectral problem can be represented in the following discrete form:

$$(3.1) \quad 4y_1 - y_2 = 0,$$

$$(3.2) \quad y_{i-2} - 4y_{i-1} + 6y_i - 4y_{i+1} + y_{i+2} + \lambda^2 h^4 y_i = 0, \quad i = 2, \dots, n-2$$

$$(3.3) \quad -2y_{n-3} + 8y_{n-2} - 10y_{n-1} + 4y_n + (\alpha\lambda h + \beta h)(y_{n-2} - 4y_{n-1} + 3y_n) = 0$$

$$(3.4) \quad 3y_{n-4} - 14y_{n-3} + 24y_{n-2} - 18y_{n-1} + 5y_n = 0,$$

where $h = \frac{1}{n}$ denote the size, and $y_i = y(ih)$, for $i = 2, \dots, n-2$.

The finite differences method look for the complex number λ , such as it exists a non zero vector $(y_1, \dots, y_n)$ satisfies the above discrete problem. Now, we consider the following matrix of order n .

$$A = \begin{pmatrix} 0 & \dots & \dots & \dots & \dots & & 0 \\ \vdots & h^4 & 0 & & & & \vdots \\ \vdots & 0 & \ddots & \ddots & & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & & \ddots & h^4 & \vdots \\ \vdots & \vdots & & & & & 0 & 0 \\ 0 & \dots & \dots & \dots & \dots & & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & & & & & & \vdots \\ \vdots & & \ddots & & & & & \vdots \\ \vdots & & & \ddots & & & & \vdots \\ \vdots & & & & \ddots & & & \vdots \\ \vdots & & & & & \ddots & & \vdots \\ \vdots & & & & & & \ddots & 0 \\ \vdots & & & & 0 & \alpha h & -4\alpha h & 3\alpha h \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 4 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ -4 & 6 & -4 & 1 & 0 & & & \vdots \\ 1 & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & 0 & 1 & -4 & 6 & -4 & 1 \\ \vdots & \cdots & \cdots & 0 & -2 & 8 + \beta h & -10 - 4\beta h & 4 + 3\beta h \\ 0 & \cdots & 0 & 3 & -14 & 24 & -18 & 5 \end{pmatrix}$$

The set of equations (3.1) – (3.4) take the following matrix equation form:

$$(3.5) \quad \lambda^2 AY + \lambda BY + CY = 0, \quad Y = (y_1, \dots, y_n)^T,$$

where the matrix A , B and C are defined as above.

Now, we introduce the following auxiliary vector:

$$Z = \lambda Y,$$

then (3.5) is equivalent to the following system

$$\begin{aligned} AZ &= \lambda AY \\ -CY &= \lambda AZ + \lambda BY. \end{aligned}$$

This is a generalized eigenvalues problem. For such a problem we may use the "QZ" method [6]

$$PU = \lambda QU; \quad U = (Z, Y)^T.$$

The matrix P and Q are defined as follows

$$P = \begin{pmatrix} A & 0_N \\ 0_N & -C \end{pmatrix} \quad Q = \begin{pmatrix} 0_N & A \\ A & B \end{pmatrix},$$

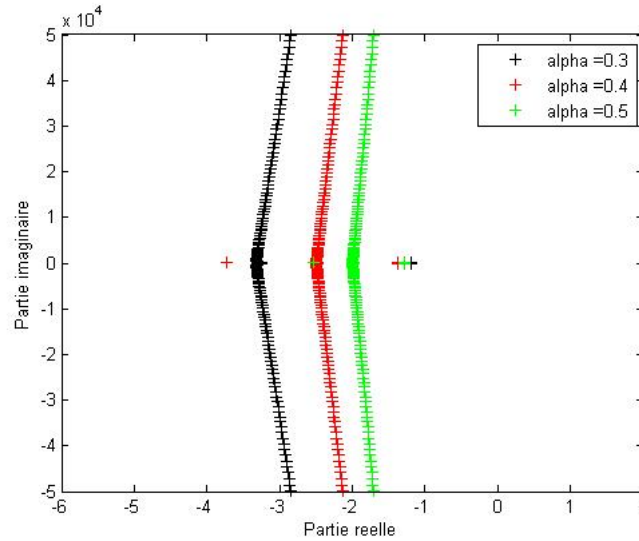
where 0_N is the zero matrix of N order.

4. NUMERICAL EXPERIMENTS

To evaluate the effect of parameters α and β on the spectrum, we give here in the same field, the graphs of the spectrum for different values of the parameters α and β , we do the study for several cases:

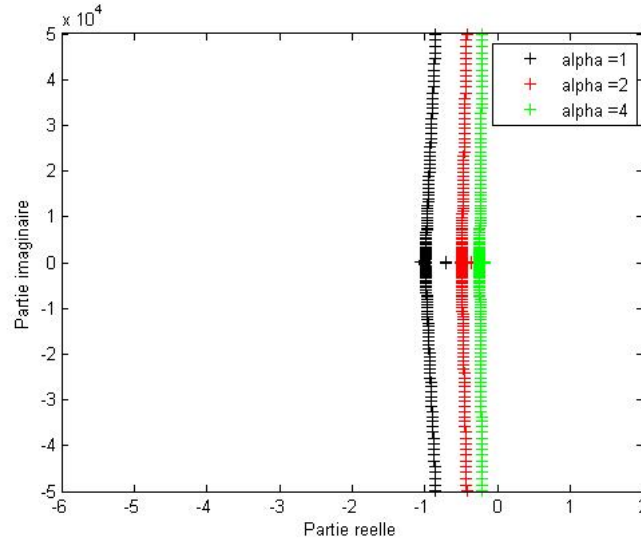
First case: $\beta = 0$

a) $\alpha \in]0; 1[$. We choose $\alpha = 0.3$, $\alpha = 0.4$ and $\alpha = 0.5$ for $n = 500$



We observe that when the parameter $\alpha \in]0; 1[$ increases, without control in rotation β the location of the spectrum moves rapidly on the left-hand side of the half plane.

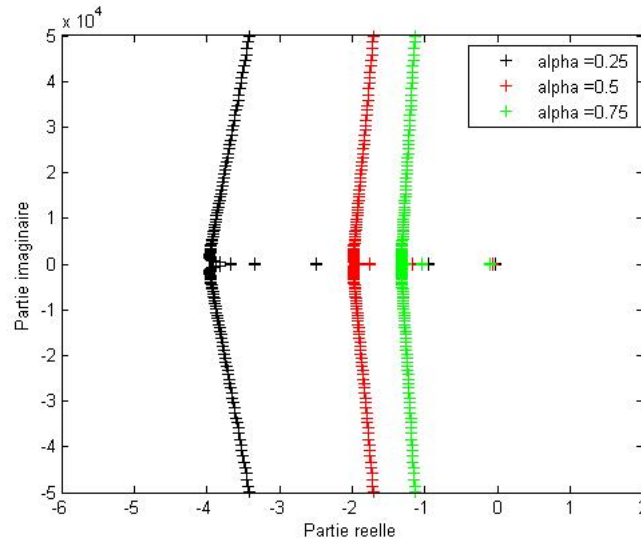
b) $\alpha \geq 1$. We choose $\alpha = 1$, $\alpha = 2$ and $\alpha = 4$ for $n = 500$



We observe that when the parameter α increases, without control in rotation β , the location of the spectrum moves rapidly on the left-hand side of the half plane.

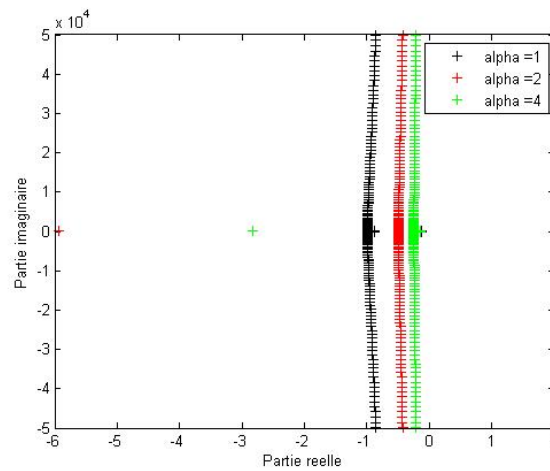
Second case: $\beta > 0$

a) $\alpha \in]0; 1[$. We fixe $\beta = 10$ take $\alpha = 0.25$, $\alpha = 0.5$ and $\alpha = 0.75$ for $n = 500$



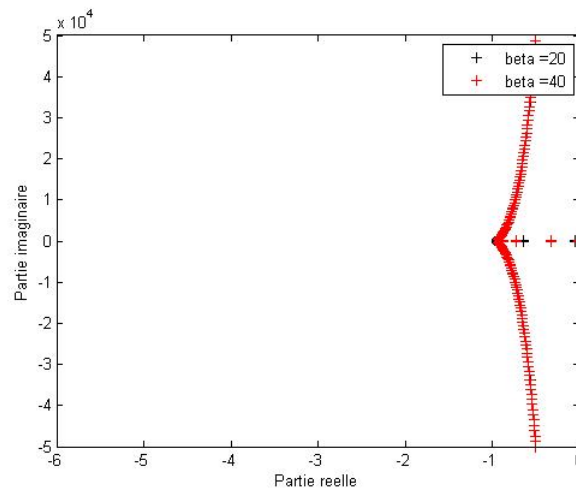
we observe that when the parameter $\alpha \in]0; 1[$ without control in rotation β the location of the spectrum moves rapidly on the left-hand side of the half plane.

b) $\alpha \geq 1$. We fixe $\beta = 10$ take $\alpha = 1$, $\alpha = 2$ and $\alpha = 4$ for $n = 500$



We observe that when the parameter α increases, without control in rotation β , the location of the spectrum moves rapidly on the left-hand side of the half plane.

Third case: We fix $\alpha = 1$ take $\beta = 20$ and $\beta = 40$ for $n = 500$



We observe that the parameter β has no effect on the spectrum for a fixed value of α and the location of the spectrum stays on the left-hand side of the left half plane.

5. CONCLUSION

From all previous observations we claim that the control feedback β in rotation has no influence on the optimal decay rate of the energy. But the control feedback in velocity rotation improves the optimal decay of the energy.

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