

DOUBLE INTEGRAL INVOLVING G-FUNCTION OF TWO VARIABLES

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ABSTRACT. In the study of certain boundary value problems integrals are useful with their connections. To obtain expansion formulae it also helps. In the study of integral equation, probability and statistical distribution, integrals are also used. To measure population density within a certain area, we can also use integrals. With integrals we can analyzed anything that changes in time. The object of this research paper is to establish a double integrals involving G-Function of two variables.

1. INTRODUCTION

The G-function of two variables was defined by Srivastava and Joshi, see [6], in terms of Mellin-Barnes type integrals as follows:

$$\begin{aligned}
 (1.1) \quad & G_{p_1, q_1; p_2, q_2; p_3, q_3}^{0, n_1; m_2, n_2; m_3, n_3} \left[\begin{matrix} x & (a_j; 1, 1)_{1, p_1} : (c_j, 1)_{1, p_2} : (e_j, 1)_{1, p_3} \\ y & (b_j; 1, 1)_{1, q_1} : (d_j, 1)_{1, q_2} : (f_j, 1)_{1, q_3} \end{matrix} \right] \\
 &= \frac{-1}{4\pi^2} \int_{L_1} \int_{L_2} \Phi_1(\xi, \eta) \theta_2(\xi) \theta_3(\eta) x^\xi y^\eta d\xi d\eta,
 \end{aligned}$$

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where

$$\begin{aligned}\Phi_1(\xi, \eta) &= \frac{\prod_{j=1}^{n_1} \Gamma(1 - a_j + \xi + \eta)}{\prod_{j=n_1+1}^{p_1} \Gamma(a_j - \xi - \eta) \prod_{j=1}^{q_1} \Gamma(1 - b_j + \xi + \eta)} \\ \theta_2(\xi) &= \frac{\prod_{j=1}^{m_2} \Gamma(d_j - \xi) \prod_{j=1}^{n_2} \Gamma(1 - c_j + \xi)}{\prod_{j=m_2+1}^{q_2} \Gamma(1 - d_j + \xi) \prod_{j=n_2+1}^{p_2} \Gamma(c_j - \xi)} \\ \theta_3(\eta) &= \frac{\prod_{j=1}^{m_3} \Gamma(f_j - \eta) \prod_{j=1}^{n_3} \Gamma(1 - e_j + \eta)}{\prod_{j=m_3+1}^{q_3} \Gamma(1 - f_j + \eta) \prod_{j=n_3+1}^{p_3} \Gamma(e_j - \eta)}\end{aligned}$$

x and y are not equal to zero, and an empty product is interpreted as unity. Further p_i, q_i, n_i and m_i are non negative integers such that $p_i \geq n_i \geq 0, q_i \geq 0, q_j \geq m_j \geq 0, i = 1, 2, 3, j = 2, 3$.

The contour L_1 is in the ξ -plane and runs from $-i\infty$ to $i\infty$, with loops, if necessary, to ensure that the poles of $\Gamma(d_j - \xi), j = 1, \dots, m_2$ lie to the right, and the poles of $\Gamma(1 - c_j + \xi), j = 1, \dots, n_2, \Gamma(1 - a_j + \xi + \eta), j = 1, \dots, n_1$ to the left of the contour.

The contour L_2 is in the η -plane and runs from $-i\infty$ to $i\infty$, with loops, if necessary, to ensure that the poles of $\Gamma(f_j - \eta), j = 1, \dots, m_3$ lie to the right, and the poles of $\Gamma(1 - e_j + \eta), j = 1, \dots, n_3, \Gamma(1 - a_j + \xi + \eta), j = 1, \dots, n_1$ to the left of the contour. The double integral converges if:

$$\begin{aligned}2(n_1 + m_2 + n_2) &> (p_1 + q_1 + p_2 + q_2) \\ 2(n_1 + m_3 + n_3) &> (p_1 + q_1 + p_3 + q_3)\end{aligned}$$

and $|\arg x| < \frac{1}{2}U\pi, |\arg y| < \frac{1}{2}V\pi$, where

$$(1.2) \quad U = \left[n_1 + m_2 + n_2 - \frac{1}{2}(p_1 + q_1 + p_2 + q_2) \right].$$

$$(1.3) \quad V = \left[n_1 + m_3 + n_3 - \frac{1}{2}(p_1 + q_1 + p_3 + q_3) \right].$$

These assumptions for the G-function of two variables will be adhered through this research work. Also, the following formulas are required in the proof, from [7]:

$$(1.4) \quad S_n^m[x] = \sum_{u=0}^{[n/m]} \frac{(-n)_{m_u}}{u!} X^u A_{n,u} \quad (n = 0, 1, 2, \dots)$$

where m is arbitrary positive integer, and the coefficients $A_{n,u}$ ($n, u \geq 0$) are arbitrary constants, real or complex.

2. INTEGRAL

Certain double integrals involving the Generalized Hypergeometric function and various polynomials have been evaluated by Kumari Shantha [3], Ayant Frederic [1], Mishra Raghunayak [4] and others from time to time. Following them, we evaluate some double integrals involving the G-function of two variables.

In this section, we shall establish the following integral:

$$\begin{aligned}
 & \int_0^1 \int_0^1 x^{\lambda-1} (1-x)^{\mu-1} y^{\rho-1} (1-y)^{a-2\rho} (1+ty)^{\rho-a-1} \\
 & \quad \cdot {}_2F_1 \left[\begin{matrix} a, b; \\ 1+a-b; \end{matrix} \frac{(1+t)y}{1+ty} \right] J_s^{(\alpha, \beta)}(1-2x; k) S_n^m [cy^\gamma (1+ty)^\gamma (1-y)^{-2\gamma}] \\
 & \quad \cdot G_{p_1, q_1; p_2, q_2; p_3, q_3}^{0, n_1; m_2, n_2; m_3, n_3} \left[\begin{matrix} z_1 x(1-x) \left\{ \frac{y(1+ty)}{(1-y)^2} \right\} \\ z_2 \end{matrix} \right] dx dy \\
 & = \frac{2^a (4+4t)^{-\rho} \Gamma(1+a/2) \Gamma(1+a-b) (a+1)_{ks}}{\sqrt{\pi} \Gamma(1+a) \Gamma(1+a/2-b) s!} \\
 & \quad \cdot \sum_{u=0}^{[n/m]} \sum_{j=0}^s \frac{(-n)_{m_u}}{u!} \frac{(\alpha + \beta + s + 1)_{kj} (-s)_j c^u}{(\alpha + 1)_{kj} j! (4+4t)^{\gamma u}} A_{n,u} \\
 (2.1) \quad & \times G_{p_1, q_1; p_2+4, q_2+3; p_3, q_3}^{0, n_1; m_2+2, n_2+3; m_3, n_3} \left[\begin{matrix} z_1 \{4(1+t)\}^{-1} \\ z_2 \end{matrix} \middle| \begin{matrix} (a_j; 1, 1)_{1, p_1} : A : (e_j, 1)_{1, p_3} \\ (b_j; 1, 1)_{1, q_1} : B : (f_j, 1)_{1, q_3} \end{matrix} \right]
 \end{aligned}$$

where

$$\begin{aligned}
 A &= (1-\mu, 1), (1-\lambda-kj), (1-\rho-\gamma u, 1), (c_j, 1)_{1, p_2}, (1+a-b-\rho-\gamma u, 1) \\
 B &= \left(\frac{1}{2} + \frac{a}{2} - \rho - \gamma u, 1\right), \left(1 + \frac{a}{2} - b - \rho - \gamma u, 1\right), (d_j, 1)_{1, q_2}, (1-\lambda-\mu-kj, 1)
 \end{aligned}$$

The integral (2.1) is valid if the following conditions hold:

- (i) m is arbitrary positive integer, and the coefficients $A_{n,u}$ ($n, u \geq 0$) are arbitrary constants, real or complex.

- (ii) $|\arg z_1| < \frac{1}{2}U\pi$, $|\arg z_2| < \frac{1}{2}V\pi$, where U and V are given by (1.2) and (1.3) respectively.
- (iii) $\operatorname{Re}(1+a-b) > 0$, $\operatorname{Re}(\alpha) > -1$, $\operatorname{Re}(\beta) > -1$, $t > -1$,
 $\gamma \geq 0$, $\operatorname{Re}(\lambda + \xi) > 0$, $\operatorname{Re}(\rho + \xi) > 0$,
 $\operatorname{Re}(\rho + \gamma u + \xi) > 0$, $\operatorname{Re}(1+a-2\rho-2\gamma u-\xi) > 0$.
- (iv) $\operatorname{Re}(1-2b) > 0$.

Proof. To establish (2.1) we will use the series representation of $S_n^m[x]$ as given by (1.4). For the G-function of two variables we will use the Mellin-Barnes types of contour integral as given in (1.1). On the left hand side of (2.1) we change the order of integration and summation, and we have for the left hand side of (2.1):

$$\begin{aligned}
 &= \sum_{u=0}^{[n/m]} \frac{(-n)_{m_u}}{u!} c^u A_{n,u} \frac{-1}{4\pi^2} \int_{L_1} \int_{L_2} \Phi_1(\xi, \eta) \theta_2(\xi) \theta_3(\eta) \\
 &\quad \cdot \int_0^1 x^{\lambda+\xi-1} (1-x)^{\mu+\xi-1} J_s^{(\alpha, \beta)}(1-2x; k) dx \\
 &\quad \cdot \int_0^1 y^{\rho+\gamma u+\xi-1} (1-y)^{a-2\rho-2\gamma u-\xi} (1+ty)^{\rho-a+\gamma u} dy \\
 &\quad \cdot {}_2F_1 \left[\begin{matrix} a, b; \\ 1+a+b; \end{matrix} \frac{(1+t)y}{1+ty} \right] z_1^\xi z_2^\eta d\xi d\eta.
 \end{aligned}$$

Now, using known results from [2] and [5], interpreting the resulting contour integral as the G-function of two variables, once we get the right hand side of (2.1). $\square$

3. SPECIAL CASE

If we put some parameters in (3.1) we get the following integral in terms of G-function of one variable:

$$\begin{aligned}
 &\int_0^1 \int_0^1 x^{\lambda-1} (1-x)^{\mu-1} y^{\rho-1} (1-y)^{a-2\rho} (1+ty)^{\rho-a-1} \\
 &\quad \cdot {}_2F_1 \left[\begin{matrix} a, b; \\ 1+a-b; \end{matrix} \frac{(1+t)y}{1+ty} \right] J_s^{(\alpha, \beta)}(1-2x; k) S_n^m[cy^\gamma (1+ty)^\gamma (1-y)^{-2\gamma}]
 \end{aligned}$$

$$\begin{aligned}
& \cdot G_{p,q}^{l,v} \left[zx(1-x) \left\{ \frac{y(1+ty)}{(1-y)^2} \right\} \middle|_{(b_j,1)_{1,q}}^{(a_j,1)_{1,p}} \right] dx dy \\
& = \frac{2^a (4+4t)^{-\rho} \Gamma(1+a/2) \Gamma(1+a-b) (a+1)_{ks}}{\sqrt{\pi} \Gamma(1+a) \Gamma(1+a/2-b) s!} \\
& \quad \cdot \sum_{u=0}^{n/m} \sum_{j=0}^s \frac{(-n)_{m_u}}{u!} \frac{(\alpha+\beta+s+1)_{kj} (-s)_j c^u}{(\alpha+1)_{kj} j! (4+4t)^{\gamma u}} A_{n,u} \\
(3.1) \quad & \times G_{p+4,q+3}^{l+2,v+3} \left[z \{4(1+t)\}^{-1} \middle|_B^A \right]
\end{aligned}$$

where

$$\begin{aligned}
A &= (1-\mu, 1), (1-\lambda-kj), (1-\rho-\gamma u, 1), (a_j, 1)_{1,p}, (1+a-b-\rho-\gamma u, 1) \\
B &= \left(\frac{1}{2} + \frac{a}{2} - \rho - \gamma u, 1\right), \left(1 + \frac{a}{2} - b - \rho - \gamma u, 1\right), (b_j, 1)_{1,q}, (1-\lambda-\mu-kj, 1)
\end{aligned}$$

The integral (3.1) is valid if the following conditions hold:

- (i) m is arbitrary positive integer, and the coefficients $A_{n,u}$ ($n, u \geq 0$) are arbitrary constants, real or complex.
- (ii) $\Omega = \sum_{j=1}^v \alpha_j - \sum_{j=v+1}^p \alpha_j + \sum_{j=1}^l \beta_j - \sum_{j=l+1}^q \beta_j > 0$ and $|\arg z| < \frac{1}{2}\Omega\pi$.
- (iii) $\operatorname{Re}(1+a-b) > 0$, $\operatorname{Re}(\alpha) > -1$, $\operatorname{Re}(\beta) > -1$, $t > -1$,
 $\gamma \geq 0$, $\operatorname{Re}(\lambda + \xi) > 0$, $\operatorname{Re}(\rho + \xi) > 0$,
 $\operatorname{Re}(\rho + \gamma u + \xi) > 0$, $\operatorname{Re}(1+a-2\rho-2\gamma u-\xi) > 0$.
- (iv) $\operatorname{Re}(1-2b) > 0$.

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