

WEAK QUASI CONTINUITY FOR MULTIFUNCTIONS IN IDEAL TOPOLOGICAL SPACES

CHAWALIT BOONPOK

ABSTRACT. The purpose of the present article is to introduce the concepts of almost quasi $\star$ -continuous and weakly quasi $\star$ -continuous multifunctions. Several characterizations of almost quasi $\star$ -continuous and weakly quasi $\star$ -continuous multifunctions are investigated.

1. INTRODUCTION

Continuity in topological spaces, as a significant and fundamental subject for the study of topology, has been researched by several mathematicians. Many authors have researched and studied several stronger and weaker forms of continuous functions and multifunctions. Marcus [12] introduced the notion of quasi continuous functions. Popa [15] introduced and investigated the notion of almost quasi continuous functions. In [16], the present authors introduced and studied the notion of weakly quasi continuous functions. The concept of quasi continuous multifunctions was firstly introduced and studied by Bânzara and Crivăţ [3]. Popa and Noiri [14] introduced the concept of almost quasi continuous multifunctions and investigated some characterizations of such multifunctions. The notion of weakly quasi continuous multifunctions

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was introduced and investigated by the present authors [13]. Several characterizations of weakly quasi continuous multifunctions have been obtained in [14].

The concept of ideals in topological spaces has been studied by Kuratowski [10] and Vaidyanathaswamy [17] which is one of the important areas of research in the branch of mathematics. Janković and Hamlett [9] introduced the notion of $\mathcal{I}$ -open sets in ideal topological spaces. Abd El-Monsef et al. [1] further investigated $\mathcal{I}$ -open sets and $\mathcal{I}$ -continuous functions. Later, several authors studied ideal topological spaces giving several convenient definitions. Some authors obtained decompositions of continuity. For instance, Açıkgöz et al. [2] introduced and investigated the notions of weakly- $\mathcal{I}$ -continuous and weak * - $\mathcal{I}$ -continuous functions in ideal topological spaces. Kuyucu et al. [11] investigated their relationships with continuous and θ -continuous functions. Hatir and Noiri [8] introduced the notions of semi- $\mathcal{I}$ -open sets, α - $\mathcal{I}$ -open sets and β - $\mathcal{I}$ -open sets via idealization and using these sets obtained new decompositions of continuity. In [7], the present authors investigated further properties of semi- $\mathcal{I}$ -open sets and semi- $\mathcal{I}$ -continuity. Hatir and Noiri [6] introduced and investigated the concepts of weakly pre- $\mathcal{I}$ -open sets and weakly pre- $\mathcal{I}$ -continuous functions.

In this paper, we introduce the concepts of almost quasi $\star$ -continuous and weakly quasi $\star$ -continuous multifunctions. Moreover, some characterizations of almost quasi $\star$ -continuous and weakly quasi $\star$ -continuous multifunctions are investigated. Furthermore, the relationships between almost quasi $\star$ -continuity and weak quasi $\star$ -continuity are discussed.

2. PRELIMINARIES

Throughout the present paper, spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. In a topological space (X, τ) , the closure and the interior of any subset A of X will denoted by $\text{Cl}(A)$ and $\text{Int}(A)$, respectively. An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X satisfying the following properties: (1) $A \in \mathcal{I}$ and $B \subseteq A$ implies $B \in \mathcal{I}$; (2) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. A topological space (X, τ) with an ideal $\mathcal{I}$ on X is called an ideal topological space and is denoted by $(X, \tau, \mathcal{I})$. For an

ideal topological space $(X, \tau, \mathcal{J})$ and a subset A of X , $A^*(\mathcal{J})$ is defined as follows: $A^*(\mathcal{J}) = \{x \in X : U \cap A \notin \mathcal{J} \text{ for every open neighbourhood } U \text{ of } x\}$. In case there is no chance for confusion, $A^*(\mathcal{J})$ is simply written as A^* . In [10], A^* is called the local function of A with respect to $\mathcal{J}$ and τ . Observe additionally that $Cl^*(A) = A^* \cup A$ defines a Kuratowski closure operator for a topology $\tau^*(\mathcal{J})$ finer than τ , generated by the base $\mathcal{B}(\mathcal{J}, \tau) = \{U - I_0 \mid U \in \tau \text{ and } I_0 \in \mathcal{J}\}$. However, $\mathcal{B}(\mathcal{J}, \tau)$ is not always a topology [17]. A subset A is said to be $\star$ -closed [9] if $A^* \subseteq A$. The complement of a $\star$ -closed set is said to be $\star$ -open. The interior of a subset A in $(X, \tau^*(\mathcal{J}))$ is denoted by $Int^*(A)$.

By a multifunction $F : X \rightarrow Y$, we mean a point-to-set correspondence from X into Y , and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \rightarrow Y$, following [4], we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$. Then F is said to be surjection if $F(X) = Y$, or equivalent, if for each $y \in Y$ there exists $x \in X$ such that $y \in F(x)$ and F is called injection if $x \neq y$ implies $F(x) \cap F(y) = \emptyset$.

Let $\mathcal{P}(X)$ be the collection of all nonempty subsets of X . For any $\star$ -open set V of an ideal topological space $(X, \tau, \mathcal{J})$, we denote

$$V^+ = \{B \in \mathcal{P}(X) \mid B \subseteq V\}$$

and $V^- = \{B \in \mathcal{P}(X) \mid B \cap V \neq \emptyset\}$.

Definition 2.1. [5] A subset A of an ideal topological space $(X, \tau, \mathcal{J})$ is said to be:

- (1) R - $\mathcal{J}^*$ -open if $A = Int^*(Cl^*(A))$;
- (2) R - $\mathcal{J}^*$ -closed if its complement is R - $\mathcal{J}^*$ -open;
- (3) $\mathcal{J}^*$ -preopen if $A \subseteq Int^*(Cl^*(A))$;
- (4) $\mathcal{J}^*$ -preclosed if its complement is $\mathcal{J}^*$ -preopen.

Definition 2.2. A subset A of an ideal topological space $(X, \tau, \mathcal{J})$ is said to be:

- (1) semi- $\mathcal{J}^*$ -open if $A \subseteq Cl^*(Int^*(A))$;
- (2) semi- $\mathcal{J}^*$ -closed if its complement is semi- $\mathcal{J}^*$ -open;
- (3) semi- $\mathcal{J}^*$ -preopen if $A \subseteq Cl^*(Int^*(Cl^*(A)))$;
- (4) semi- $\mathcal{J}^*$ -preclosed if its complement is semi- $\mathcal{J}^*$ -preopen.

For a subset A of an ideal topological space $(X, \tau, \mathcal{I})$, the intersection of all semi- $\mathcal{I}^*$ -closed sets containing A is called the *semi- $\mathcal{I}^*$ -closure* of A and is denoted by $sCl_{\mathcal{I}^*}(A)$. The union of all semi- $\mathcal{I}^*$ -open sets contained in A is called the *semi- $\mathcal{I}^*$ -interior* of A and is denoted by $sInt_{\mathcal{I}^*}(A)$.

Proposition 2.1. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $\{A_\gamma \mid \gamma \in \Gamma\}$ be a family of subsets of X . Then, the following properties hold:*

- (1) *If A_γ is semi- $\mathcal{I}^*$ -open for each $\gamma \in \Gamma$, then $\bigcup_{\gamma \in \Gamma} A_\gamma$ is semi- $\mathcal{I}^*$ -open.*
- (2) *If A_γ is semi- $\mathcal{I}^*$ -closed for each $\gamma \in \Gamma$, then $\bigcap_{\gamma \in \Gamma} A_\gamma$ is semi- $\mathcal{I}^*$ -closed.*

Proof. (1) Suppose that A_γ is semi- $\mathcal{I}^*$ -open for each $\gamma \in \Gamma$. Then $A_\gamma \subseteq Cl^*(Int^*(A_\gamma)) \subseteq Cl^*(Int^*(\bigcup_{\gamma \in \Gamma} A_\gamma))$ and hence $\bigcup_{\gamma \in \Gamma} A_\gamma \subseteq Cl^*(Int^*(\bigcup_{\gamma \in \Gamma} A_\gamma))$. Consequently, we obtain $\bigcup_{\gamma \in \Gamma} A_\gamma$ is semi- $\mathcal{I}^*$ -open.

(2) Suppose that A_γ is semi- $\mathcal{I}^*$ -closed for each $\gamma \in \Gamma$. Then $X - A_\gamma$ is semi- $\mathcal{I}^*$ -open for each $\gamma \in \Gamma$. By (1), we have $\bigcup_{\gamma \in \Gamma} (X - A_\gamma) = X - \bigcap_{\gamma \in \Gamma} A_\gamma$ is semi- $\mathcal{I}^*$ -open and hence $\bigcap_{\gamma \in \Gamma} A_\gamma$ is semi- $\mathcal{I}^*$ -closed.

□

Proposition 2.2. *For a subset A of an ideal topological space $(X, \tau, \mathcal{I})$, the following properties hold:*

- (1) *$sInt_{\mathcal{I}^*}(A)$ is semi- $\mathcal{I}^*$ -open.*
- (2) *$sCl_{\mathcal{I}^*}(A)$ is semi- $\mathcal{I}^*$ -closed.*
- (3) *A is semi- $\mathcal{I}^*$ -open if and only if $A = sInt_{\mathcal{I}^*}(A)$.*
- (4) *A is semi- $\mathcal{I}^*$ -closed if and only if $A = sCl_{\mathcal{I}^*}(A)$.*

Proof. (1) and (2) follows from Proposition 2.1.

(3) and (4) follows from (1) and (2).

□

Proposition 2.3. *For a subset A of an ideal topological space $(X, \tau, \mathcal{I})$, $x \in sCl_{\mathcal{I}^*}(A)$ if and only if $U \cap A \neq \emptyset$ for every semi- $\mathcal{I}^*$ -open set U containing x .*

Proof. Let $x \in sCl_{\mathcal{I}^*}(A)$. Suppose that $U \cap A = \emptyset$ for some semi- $\mathcal{I}^*$ -open set U containing x . Then $A \subseteq X - U$ and $X - U$ is semi- $\mathcal{I}^*$ -closed. Since $x \in sCl_{\mathcal{I}^*}(A)$, we have $x \in sCl_{\mathcal{I}^*}(X - U) = X - U$; hence $x \notin U$, which is a contradiction that $x \in U$. Therefore, $U \cap A \neq \emptyset$.

Conversely, assume that $U \cap A \neq \emptyset$ for every semi- $\mathcal{I}^*$ -open set U containing x . We shall show that $x \in sCl_{\mathcal{I}^*}(A)$. Suppose that $x \notin sCl_{\mathcal{I}^*}(A)$. Then, there

exists a semi- $\mathcal{J}^*$ -closed set F such that $A \subseteq F$ and $x \notin F$. Therefore, we obtain $X - F$ is a semi- $\mathcal{J}^*$ -open set containing x such that $(X - F) \cap A = \emptyset$. This a contradiction to $U \cap A \neq \emptyset$; hence $x \in sCl_{\mathcal{J}^*}(A)$. $\square$

Proposition 2.4. *For a subset A of an ideal topological space $(X, \tau, \mathcal{J})$, the following properties hold:*

- (1) $X - sCl_{\mathcal{J}^*}(A) = sInt_{\mathcal{J}^*}(X - A)$.
- (2) $X - sInt_{\mathcal{J}^*}(A) = sCl_{\mathcal{J}^*}(X - A)$.

Proof. (1) Let $x \in X - sCl_{\mathcal{J}^*}(A)$. Then $x \notin sCl_{\mathcal{J}^*}(A)$, there exists a semi- $\mathcal{J}^*$ -open set V containing x such that $V \cap A = \emptyset$. Then $V \subseteq X - A$ and hence $x \in sInt_{\mathcal{J}^*}(X - A)$. Thus, $X - sCl_{\mathcal{J}^*}(A) \subseteq sInt_{\mathcal{J}^*}(X - A)$. On the other hand, let $x \in sInt_{\mathcal{J}^*}(X - A)$. Then, there exists a semi- $\mathcal{J}^*$ -open set V containing x such that $V \subseteq X - A$ and hence $V \cap A = \emptyset$. By Proposition 2.3, we have $x \notin sCl_{\mathcal{J}^*}(A)$; hence $x \in X - sCl_{\mathcal{J}^*}(A)$. Therefore, $sInt_{\mathcal{J}^*}(X - A) \subseteq X - sCl_{\mathcal{J}^*}(A)$. Consequently, we obtain $X - sCl_{\mathcal{J}^*}(A) = sInt_{\mathcal{J}^*}(X - A)$.

- (2) This follows from (1). $\square$

Definition 2.3. *A point x in an ideal topological space $(X, \tau, \mathcal{J})$ is called a $\star_\theta$ -cluster point of A if $Cl^*(U) \cap A \neq \emptyset$ for every $\star$ -open set U of X containing x . The set of all $\star_\theta$ -cluster points of A is called the $\star_\theta$ -closure of A and is denoted by $\star_\theta Cl(A)$.*

Definition 2.4. *A subset A of an ideal topological space $(X, \tau, \mathcal{J})$ is said to be:*

- (1) $\star_\theta$ -closed if $\star_\theta Cl(A) = A$;
- (2) $\star_\theta$ -open if its complement is $\star_\theta$ -closed.

Lemma 2.1. *For subset A of an ideal topological space $(X, \tau, \mathcal{J})$, the following properties hold:*

- (1) If A is $\star$ -open in X , then $Cl^*(A) = \star_\theta Cl(A)$.
- (2) $\star_\theta Cl(A)$ is $\star$ -closed in X .

Proof. (1) Suppose that $x \notin Cl^*(A)$. There exists a $\star$ -open set U containing x such that $U \cap A = \emptyset$; hence $Cl^*(U) \cap A = \emptyset$. This shows that $x \notin \star_\theta Cl(A)$ and so $\star_\theta Cl(A) \subseteq Cl^*(A)$. On the other hand, we have $Cl^*(A) \subseteq \star_\theta Cl(A)$. Therefore, $Cl^*(A) = \star_\theta Cl(A)$.

- (2) Let $x \in X - \star_\theta \text{Cl}(A)$. Then, we have $x \notin \star_\theta \text{Cl}(A)$ and so there exists a $\star$ -open set U_x containing x such that $\text{Cl}^*(U_x) \cap A = \emptyset$. Thus, $\star_\theta \text{Cl}(A) \cap U_x = \emptyset$ and hence $x \in U_x \subseteq X - \star_\theta \text{Cl}(A)$. Therefore, we obtain $X - \star_\theta \text{Cl}(A) = \bigcup_{x \in X - \star_\theta \text{Cl}(A)} U_x$. This shows that $\star_\theta \text{Cl}(A)$ is $\star$ -closed. $\square$

Lemma 2.2. *For a subset A of an ideal topological space $(X, \tau, \mathcal{J})$, the following properties hold:*

- (1) $s\text{Cl}_{\mathcal{J}^*}(A) = A \cup \text{Int}^*(\text{Cl}^*(A))$.
 (2) $s\text{Int}_{\mathcal{J}^*}(A) = A \cap \text{Cl}^*(\text{Int}^*(A))$.

Proof. (1) To begin, observe that

$$\begin{aligned} \text{Int}^*(\text{Cl}^*(A \cup \text{Int}^*(\text{Cl}^*(A)))) &\subseteq \text{Int}^*(\text{Cl}^*(\text{Cl}^*(A) \cup \text{Int}^*(\text{Cl}^*(A)))) \\ &= \text{Int}^*(\text{Cl}^*(A)) \subseteq A \cup \text{Int}^*(\text{Cl}^*(A)). \end{aligned}$$

Thus, $A \cup \text{Int}^*(\text{Cl}^*(A))$ is semi- $\mathcal{J}^*$ -closed and so $s\text{Cl}_{\mathcal{J}^*}(A) \subseteq A \cup \text{Int}^*(\text{Cl}^*(A))$. On the other hand, since $s\text{Cl}_{\mathcal{J}^*}(A)$ is semi- $\mathcal{J}^*$ -closed, we have

$$\text{Int}^*(\text{Cl}^*(A)) \subseteq \text{Int}^*(\text{Cl}^*(s\text{Cl}_{\mathcal{J}^*}(A))) \subseteq s\text{Cl}_{\mathcal{J}^*}(A)$$

and hence $A \cup \text{Int}^*(\text{Cl}^*(A)) \subseteq s\text{Cl}_{\mathcal{J}^*}(A)$. Therefore, $s\text{Cl}_{\mathcal{J}^*}(A) = A \cup \text{Int}^*(\text{Cl}^*(A))$.

- (2) This follows from (1). $\square$

3. ALMOST QUASI $\star$ -CONTINUOUS MULTIFUNCTIONS

In this section, we introduce the notion of almost quasi $\star$ -continuous multifunctions. Furthermore, several characterizations of almost quasi $\star$ -continuous multifunctions are discussed.

Definition 3.1. A multifunction $F : (X, \tau, \mathcal{J}) \rightarrow (Y, \sigma, \mathcal{J})$ is said to be almost quasi $\star$ -continuous at a point $x \in X$ if for any $\star$ -open sets V_1, V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$ and each $\star$ -open set U containing x , there exists a nonempty $\star$ -open set G of X such that $G \subseteq U$, $F(G) \subseteq s\text{Cl}_{\mathcal{J}^*}(V_1)$ and $F(z) \cap s\text{Cl}_{\mathcal{J}^*}(V_2) \neq \emptyset$ for every $z \in G$. A multifunction $F : (X, \tau, \mathcal{J}) \rightarrow (Y, \sigma, \mathcal{J})$ is said to be almost quasi $\star$ -continuous if F is almost quasi $\star$ -continuous at each point of X .

Theorem 3.1. For a multifunction $F : (X, \tau, \mathcal{J}) \rightarrow (Y, \sigma, \mathcal{J})$, the following properties are equivalent:

- (1) F is almost quasi $\star$ -continuous at $x \in X$;
- (2) for every $\star$ -open sets V_1, V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$, there exists a semi- $\mathcal{J}^\star$ -open set U of X containing x such that $F(U) \subseteq sCl_{\mathcal{J}^\star}(V_1)$ and $F(z) \cap sCl_{\mathcal{J}^\star}(V_2) \neq \emptyset$ for every $z \in U$;
- (3) $x \in sInt_{\mathcal{J}^\star}(F^+(sCl_{\mathcal{J}^\star}(V_1)) \cap F^-(sCl_{\mathcal{J}^\star}(V_2)))$ for every $\star$ -open sets V_1, V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$;
- (4) $x \in Cl^\star(Int^\star(F^+(sCl_{\mathcal{J}^\star}(V_1)) \cap F^-(sCl_{\mathcal{J}^\star}(V_2))))$ for every $\star$ -open sets V_1, V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$.

Proof. (1) $\Rightarrow$ (2): Let $\mathcal{U}(x)$ the family of all $\star$ -open sets of X containing x . Let V_1, V_2 be any $\star$ -open sets of Y such that $F(x) \in V_1^+ \cap V_2^-$. For each $H \in \mathcal{U}(x)$, there exists a nonempty $\star$ -open set G_H such that $G_H \subseteq H$, $F(G_H) \subseteq sCl_{\mathcal{J}^\star}(V_1)$ and $F(y) \cap sCl_{\mathcal{J}^\star}(V_2) \neq \emptyset$ for every $y \in G_H$. Let $W = \cup\{G_H \mid H \in \mathcal{U}(x)\}$. Then W is $\star$ -open in X , $x \in Cl^\star(W)$, $F(W) \subseteq sCl_{\mathcal{J}^\star}(V_1)$ and $F(w) \cap sCl_{\mathcal{J}^\star}(V_2) \neq \emptyset$ for every $w \in W$. Put $U = W \cup \{x\}$, then $W \subseteq U \subseteq Cl^\star(W)$. Therefore, we obtain U is a semi- $\mathcal{J}^\star$ -open set of X containing x such that $F(U) \subseteq sCl_{\mathcal{J}^\star}(V_1)$ and $F(z) \cap sCl_{\mathcal{J}^\star}(V_2) \neq \emptyset$ for every $z \in U$.

(2) $\Rightarrow$ (3): Let V_1, V_2 be any $\star$ -open sets of Y such that $F(x) \in V_1^+ \cap V_2^-$. Then, there exists a semi- $\mathcal{J}^\star$ -open set U of X containing x such that $F(U) \subseteq sCl_{\mathcal{J}^\star}(V_1)$ and $F(z) \cap sCl_{\mathcal{J}^\star}(V_2) \neq \emptyset$ for every $z \in U$. Thus,

$$x \in U \subseteq F^+(sCl_{\mathcal{J}^\star}(V_1)) \cap F^-(sCl_{\mathcal{J}^\star}(V_2))$$

and hence $x \in U \subseteq sInt_{\mathcal{J}^\star}(F^+(sCl_{\mathcal{J}^\star}(V_1)) \cap F^-(sCl_{\mathcal{J}^\star}(V_2)))$.

(3) $\Rightarrow$ (4): Let V_1, V_2 be any $\star$ -open sets of Y such that $F(x) \in V_1^+ \cap V_2^-$. By (3), $x \in sInt_{\mathcal{J}^\star}(F^+(sCl_{\mathcal{J}^\star}(V_1)) \cap F^-(sCl_{\mathcal{J}^\star}(V_2)))$. Now, put

$$U = sInt_{\mathcal{J}^\star}(F^+(sCl_{\mathcal{J}^\star}(V_1)) \cap F^-(sCl_{\mathcal{J}^\star}(V_2))).$$

Then, we have U is semi- $\mathcal{J}^\star$ -open in X and by Lemma 2.2,

$$x \in U \subseteq Cl^\star(Int^\star(U)) \subseteq Cl^\star(Int^\star(F^+(sCl_{\mathcal{J}^\star}(V_1)) \cap F^-(sCl_{\mathcal{J}^\star}(V_2)))).$$

(4) $\Rightarrow$ (1): Let U be any $\star$ -open set of X containing x and V_1, V_2 be any $\star$ -open sets of Y such that $F(x) \in V_1^+ \cap V_2^-$. Then, we have

$$x \in Cl^\star(Int^\star(F^+(sCl_{\mathcal{J}^\star}(V_1)) \cap F^-(sCl_{\mathcal{J}^\star}(V_2)))))$$

and hence $U \cap \text{Int}^*(F^+(s\text{Cl}_{\mathcal{J}^*}(V_1)) \cap F^-(s\text{Cl}_{\mathcal{J}^*}(V_2))) \neq \emptyset$. Put

$$W = U \cap \text{Int}^*(F^+(s\text{Cl}_{\mathcal{J}^*}(V_1)) \cap F^-(s\text{Cl}_{\mathcal{J}^*}(V_2))).$$

Then, we have W is a nonempty $\star$ -open set of X such that $W \subseteq U$, $F(W) \subseteq s\text{Cl}_{\mathcal{J}^*}(V_1)$ and $F(w) \cap s\text{Cl}_{\mathcal{J}^*}(V_2) \neq \emptyset$ for every $w \in W$. This shows that F is almost quasi $\star$ -continuous at x . $\square$

The following results give some characterizations of almost quasi $\star$ -continuous multifunctions.

Theorem 3.2. *For a multifunction $F : (X, \tau, \mathcal{J}) \rightarrow (Y, \sigma, \mathcal{J})$, the following properties are equivalent:*

- (1) F is almost quasi $\star$ -continuous;
- (2) for each $x \in X$ and every $\star$ -open sets V_1, V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$, there exists a semi- $\mathcal{J}^*$ -open set U of X containing x such that $F(U) \subseteq s\text{Cl}_{\mathcal{J}^*}(V_1)$ and $F(z) \cap s\text{Cl}_{\mathcal{J}^*}(V_2) \neq \emptyset$ for every $z \in U$;
- (3) $F^+(V_1) \cap F^-(V_2)$ is semi- $\mathcal{J}^*$ -open in X for every R - $\mathcal{J}^*$ -open sets V_1, V_2 of Y ;
- (4) $F^+(V_1) \cap F^-(V_2) \subseteq s\text{Int}_{\mathcal{J}^*}(F^+(s\text{Cl}_{\mathcal{J}^*}(V_1)) \cap F^-(s\text{Cl}_{\mathcal{J}^*}(V_2)))$ for every $\star$ -open sets V_1, V_2 of Y ;
- (5)

$$\begin{aligned} & s\text{Cl}_{\mathcal{J}^*}(F^-(\text{Cl}^*(\text{Int}^*(\text{Cl}^*(B_1)))) \cup F^+\text{Cl}^*(\text{Int}^*(\text{Cl}^*(B_2)))) \\ & \subseteq F^-(\text{Cl}^*(B_1)) \cup F^+(\text{Cl}^*(B_2)) \end{aligned}$$

for every subsets B_1, B_2 of Y ;

- (6) $F^+(V_1) \cap F^-(V_2) \subseteq \text{Cl}^*(\text{Int}^*(F^+(s\text{Cl}_{\mathcal{J}^*}(V_1)) \cap F^-(s\text{Cl}_{\mathcal{J}^*}(V_2))))$ for every $\star$ -open sets V_1, V_2 of Y .

Proof. (1) $\Rightarrow$ (2): The proof follows from Theorem 3.1.

(2) $\Rightarrow$ (3): Let V_1, V_2 be any R - $\mathcal{J}^*$ -open sets of Y and $x \in F^+(V_1) \cap F^-(V_2)$. Then, we have $F(x) \in V_1^+ \cap V_2^-$ and there exists a semi- $\mathcal{J}^*$ -open set U of X containing x such that $F(U) \subseteq V_1$ and $F(z) \cap V_2 \neq \emptyset$ for every $z \in U$. Therefore, $x \in U \subseteq F^+(V_1) \cap F^-(V_2)$ and hence $x \in s\text{Int}_{\mathcal{J}^*}(F^+(V_1) \cap F^-(V_2))$. Thus, $F^+(V_1) \cap F^-(V_2) \subseteq s\text{Int}_{\mathcal{J}^*}(F^+(V_1) \cap F^-(V_2))$. This shows that $F^+(V_1) \cap F^-(V_2)$ is semi- $\mathcal{J}^*$ -open in X .

(3) $\Rightarrow$ (4): Let V_1, V_2 be any $\star$ -open sets of Y such that $x \in F^+(V_1) \cap F^-(V_2)$. Then, we have $F(x) \subseteq V_1 \subseteq s\text{Cl}_{\mathcal{J}^*}(V_1)$ and $\emptyset \neq F(x) \cap V_2 \subseteq F(x) \cap s\text{Cl}_{\mathcal{J}^*}(V_2)$.

Thus, $x \in F^+(sCl_{\mathcal{J}^*}(V_1))$ and $x \in F^-(sCl_{\mathcal{J}^*}(V_2))$. By Lemma 2.2, $sCl_{\mathcal{J}^*}(V_1)$ and $sCl_{\mathcal{J}^*}(V_2)$ are R - $\mathcal{J}^*$ -open sets and by (3), $F^+(sCl_{\mathcal{J}^*}(V_1)) \cap F^-(sCl_{\mathcal{J}^*}(V_2))$ is semi- $\mathcal{J}^*$ -open in X and $x \in sInt_{\mathcal{J}^*}(F^+(sCl_{\mathcal{J}^*}(V_1)) \cap F^-(sCl_{\mathcal{J}^*}(V_2)))$. Consequently, we obtain $F^+(V_1) \cap F^-(V_2) \subseteq sInt_{\mathcal{J}^*}(F^+(sCl_{\mathcal{J}^*}(V_1)) \cap F^-(sCl_{\mathcal{J}^*}(V_2)))$.

(4) $\Rightarrow$ (5): Let B_1, B_2 be any subsets of Y . Then, we have $Y - Cl^*(B_1)$ and $Y - Cl^*(B_2)$ are $\star$ -open sets of Y . By (4) and Lemma 2.2,

$$\begin{aligned} & X - (F^-(Cl^*(B_1)) \cup F^+(Cl^*(B_2))) \\ &= (X - F^-(Cl^*(B_1))) \cap (X - F^+(Cl^*(B_2))) \\ &= F^+(Y - Cl^*(B_1)) \cap F^-(Y - Cl^*(B_2)) \\ &\subseteq sInt_{\mathcal{J}^*}[F^+(sCl_{\mathcal{J}^*}(Y - Cl^*(B_1))) \cap F^-(sCl_{\mathcal{J}^*}(Y - Cl^*(B_2)))] \\ &= sInt_{\mathcal{J}^*}[(X - F^-(Cl^*(Int^*(Cl^*(B_1)))) \cap (X - F^+(Cl^*(Int^*(Cl^*(B_1))))) \\ &= X - sCl_{\mathcal{J}^*}[F^-(Cl^*(Int^*(Cl^*(B_1)))) \cup F^+(Cl^*(Int^*(Cl^*(B_2)))] \end{aligned}$$

and hence

$$\begin{aligned} & sCl_{\mathcal{J}^*}(F^-(Cl^*(Int^*(Cl^*(V_1)))) \cup F^+(Cl^*(Int^*(Cl^*(V_2)))) \\ &\subseteq F^-(Cl^*(V_1)) \cup F^+(Cl^*(V_2)). \end{aligned}$$

(5) $\Rightarrow$ (6): Let V_1, V_2 be any $\star$ -open sets of Y . Then $Y - V_1$ and $Y - V_2$ are $\star$ -closed sets of Y . By (5) and Lemma 2.2, we have

$$\begin{aligned} & Int^*(Cl^*(F^-(Cl^*(Int^*(Y - V_1))) \cup F^+(Cl^*(Int^*(Y - V_2))))) \\ &\subseteq F^-(Y - V_1) \cup F^+(Y - V_2) \\ &= (X - F^+(V_1)) \cup (X - F^-(V_2)) \\ &= X - (F^+(V_1) \cap F^-(V_2)). \end{aligned}$$

Moreover, we have

$$\begin{aligned} & Int^*(Cl^*(F^-(Cl^*(Int^*(Y - V_1))) \cup F^+(Cl^*(Int^*(Y - V_2))))) \\ &= Int^*(Cl^*(F^-(Y - Int^*(Cl^*(V_1))) \cup F^+(Y - Int^*(Cl^*(V_2))))) \\ &= Int^*(Cl^*((X - F^+(sCl_{\mathcal{J}^*}(V_1))) \cup (X - F^-(sCl_{\mathcal{J}^*}(V_2))))) \\ &= Int^*(Cl^*(X - (F^+(sCl_{\mathcal{J}^*}(V_1)) \cap F^-(sCl_{\mathcal{J}^*}(V_2))))) \\ &= X - Cl^*(Int^*(F^+(sCl_{\mathcal{J}^*}(V_1)) \cap F^-(sCl_{\mathcal{J}^*}(V_2)))). \end{aligned}$$

Thus, $F^+(V_1) \cap F^-(V_2) \subseteq Cl^*(Int^*(F^+(sCl_{\mathcal{J}^*}(V_1)) \cap F^-(sCl_{\mathcal{J}^*}(V_2))))$.

(6) $\Rightarrow$ (1): Let $x \in X$ and V_1, V_2 be any $\star$ -open sets of Y such that

$$F(x) \in V_1^+ \cap V_2^-.$$

By (6), we have

$$x \in F^+(V_1) \cap F^-(V_2) \subseteq \text{Cl}^*(\text{Int}^*(F^+(s\text{Cl}_{\mathcal{J}^*}(V_1)) \cap F^-(s\text{Cl}_{\mathcal{J}^*}(V_2))))$$

and by Lemma 2.2,

$$x \in F^+(V_1) \cap F^-(V_2) \subseteq s\text{Int}_{\mathcal{J}^*}(F^+(s\text{Cl}_{\mathcal{J}^*}(V_1)) \cap F^-(s\text{Cl}_{\mathcal{J}^*}(V_2))).$$

Put $U = s\text{Int}_{\mathcal{J}^*}(F^+(s\text{Cl}_{\mathcal{J}^*}(V_1)) \cap F^-(s\text{Cl}_{\mathcal{J}^*}(V_2)))$, then U is semi- $\mathcal{J}^*$ -open set of X containing x such that $F(U) \subseteq s\text{Cl}_{\mathcal{J}^*}(V_1)$ and $F(z) \cap s\text{Cl}_{\mathcal{J}^*}(V_2) \neq \emptyset$ for every $z \in U$. This shows that F is almost quasi $\star$ -continuous. $\square$

Theorem 3.3. For a multifunction $F : (X, \tau, \mathcal{J}) \rightarrow (Y, \sigma, \mathcal{J})$, the following properties are equivalent:

- (1) F is almost quasi $\star$ -continuous;
- (2) $s\text{Cl}_{\mathcal{J}^*}(F^-(V_1) \cap F^+(V_2)) \subseteq F^-(\text{Cl}^*(V_1)) \cup F^+(\text{Cl}^*(V_2))$ for every semi- $\mathcal{J}^*$ -preopen sets V_1, V_2 of Y ;
- (3) $s\text{Cl}_{\mathcal{J}^*}(F^-(V_1) \cap F^+(V_2)) \subseteq F^-(\text{Cl}^*(V_1)) \cup F^+(\text{Cl}^*(V_2))$ for every semi- $\mathcal{J}^*$ -open sets V_1, V_2 of Y ;
- (4) $F^+(V_1) \cap F^-(V_2) \subseteq s\text{Int}_{\mathcal{J}^*}[F^+(\text{Int}^*(\text{Cl}^*(V_1))) \cap F^-(\text{Int}^*(\text{Cl}^*(V_2)))]$ for every $\mathcal{J}^*$ -preopen sets V_1, V_2 of Y .

Proof. The proof follows from Theorem 3.2. $\square$

4. WEAKLY QUASI $\star$ -CONTINUOUS MULTIFUNCTIONS

In this section, we introduce the notion of weakly quasi $\star$ -continuous multifunctions and investigate some characterizations of such multifunctions.

Definition 4.1. A multifunction $F : (X, \tau, \mathcal{J}) \rightarrow (Y, \sigma, \mathcal{J})$ is said to be weakly quasi $\star$ -continuous at a point $x \in X$ if for any $\star$ -open sets V_1, V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$ and each $\star$ -open set U containing x , there exists a nonempty $\star$ -open set G of X such that $G \subseteq U$, $F(G) \subseteq \text{Cl}^*(V_1)$ and $F(z) \cap \text{Cl}^*(V_2) \neq \emptyset$ for every $z \in G$. A multifunction $F : (X, \tau, \mathcal{J}) \rightarrow (Y, \sigma, \mathcal{J})$ is said to be weakly quasi $\star$ -continuous if F is weakly quasi $\star$ -continuous at each point of X .

The following results give some characterizations of weakly quasi $\star$ -continuous multifunctions.

Theorem 4.1. *For a multifunction $F : (X, \tau, \mathcal{J}) \rightarrow (Y, \sigma, \mathcal{J})$, the following properties are equivalent:*

- (1) F is weakly quasi $\star$ -continuous;
- (2) for each $x \in X$ and every $\star$ -open sets V_1, V_2 of Y such that

$$F(x) \in V_1^+ \cap V_2^-,$$

there exists a semi- $\mathcal{J}^\star$ -open set of X containing x such that $F(U) \subseteq \text{Cl}^\star(V_1)$ and $F(z) \cap \text{Cl}^\star(V_2) \neq \emptyset$ for every $z \in U$;

- (3) $\text{Int}^\star(\text{Cl}^\star(F^-(\text{Int}^\star(K_1)) \cup F^+(\text{Int}^\star(K_2)))) \subseteq F^-(K_1) \cup F^+(K_2)$ for every $\star$ -closed sets K_1, K_2 of Y ;
- (4) $F^+(V_1) \cap F^-(V_2) \subseteq s\text{Int}_{\mathcal{J}^\star}(F^+(\text{Cl}^\star(V_1)) \cap F^-(\text{Cl}^\star(V_2)))$ for every $\star$ -open sets V_1, V_2 of Y ;
- (5) $s\text{Cl}_{\mathcal{J}^\star}(F^-(V_1) \cup F^+(V_2)) \subseteq F^-(\text{Cl}^\star(V_1)) \cup F^+(\text{Cl}^\star(V_2))$ for every $\star$ -open sets V_1, V_2 of Y .

Proof. (1) $\Rightarrow$ (2): The proof follows from Theorem 3.2.

(2) $\Rightarrow$ (4): Let V_1, V_2 be any $\star$ -open sets of Y and $x \in F^+(V_1) \cap F^-(V_2)$. Then $F(x) \in V_1^+ \cap V_2^-$ and there exists a semi- $\mathcal{J}^\star$ -open set U of X containing x such that $F(U) \subseteq \text{Cl}^\star(V_1)$ and $F(z) \cap \text{Cl}^\star(V_2) \neq \emptyset$ for every $z \in U$. Therefore, we have $x \in U \subseteq s\text{Int}_{\mathcal{J}^\star}(F^+(\text{Cl}^\star(V_1)) \cap F^-(\text{Cl}^\star(V_2)))$ and so $F^+(V_1) \cap F^-(V_2) \subseteq s\text{Int}_{\mathcal{J}^\star}(F^+(\text{Cl}^\star(V_1)) \cap F^-(\text{Cl}^\star(V_2)))$.

(4) $\Rightarrow$ (5): Let V_1, V_2 be any $\star$ -open sets of Y . Then by (4), we have

$$\begin{aligned} & X - [F^-(\text{Cl}^\star(V_1)) \cup F^+(\text{Cl}^\star(V_2))] \\ &= [X - F^-(\text{Cl}^\star(V_1))] \cap [X - F^+(\text{Cl}^\star(V_2))] \\ &= F^+(Y - \text{Cl}^\star(V_1)) \cap F^-(Y - \text{Cl}^\star(V_2)) \\ &\subseteq s\text{Int}_{\mathcal{J}^\star}[F^+(\text{Cl}^\star(Y - \text{Cl}^\star(V_1))) \cap F^-(\text{Cl}^\star(Y - \text{Cl}^\star(V_2)))] \\ &= s\text{Int}_{\mathcal{J}^\star}[F^+(Y - \text{Int}^\star(\text{Cl}^\star(V_1))) \cap F^-(Y - \text{Int}^\star(\text{Cl}^\star(V_2)))] \\ &\subseteq s\text{Int}_{\mathcal{J}^\star}(F^+(Y - V_1) \cap F^-(Y - V_2)) \\ &= s\text{Int}_{\mathcal{J}^\star}([X - F^-(V_1)] \cap [X - F^+(V_2)]) \end{aligned}$$

$$\begin{aligned}
&= s\text{Int}_{\mathcal{J}^*}(X - (F^-(V_1) \cup F^+(V_2))) \\
&= X - s\text{Cl}_{\mathcal{J}^*}(F^-(V_1) \cup F^+(V_2))
\end{aligned}$$

and hence $s\text{Cl}_{\mathcal{J}^*}(F^-(V_1) \cup F^+(V_2)) \subseteq F^-(\text{Cl}^*(V_1)) \cup F^+(\text{Cl}^*(V_2))$.

(5) $\Rightarrow$ (3): Let K_1, K_2 be any $\star$ -closed sets of Y . By (5) and Lemma 2.2,

$$\begin{aligned}
&\text{Int}^*(\text{Cl}^*(F^-(\text{Int}^*(K_1)) \cup F^+(\text{Int}^*(K_2)))) \\
&\subseteq s\text{Cl}_{\mathcal{J}^*}(F^-(\text{Int}^*(K_1)) \cup F^+(\text{Int}^*(K_2))) \\
&\subseteq F^-(\text{Cl}^*(\text{Int}^*(K_1))) \cup F^+(\text{Cl}^*(\text{Int}^*(K_2))) \\
&\subseteq F^-(\text{Cl}^*(K_1)) \cup F^+(\text{Cl}^*(K_2)) \\
&= F^-(K_1) \cup F^+(K_2).
\end{aligned}$$

(3) $\Rightarrow$ (4): Let V_1, V_2 be any $\star$ -open sets of Y . By (3) and Lemma 2.2,

$$\begin{aligned}
&X - s\text{Int}_{\mathcal{J}^*}[F^+(\text{Cl}^*(V_1) \cap F^-(\text{Cl}^*(V_2)))] \\
&= s\text{Cl}_{\mathcal{J}^*}[F^-(Y - \text{Cl}^*(V_1)) \cup F^+(Y - \text{Cl}^*(V_2))] \\
&\subseteq F^-(\text{Cl}^*(Y - \text{Cl}^*(V_1))) \cup F^+(\text{Cl}^*(Y - \text{Cl}^*(V_2))) \\
&= F^-(Y - \text{Int}^*(\text{Cl}^*(V_1))) \cup F^+(Y - \text{Int}^*(\text{Cl}^*(V_2))) \\
&\subseteq F^-(Y - V_1) \cup F^+(Y - V_2) \\
&= (X - F^+(V_1)) \cup (X - F^-(V_2)) \\
&= X - (F^+(V_1) \cap F^-(V_2))
\end{aligned}$$

and hence $F^+(V_1) \cap F^-(V_2) \subseteq s\text{Int}_{\mathcal{J}^*}(F^+(\text{Cl}^*(V_1)) \cap F^-(\text{Cl}^*(V_2)))$.

(4) $\Rightarrow$ (1): Let $x \in X$ and V_1, V_2 be any $\star$ -open sets of Y such that

$$F(x) \in V_1^+ \cap V_2^-.$$

By (4), we have $F^+(V_1) \cap F^-(V_2) \subseteq s\text{Int}_{\mathcal{J}^*}(F^+(\text{Cl}^*(V_1)) \cap F^-(\text{Cl}^*(V_2)))$. Put $U = s\text{Int}_{\mathcal{J}^*}(F^+(\text{Cl}^*(V_1)) \cap F^-(\text{Cl}^*(V_2)))$, then U is semi- $\mathcal{J}^*$ -open set of X containing x such that $F(U) \subseteq \text{Cl}^*(V_1)$ and $F(z) \cap \text{Cl}^*(V_2) \neq \emptyset$ for every $z \in U$. Consequently, we obtain F is weakly quasi $\star$ -continuous. $\square$

Theorem 4.2. For a multifunction $F : (X, \tau, \mathcal{J}) \rightarrow (Y, \sigma, \mathcal{J})$, the following properties are equivalent:

- (1) F is weakly quasi $\star$ -continuous;

(2)

$$\begin{aligned} & sCl_{\mathcal{J}^*}(F^-(Int^*(\star_{\theta}Cl(B_1))) \cup F^+(Int^*(\star_{\theta}Cl(B_2)))) \\ & \subseteq F^-(\star_{\theta}Cl(B_1)) \cup F^+(\star_{\theta}Cl(B_2)) \end{aligned}$$

for every subsets B_1, B_2 of Y ;

(3)

$$\begin{aligned} & sCl_{\mathcal{J}^*}(F^-(Int^*(Cl^*(B_1))) \cup F^+(Int^*(Cl^*(B_2)))) \\ & \subseteq F^-(\star_{\theta}Cl(B_1)) \cup F^+(\star_{\theta}Cl(B_2)) \end{aligned}$$

for every subsets B_1, B_2 of Y ;

$$(4) \quad sCl_{\mathcal{J}^*}(F^-(Int^*(Cl^*(V_1))) \cup F^+(Int^*(Cl^*(V_2)))) \subseteq F^-(Cl^*(V_1)) \cup F^+(Cl^*(V_2))$$

for every $\star$ -open sets V_1, V_2 of Y ;

$$(5) \quad sCl_{\mathcal{J}^*}(F^-(Int^*(Cl^*(V_1))) \cup F^+(Int^*(Cl^*(V_2)))) \subseteq F^-(Cl^*(V_1)) \cup F^+(Cl^*(V_2))$$

for every $\mathcal{J}^*$ -preopen sets V_1, V_2 of Y ;

$$(6) \quad sCl_{\mathcal{J}^*}(F^-(Int^*(K_1)) \cup F^+(Int^*(K_2))) \subseteq F^-(K_1) \cup F^+(K_2) \text{ for every } R\text{-}\mathcal{J}^*\text{-closed sets } K_1, K_2 \text{ of } Y.$$

Proof. (1) $\Rightarrow$ (2): Let B_1, B_2 be any subsets of Y . Since $\star_{\theta}Cl(B_1)$ and $\star_{\theta}Cl(B_2)$ are $\star$ -closed in Y , by Theorem 4.1,

$$\begin{aligned} & Int^*[Cl^*[F^-(Int^*(\star_{\theta}Cl(B_1))) \cup F^+(Int^*(\star_{\theta}Cl(B_2)))] \\ & \subseteq F^-(\star_{\theta}Cl(B_1)) \cup F^+(\star_{\theta}Cl(B_2)) \end{aligned}$$

and by Lemma 2.2, we have

$$\begin{aligned} & sCl_{\mathcal{J}^*}[F^-(Int^*(\star_{\theta}Cl(B_1))) \cup F^+(Int^*(\star_{\theta}Cl(B_2)))] \\ & \subseteq F^-(\star_{\theta}Cl(B_1)) \cup F^+(\star_{\theta}Cl(B_2)). \end{aligned}$$

(2) $\Rightarrow$ (3): This is obvious since $Cl^*(B) \subseteq \star_{\theta}Cl(B)$ for every subset B of Y .

(3) $\Rightarrow$ (4): This is obvious since $Cl^*(V) = \star_{\theta}Cl(V)$ for every $\star$ -open set V of Y .

(4) $\Rightarrow$ (5): Let V_1, V_2 be any $\mathcal{J}^*$ -preopen sets of Y . Then, we have $V_i \subseteq Int^*(Cl^*(V_i))$ and $Cl^*(V_i) = Cl^*(Int^*(Cl^*(V_i)))$ for $i = 1, 2$. Now, put $G_i = Int^*(Cl^*(V_i))$, then G_i is $\star$ -open in Y and $Cl^*(G_i) = Cl^*(V_i)$. Therefore, by (4), we obtain

$$sCl_{\mathcal{J}^*}[F^-(Int^*(Cl^*(V_1))) \cup F^+(Int^*(Cl^*(V_2)))] \subseteq F^-(Cl^*(V_1)) \cup F^+(Cl^*(V_2)).$$

(5) $\Rightarrow$ (6): Let K_1, K_2 be any R - $\mathcal{J}^*$ -closed sets of Y . Since $\text{Int}^*(K_1)$ and $\text{Int}^*(K_2)$ are $\mathcal{J}^*$ -preopen in Y , by (5), we have

$$\begin{aligned} & s\text{Cl}_{\mathcal{J}^*}[F^-(\text{Int}^*(K_1)) \cup F^+(\text{Int}^*(K_2))] \\ &= s\text{Cl}_{\mathcal{J}^*}[F^-(\text{Int}^*(\text{Cl}^*(\text{Int}^*(K_1)))) \cup F^+(\text{Int}^*(\text{Cl}^*(\text{Int}^*(K_2))))] \\ &\subseteq F^-(\text{Cl}^*(\text{Int}^*(K_1))) \cup F^+(\text{Cl}^*(\text{Int}^*(K_2))) \\ &= F^-(K_1) \cup F^+(K_2). \end{aligned}$$

(6) $\Rightarrow$ (1): Let V_1, V_2 be any $\star$ -open sets of Y . Then $\text{Cl}^*(V_1)$ and $\text{Cl}^*(V_2)$ are R - $\mathcal{J}^*$ -closed in Y and by (6),

$$\begin{aligned} s\text{Cl}_{\mathcal{J}^*}(F^-(V_1) \cup F^+(V_2)) &\subseteq s\text{Cl}_{\mathcal{J}^*}[F^-(\text{Int}^*(\text{Cl}^*(V_1))) \cup F^+(\text{Int}^*(\text{Cl}^*(V_2)))] \\ &\subseteq F^-(\text{Cl}^*(V_1)) \cup F^+(\text{Cl}^*(V_2)). \end{aligned}$$

It follows from Theorem 4.1 that F is weakly quasi $\star$ -continuous. $\square$

Theorem 4.3. For a multifunction $F : (X, \tau, \mathcal{J}) \rightarrow (Y, \sigma, \mathcal{J})$, the following properties are equivalent:

- (1) F is weakly quasi $\star$ -continuous;
- (2) $s\text{Cl}_{\mathcal{J}^*}(F^-(\text{Int}^*(\text{Cl}^*(V_1))) \cap F^+(\text{Int}^*(\text{Cl}^*(V_2)))) \subseteq F^-(\text{Cl}^*(V_1)) \cup F^+(\text{Cl}^*(V_2))$ for every semi- $\mathcal{J}^*$ -preopen sets V_1, V_2 of Y ;
- (3) $s\text{Cl}_{\mathcal{J}^*}(F^-(\text{Int}^*(\text{Cl}^*(V_1))) \cap F^+(\text{Int}^*(\text{Cl}^*(V_2)))) \subseteq F^-(\text{Cl}^*(V_1)) \cup F^+(\text{Cl}^*(V_2))$ for every semi- $\mathcal{J}^*$ -open sets V_1, V_2 of Y ;
- (4) $s\text{Cl}_{\mathcal{J}^*}(F^-(\text{Int}^*(\text{Cl}^*(V_1))) \cap F^+(\text{Int}^*(\text{Cl}^*(V_2)))) \subseteq F^-(\text{Cl}^*(V_1)) \cup F^+(\text{Cl}^*(V_2))$ for every $\mathcal{J}^*$ -preopen sets V_1, V_2 of Y .

Proof. (1) $\Rightarrow$ (2): Let V_1, V_2 be any semi- $\mathcal{J}^*$ -peropen sets of Y . Then, we have $V_i \subseteq \text{Cl}^*(\text{Int}^*(\text{Cl}^*(V_i)))$ and hence $\text{Cl}^*(V_i) = \text{Cl}^*(\text{Int}^*(\text{Cl}^*(V_i)))$ for $i = 1, 2$. Since $\text{Cl}^*(V_1)$ and $\text{Cl}^*(V_2)$ are R - $\mathcal{J}^*$ -closed sets, by Theorem 4.2,

$$s\text{Cl}_{\mathcal{J}^*}[F^-(\text{Int}^*(\text{Cl}^*(V_1))) \cup F^+(\text{Int}^*(\text{Cl}^*(V_2)))] \subseteq F^-(\text{Cl}^*(V_1)) \cup F^+(\text{Cl}^*(V_2)).$$

(2) $\Rightarrow$ (3): This is obvious since every semi- $\mathcal{J}^*$ -open set is semi- $\mathcal{J}^*$ -preopen.

(3) $\Rightarrow$ (4): For any $\mathcal{J}^*$ -preopen set V of Y , $\text{Cl}^*(V)$ is R - $\mathcal{J}^*$ -closed and $\text{Cl}^*(V)$ is semi- $\mathcal{J}^*$ -open in Y .

(4) $\Rightarrow$ (1): Let V_1, V_2 be any $\star$ -open sets of Y . Then V_1 and V_2 are $\mathcal{J}^*$ -preopen in Y . By (4), we have

$$s\text{Cl}_{\mathcal{J}^*}[F^-(\text{Int}^*(\text{Cl}^*(V_1))) \cup F^+(\text{Int}^*(\text{Cl}^*(V_2)))] \subseteq F^-(\text{Cl}^*(V_1)) \cup F^+(\text{Cl}^*(V_2)).$$

It follows from Theorem 4.2 that F is weakly quasi $\star$ -continuous. $\square$

Theorem 4.4. *For a multifunction $F : (X, \tau, \mathcal{J}) \rightarrow (Y, \sigma, \mathcal{J})$, the following properties are equivalent:*

- (1) F is weakly quasi $\star$ -continuous;
- (2) $\text{Int}^*(\text{Cl}^*(F^-(V_1) \cup F^+(V_2))) \subseteq F^-(\text{Cl}^*(V_1)) \cup F^+(\text{Cl}^*(V_2))$ for every $\mathcal{J}^*$ -preopen sets V_1, V_2 of Y ;
- (3) $s\text{Cl}_{\mathcal{J}^*}(F^-(V_1) \cup F^+(V_2)) \subseteq F^-(\text{Cl}^*(V_1)) \cup F^+(\text{Cl}^*(V_2))$ for every $\mathcal{J}^*$ -preopen sets V_1, V_2 of Y ;
- (4) $F^+(V_1) \cap F^-(V_2) \subseteq s\text{Int}_{\mathcal{J}^*}(F^+(\text{Cl}^*(V_1)) \cap F^-(\text{Cl}^*(V_2)))$ for every $\mathcal{J}^*$ -preopen sets V_1, V_2 of Y .

Proof. (1) $\Rightarrow$ (2): Let V_1, V_2 be any $\mathcal{J}^*$ -preopen sets of Y . Since F is weakly quasi $\star$ -continuous, by Theorem 4.2, we obtain $\text{Int}^*(\text{Cl}^*(F^-(V_1) \cup F^+(V_2))) \subseteq \text{Int}^*[\text{Cl}^*[F^-(\text{Int}^*(\text{Cl}^*(V_1))) \cup F^+(\text{Int}^*(\text{Cl}^*(V_2)))] \subseteq F^-(\text{Cl}^*(V_1)) \cup F^+(\text{Cl}^*(V_2))$.

(2) $\Rightarrow$ (3): Let V_1, V_2 be any $\mathcal{J}^*$ -preopen sets of Y . By (2) and Lemma 2.2, we have

$$\begin{aligned} s\text{Cl}_{\mathcal{J}^*}(F^-(V_1) \cup F^+(V_2)) &= [F^-(V_1) \cup F^+(V_2)] \cup \text{Int}^*(\text{Cl}^*(F^-(V_1) \cup F^+(V_2))) \\ &\subseteq F^-(\text{Cl}^*(V_1)) \cup F^+(\text{Cl}^*(V_2)). \end{aligned}$$

(3) $\Rightarrow$ (4): Let V_1, V_2 be any $\mathcal{J}^*$ -preopen sets of Y . Then by (3), we have

$$\begin{aligned} X - s\text{Int}_{\mathcal{J}^*}(F^+(\text{Cl}^*(V_1)) \cap F^-(\text{Cl}^*(V_2))) &= s\text{Cl}_{\mathcal{J}^*}(X - [F^+(\text{Cl}^*(V_1)) \cap F^-(\text{Cl}^*(V_2))]) \\ &= s\text{Cl}_{\mathcal{J}^*}([X - F^+(\text{Cl}^*(V_1))] \cup [X - F^-(\text{Cl}^*(V_2))]) \\ &= s\text{Cl}_{\mathcal{J}^*}(F^-(Y - \text{Cl}^*(V_1)) \cup F^+(\text{Cl}^*(V_2))) \\ &\subseteq F^-(\text{Cl}^*(Y - \text{Cl}^*(V_1))) \cup F^+(\text{Cl}^*(Y - \text{Cl}^*(V_2))) \\ &= [X - F^+(\text{Int}^*(\text{Cl}^*(V_1)))] \cup [X - F^-(\text{Int}^*(\text{Cl}^*(V_2)))] \\ &= X - [F^+(\text{Int}^*(\text{Cl}^*(V_1))) \cap F^-(\text{Int}^*(\text{Cl}^*(V_2)))] \\ &\subseteq X - (F^+(V_1) \cap F^-(V_2)) \end{aligned}$$

and hence $F^+(V_1) \cap F^-(V_2) \subseteq s\text{Int}_{\mathcal{J}^*}(F^+(\text{Cl}^*(V_1)) \cap F^-(\text{Cl}^*(V_2)))$.

(4) $\Rightarrow$ (1): Since every $\star$ -open set is $\mathcal{J}^*$ -preopen, this follows from Theorem 4.1. $\square$

Remark 4.1. For a multifunction $F : (X, \tau, \mathcal{J}) \rightarrow (Y, \sigma, \mathcal{J})$, the following implication holds:

$$\text{almost quasi } \star\text{-continuity} \Rightarrow \text{weak quasi } \star\text{-continuity}.$$

The converse of the implication is not true in general. We give example for the implication as follows.

Example 1. Let $X = \{1, 2, 3\}$ with topology $\tau = \{\emptyset, \{1\}, \{2, 3\}, X\}$ and ideal $\mathcal{J} = \{\emptyset\}$. Let $Y = \{a, b, c\}$ with topology $\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Y\}$ and ideal $\mathcal{J} = \{\emptyset\}$. A multifunction $F : (X, \tau, \mathcal{J}) \rightarrow (Y, \sigma, \mathcal{J})$ is defined as follows: $F(1) = \{a\}$, $F(2) = \{b\}$ and $F(3) = \{b, c\}$. Then F is weakly quasi $\star$ -continuous but F is not almost quasi $\star$ -continuous.

Theorem 4.5. If $F : (X, \tau, \mathcal{J}) \rightarrow (Y, \sigma, \mathcal{J})$ is weakly quasi $\star$ -continuous and $F(x)$ is $\star$ -open in Y for each $x \in X$, then F is almost quasi $\star$ -continuous.

Proof. Let $x \in X$ and V_1, V_2 be any $\star$ -open set of Y such that

$$x \in F^+(V_1) \cap F^-(V_2).$$

Then, we have $F(x) \subseteq V_1$ and $F(x) \cap V_2 \neq \emptyset$. Since F is weakly quasi $\star$ -continuous, by Theorem 4.1, there exists a semi- $\mathcal{J}^*$ -open set U of X containing x such that $F(U) \subseteq \text{Cl}^*(V_1)$ and $F(z) \cap \text{Cl}^*(V_2) \neq \emptyset$ for every $z \in U$. Since $F(x)$ is $\star$ -open for each $x \in X$, $F(U)$ is $\star$ -open and $F(U) \subseteq \text{Int}^*(\text{Cl}^*(V_1)) = s\text{Cl}_{\mathcal{J}^*}(V_1)$. Moreover, $F(z) \cap \text{Cl}^*(V_2) \neq \emptyset$ and hence $F(z) \cap \text{Int}^*(\text{Cl}^*(V_2)) = F(z) \cap s\text{Cl}_{\mathcal{J}^*}(V_2)$. Thus, F is almost quasi $\star$ -continuous by Theorem 3.2. $\square$

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MATHEMATICS AND APPLIED MATHEMATICS RESEARCH UNIT
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
MAHASARAKHAM UNIVERSITY
MAHASARAKHAM, 44150
THAILAND
E-mail address: chawalit.b@msu.ac.th