

VALUED FIELD GRAPH AND SOME RELATED PARAMETERS

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ABSTRACT. Let K be any finite field. For any prime p , the p -adic valuation map is given by $\psi_p : K/\{0\} \rightarrow \mathbb{R}^+ \cup \{0\}$ is given by $\psi_p(r) = n$ where $r = p^n \frac{a}{b}$, where p, a, b are relatively prime. The field K together with a valuation is called valued field. Also, any field K has the trivial valuation determined by $\psi(K) = \{0, 1\}$. Through out the paper K represents $\mathbb{Z}_q$. In this paper, we construct the graph corresponding to the valuation map called the valued field graph, denoted by $VFG_p(\mathbb{Z}_q)$ whose vertex set is $\{v_0, v_1, v_2, \dots, v_{q-1}\}$ where two vertices v_i and v_j are adjacent if $\psi_p(i) = j$ or $\psi_p(j) = i$. Here, we tried to characterize the valued field graph in $\mathbb{Z}_q$. Also we analyse various graph theoretical parameters such as diameter, independence number etc.

1. INTRODUCTION

The theory of valuations propounded in 1912 by the Hungarian Mathematician Josepf Kurschak, also see [1–4]. It plays a fundamental role in the study of algebraic function field. In this paper, using the concept of p -adic valuations in a finite field, we construct a graph and study its characteristics.

Throughout the paper, β_0 represents independence number, β_1 represents the matching number and χ represents chromatic number of a graph.

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2. PRELIMINARIES AND BASIC RESULTS

Definition 2.1. Let K be a field. A mapping $\psi_p : K \rightarrow \mathbb{R}^+ \cup \{0\}$ is called a valuation of K if it satisfies the following properties:

- (1) $\psi(x) = 0 \implies x = 0$
- (2) $\psi(xy) = \psi(x)\psi(y) \forall x, y \in K$
- (3) $\psi(x + y) \leq \psi(x) + \psi(y) \forall x, y \in K$

where $\mathbb{R}^+$ is the set of all non-negative real numbers.

Definition 2.2. Let $\psi : K \rightarrow \mathbb{R}^+ \cup \{0\}$ be a valuation map defined by $\psi(K) = \{0, 1\}$, then ψ is said to be a trivial valuation.

Definition 2.3. The field K together with a valuation is called valued field. That is, let K be a field and ψ be a valuation, then the pair (K, ψ) is called a valued field.

Definition 2.4. For every prime p and a non-zero element x , $\psi_p(x)$ is the exponent of p in the factorization of x into the product of prime powers, then $\psi_p(x)$ is called the p -adic valuation of a finite field K .

Definition 2.5. A graph G is a pair $G = (V, E)$ consisting of a finite set V and a set E of 2-element subsets of V . The set V is called the vertex set of G and E as the edge set of G . If $\{u, v\}$ is a member of E then we say u and v are adjacent and this edge is denoted as uv . Two edges are said to be adjacent if they have a common vertex. If the edge set is empty it is called null (empty or void) graph.

Definition 2.6. The distance between two vertices u and v is the length of a shortest path joining them, and is denoted by $d(u, v)$. The diameter of a graph G , denoted by $d(G)$ is defined by $d(G) = \max_{u, v \in V(G)} d(u, v)$

Definition 2.7. Let G be a simple graph. $M \subseteq E(G)$ is said to be a matching in G if no two elements of M are adjacent in G . M is said to be a maximum matching if there exists no matching M' of G with $|M'| > |M|$. Number of edges in a maximum matching is known as matching number or edge independent number which is denoted by β_1

Definition 2.8. A set $S \subseteq V(G)$ is an independent set of G if no two vertices of S are adjacent in G . An independent set is maximum, if G has no independent set S' with $|S'| > |S|$. The number of vertices in a maximum independent set of G is called an independent number of G and is denoted by β_0 .

Definition 2.9. Let G be a loopless graph. A k -colouring of G is an assignment of k colours to the vertices of G in such a way that adjacent vertices are received different colours. If G has a k -colouring, then G is said to be k -colourable. The chromatic number $\chi(G)$ of a graph G is the smallest number k for which G is k -colourable.

3. MAIN RESULTS

Definition 3.1. Let $\mathbb{Z}_q$ be a finite field. We define a new graph associated with a p -adic valuation ψ_p of a field denoted by $VFG_p(\mathbb{Z}_q)$ where p and q are prime numbers, is called valued field graph. The vertices of $VFG_p(\mathbb{Z}_q)$ are the elements of $\mathbb{Z}_q$. i.e, v_i denotes the vertex corresponds to the element $i \in \mathbb{Z}_q$. where two vertices v_i and v_j are adjacent if $\psi_p(i) = j$ or $\psi_p(j) = i$

Example 1. Consider the field $(\mathbb{Z}_5, +, \cdot)$. Let ψ_2 be a valuation from $\mathbb{Z}_5/\{0\} \rightarrow \mathbb{R}^+ \cup \{0\}$. Since $1 = 2^0 \cdot 1$, $\psi_2(1) = 0$. Therefore the vertices v_0 and v_1 are adjacent in $VFG_2(\mathbb{Z}_5)$. Also $2 = 2^1 \cdot 1$, $\psi_2(2) = 1$. Therefore, the vertices v_1 and v_2 are adjacent. Similarly the other pair of vertices and the corresponding graph $VFG_2(\mathbb{Z}_5)$ is shown on Figure 1:

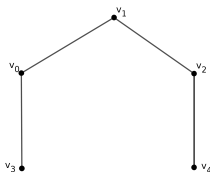


FIGURE 1

Example 2. Consider the field $(\mathbb{Z}_{19}, +, \cdot)$. Let ψ_p be a valuation from $\mathbb{Z}_{19} \rightarrow \mathbb{R}^+ \cup \{0\}$. The two vertices v_i and v_j are adjacent in $VFG_p(\mathbb{Z}_{19})$ if $\psi_p(i) = j$. The graph $VFG_p(\mathbb{Z}_{19})$ is shown on Figure 2 for a prime $p = 3$:

- Observation 1.**
- (1) The only complete valued field graph is $VFG_p(\mathbb{Z}_2)$
 - (2) For $p < q$, $VFG_p(\mathbb{Z}_q)$ is a tree.
 - (3) For $p = q$, $VFG_p(\mathbb{Z}_q)$ is a star graph.
 - (4) For any prime p , $VFG_p(\mathbb{Z}_3)$ is the star graph $\mathcal{K}_{1,2}$

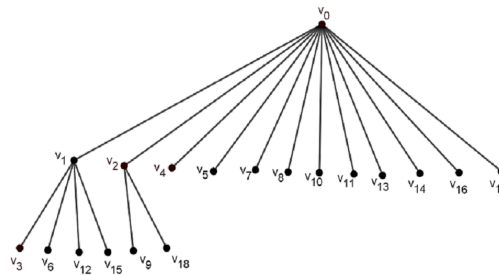


FIGURE 2

Theorem 3.1. Let p be a fixed prime. Then for any two primes q_1 and q_2 such that for $q_1 < q_2$, $VFG_p(\mathbb{Z}_{q_2})$ contains a subgraph isomorphic to $VFG_p(\mathbb{Z}_{q_1})$

Definition 3.2. The graph which corresponds to the trivial valuation in which all the vertices are adjacent to either v_0 or v_1 is called trivial valued field graph. Otherwise, it is called non-trivial valued field graph.

Example 3. $VFG_5(\mathbb{Z}_{11})$, is a trivial valued field graph.

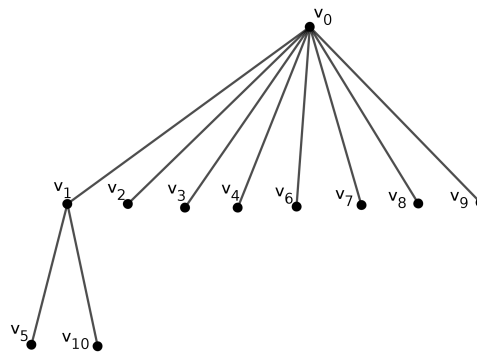


FIGURE 3

Theorem 3.2. In $VFG_p(\mathbb{Z}_q)$, all graphs are non-trivial valued field graphs if and only if $p \leq \lfloor \sqrt{q} \rfloor$, the greatest integer function of $\sqrt{q}$

Proof. For $p \leq \lfloor \sqrt{q} \rfloor$, suppose on the contrary let us assume that $VFG_p(\mathbb{Z}_q)$ is a trivial valued field graphs. Therefore all vertices except v_0 and v_1 are adjacent to only v_0 and v_1 . And the maximum level of a trivial valued field graph is

2. But for $p \leq \lfloor \sqrt{q} \rfloor$, if level of a graph is 2, then its branches increases. for example, the vertex v_2 is adjacent to v_{p^2} , v_3 is adjacent to v_{p^3} etc. If level of a graph is greater than 2, then clearly the vertex v_i is adjacent to v_j for $i, j \neq 0$ and 1. Hence $p \leq \lfloor \sqrt{q} \rfloor$, all the valued field graphs are non trivial valued field graph. $\square$

Corollary 3.1. *The total number of non-trivial valued field graph in $VFG_p(\mathbb{Z}_q)$ is $\phi[\sqrt{q}]$, where ϕ represents the Euler-Totient function*

Proposition 3.1. *For $p < q$ and $q > 3$, the centre of a trivial valued field graph is $\{v_0, v_1\}$ with diameter 3 and radius 2.*

Proposition 3.2. $\chi(VFG_p(\mathbb{Z}_q)) = 2$

Proof. Since $VFG_p(\mathbb{Z}_q)$ is a tree for all values of p and q , and the chromatic number of a tree is 2, the proof follows immediately. $\square$

Proposition 3.3. *In a trivial valued field graph*

$$\beta_0(VFG_p(\mathbb{Z}_q)) = \begin{cases} q - 2 & \text{for } p < q, \\ q - 1 & \text{for } p = q. \end{cases}$$

Proof. For $p = q$, $VFG_p(\mathbb{Z}_q)$ is a star graph. The independence number of a star graph with q vertices is $q - 1$.

From the definition of a trivial valued field graph, all the vertices except v_0 and v_1 are adjacent to either v_0 or v_1 . Therefore, excluding the vertices v_0 and v_1 , the collection of other vertices forms maximum independent set. Hence, in trivial valued field graph with q vertices, $q - 2$ vertices form a maximum independent set. Therefore, independence number of trivial valued field graph is $q - 2$. $\square$

Proposition 3.4. *In a trivial valued field graph*

$$\beta_1(VFG_p(\mathbb{Z}_q)) = \begin{cases} 2 & \text{for } p < q, \\ 1 & \text{for } p = q. \end{cases}$$

Proof. For $p = q$, $VFG_p(\mathbb{Z}_q)$ is a star graph. Hence the matching number is 1. By Proposition 3.1 for $p < q$, the diameter of trivial valued field graph is 3. Hence the matching number of trivial valued field graph is 2. $\square$

Proposition 3.5. *For any prime p , the vertices in $VFG_p(\mathbb{Z}_q)$ are parent vertices if and only if the elements corresponding to those vertices belong to the range set of $\psi_p(\mathbb{Z}_q/\{0\})$.*

Proof. **Case 1.** Suppose $VFG_p(\mathbb{Z}_q)$ is a trivial valued field graph. Then all the vertices are adjacent to v_0 or v_1 only. Hence all other vertices are pendant vertices. And by Theorem 3.2, we know that the total number of levels of a trivial valued field graph is 2. Since all vertices except v_0 and v_1 are pendant, the only parent vertices in $VFG_p(\mathbb{Z}_q)$ are v_0 and v_1 where v_0 and v_1 corresponds to the elements 0 and 1 in $\psi_p(\mathbb{Z}_q/\{0\})$.

Case 2. Suppose $VFG_p(\mathbb{Z}_q)$ is a non-trivial valued field graph. Assume that for some j , v_j is a parent vertex. Then by definition 3.2, we have for $i, j \neq 0, 1$, some vertex v_i is adjacent to v_j . This is possible only when $\psi_p(i) = j$, i.e., $j \in \psi_p(\mathbb{Z}_q/\{0\})$, where v_j is the vertex corresponding to the element j in $\psi_p(\mathbb{Z}_q/\{0\})$. Hence the proof. $\square$

Theorem 3.3. *If two fields are isomorphic then the corresponding valued field graphs are isomorphic. But the converse need not be true.*

Proof. Suppose K_1 and K_2 are two fields which are isomorphic. Then they are structurely same. Therefore the corresponding valued field graphs of these fields are same. Hence $VFG_p(K_1) \cong VFG_p(K_2)$

Conversely, consider two valued field graphs are isomorphic. For example, we have $VFG_p(\mathbb{Z}_q) \cong VFG_p(\mathbb{Z}_q^*)$, where $\mathbb{Z}_q^*$ is the multiplicative group of $\mathbb{Z}_q$. Since the domain of the valuation map from $\mathbb{Z}_q$ and $\mathbb{Z}_q^*$ are same, the corresponding vertices in $VFG_p(\mathbb{Z}_q)$ and $VFG_p(\mathbb{Z}_q^*)$ are same. Hence the two vertices in $VFG_p(\mathbb{Z}_q)$ is also adjacent in $VFG_p(\mathbb{Z}_q^*)$. That is if v_0 and v_1 are adjacent vertices in $VFG_p(\mathbb{Z}_q)$ then they are also adjacent in $VFG_p(\mathbb{Z}_q^*)$, and the degree of vertices in both the graphs are same. Therefore, $VFG_p(\mathbb{Z}_q) \cong VFG_p(\mathbb{Z}_q^*)$.

But we have, $\mathbb{Z}_q$ is not isomorphic to $\mathbb{Z}_q^*$.

Hence the valued field graphs of two fields are isomorphic does not imply the fields are isomorphic. $\square$

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