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PRIME IDEALS OF MΓ-GROUPS

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ABSTRACT. In this paper we consider the algebraic system MΓ-group, a generalization of the concept module over a nearring. We define prime ideal of MΓ-group and obtain some equivalent conditions for a prime ideal of an MΓ-group. Some related fundamental results and examples are also presented.

1. INTRODUCTION

In this section we provide elementary definition and examples from Satyanarayana [11, 13] for the sake of completeness.

Let (M, +) be a group (not necessarily Abelian) and Γ a non-empty set. Then M is said to be a Γ-nearring if there exists a mapping M × Γ × M → M (denote the image of (m_1, α_1, m_2) by $m_1\alpha_1 m_2$ for m_1, $m_2 \in$ M and $\alpha_1 \in \Gamma$) satisfying the following conditions:

(1) $(m_1 + m_2)\alpha_1 m_3 = m_1\alpha_1 m_3 + m_2\alpha_1 m_3$ and
(2) $(m_1\alpha_1 m_2)\alpha_2 m_3 = m_1\alpha_1(m_2\alpha_2 m_3)$

for all $m_1, m_2, m_3 \in$ M and for all $\alpha_1, \alpha_2 \in \Gamma$.

Furthermore, M is said to be a **zero-symmetric** **Γ-nearring** if $m\alpha 0 = 0$ for all $m \in$ M, $\alpha \in \Gamma$ (where '0' is the additive identity in M).

Consider an example, take $\mathbb{Z}_8 = \{0, 1, 2, 3, \dots, 7\}$, the group of integers modulo 8 and a set $X = \{a, b\}$. Write M $= \{f | f : X \to \mathbb{Z}_8$ and $f(a) = 0\}$.

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Then M $= \{f_0, f_1, f_2, \ldots, f_7\}$ where f_i is defined by $f_i(a) = 0$ and $f_i(b) = i$ for $0 \leq i \leq 7$. Now define two mappings $g_0, g_1 : \mathbb{Z}_8 \to X$ by setting $g_0(i) = a$ for all $i \in \mathbb{Z}_8$ and $g_1(i) = a$ if $i \notin \{3, 7\}$, $g_1(i) = b$ if $i \in \{3, 7\}$. Write $\Gamma = \{g_0, g_1\}$, $\Gamma^* = \{g_0\}$. Then M is a Γ-nearring and a Γ^*-nearring.

Let M be a Γ-nearring. An additive group G is said to be an **MΓ-group** if there exists a mapping G $\times \Gamma \times$ G $\to$ G (denote the image of (m, α, g) by $m\alpha g$ for $m \in$ M, $\alpha \in \Gamma$, $g \in G$) satisfying the conditions:

(1) $(m_1 + m_2)\alpha_1 g = m_1\alpha_1 g + m_2\alpha_1 g$ and

(2) $(m_1\alpha_1 m_2)\alpha_2 g = m_1\alpha_1(m_2\alpha_2 g)$

for all $m_1, m_2 \in$ M, $\alpha_1, \alpha_2 \in \Gamma$ and $g \in G$.

Satyanarayana [6, 7, 12] introduced and studied the concepts like f-prime ideals and corresponding f-prime radical in Γ-near-rings. Further Satyanarayana [13] generalized the notion of module over nearring to module over gamma nearrings and established fundamental structure theorems. Radical of gamma nearrings also studied by Booth [1–3]. The concept of equiprime ideal of a gamma nearring is a generalization of equiprime ideal of a nearring which was studied in Booth and Groenewald [4]. Satyanarayana and Syam Prasad [8, 15] studied fuzzy aspects of gamma nearrings.

For standard notations, elementary definitions and results on nearrings, we refer Pilz [5], Satyanarayana and Syam Prasad [9]. Throughout, we denote M for a Γ-nearring and G for an MΓ-group.

2. Subgroups and ideals of MΓ-group:

Definition 1 (Satyanarayana [11, 13])**.** *An additive subgroup H of G is said to be MΓ-subgroup if $m\alpha h \in H$ for all $m \in$ M, $\alpha \in \Gamma$ and $h \in H$. Note that (0) and G are the trivial MΓ-subgroups. A normal subgroup H of G is said to be an ideal of G if $m\alpha(g + h) - m\alpha g \in H$ for $m \in M$, $\alpha \in \Gamma$, $g \in G$ and $h \in H$. Moreover, a subgroup A of M is said to be an MΓ-subgroup of M if MΓA $\subseteq$A.*

Note 1. *If M is zero-symmetric then every ideal is a MΓ-subgroup. However, the converse need not be true. Consider the following example.*

Example 1. *Let $G = \mathbb{Z}_4 = \{0, 1, 2, 3\}$, the ring of integers modulo 4 and $X = \{a, b\}$. Write $M = \{g | g : X \to G, g(a) = 0\} = \{g_0, g_1, g_2, g_3\}$, where $g_i(a) = 0$, $g_i(b) = i$ for $0 \leq i \leq 3$. Let $\Gamma = \{f_1, f_2, f_3, f_4\}$ where each $f_i : G \to X$ defined by*

$f_1(i) = a(0 \leq i \leq 3)$, $f_2(i) = a(0 \leq i \leq 2)$, $f_2(3) = b$, $f_3(i) = a$ for $i \in \{0, 2, 3\}$, $f_3(1) = b$, $f_4(i) = a$ if $i \in \{0, 2\}$ and $f_4(i) = b$ if $i \notin \{0, 2\}$.
For $g \in M$, $f \in \Gamma$, $x \in G$, write $gfx = g(f(x))$. Now G becomes an MΓ-group. Further, $Y = \{0, 2\}$ is only the nontrivial subgroup and also MΓ-subgroup, but not an ideal of G (since $3 \notin Y$ and $g_3f_2(1 + 2) - g_3f_2(1) = 3$).

Notation 1. *Let P be an ideal of G. We define $(P : \Gamma G) = \{x \in M | x\Gamma G \subseteq P\}$.*

Lemma 1. *Let M be zero-symmetric and let P be an ideal of MΓ- group G and B be an MΓ- subgroup of G. If $B \subsetneq P$ then $(P : \Gamma B) = (P : \Gamma(P + B))$.*

Proof. Clearly $B \subseteq P + B$. Take $a \in (P : \Gamma(P + B))$. Then $a\gamma(p + b) \in P$ for all $b \in B$, $p \in P$. Now $a\gamma b = a\gamma(0 + b) - a\gamma 0 \in P$ (since $P \trianglelefteq G$ and M is zero-symmetric). This implies $a \in (P : \Gamma B)$. Therefore $(P : \Gamma(P + B) \subseteq (P : \Gamma B)$. Conversely, take $a \in (P : \Gamma B)$. To show $a\gamma(p + b) \in P$ for all $p \in P$, $b \in B$ and $\gamma \in \Gamma$.
Now $a\gamma(p + b) = (a\gamma(p + b)a\gamma b) + (a\gamma b) \in P$ (since $P \trianglelefteq G$ and $a\gamma b \in P$ (by converse hypo.)). This implies $a \in (P : \Gamma(P + B))$. Hence $(P : \Gamma B) = (P : \Gamma(P + B))$. □

Notation 2. *As the notation given in Reddy and Satyanarrayana [10], for any non-empty subset A of G we write*

(1) *$A^0 = \{x - y | x, y \in A\}$; and*
(2) *$A^{\#} = \{n\gamma g | n \in M, g \in A, \gamma \in \Gamma\}$.*

Let X be a non-empty subset of G and write $X_0 = X$, and $X_{i+1} = X_i^0 \bigcup X_i^{\#}$ for all integers $i \geq 0$. Then $X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots$ and clearly $\bigcup_{i=0}^{\infty} X_i$ is the MΓ-subgroup generated by X.

Notation 3. *For any $b \in M$, we denote $[b]_M$ as the MΓ-subgroup of M generated by b.*

Proposition 1. *Let P be an ideal of G. Suppose that for any MΓ-subgroup H of G such that $P \subset H$, we have $(P : \Gamma G) = (P : \Gamma H)$. Then for all $a \in M$ and $b \in G$, $a\Gamma[b]_M \subseteq P$ implies $a\Gamma G \subseteq P$ or $b \in P$.*

Proof. Take $a \in M$, $b \in G$ such that $a\Gamma[b]_M \subseteq P$. Suppose $b \notin P$. Then we have the following cases.

Case 1: $P \subsetneq [b]_M$. Now $a\Gamma[b]_M \subseteq P$. This implies $a \in (P : \Gamma[b]_M) = (P : \Gamma M)$ (by hypothesis) $= (P : \Gamma G)$ (we considered with respect to M). This implies $a\Gamma G \subseteq P$.

Case 2: $[b]_M \subsetneq P$. Then there exists $x \in P$ such that $x \notin [b]_M$. This implies $P \subsetneq (P + [b]_M)$. By hypothesis $(P : \Gamma M) = (P : \Gamma(P + [b]_M))$. Now $a\Gamma[b]_M \subseteq P \implies a \in (P : \Gamma[b]_M) = (P : \Gamma G)$. This implies $a\Gamma G \subseteq P$.

□

Note 2. *Let G be an MΓ-group. Then a subgroup of G need not be an MΓ-group, in general.*

Consider the following example:

Example 2. *Take $M = \{0, a, b, c\}$, $\Gamma = \{\gamma_1, \gamma_2\}$ and $G = M$ with the following binary operations.*

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

γ_1	0	a	b	c
0	0	0	0	0
a	0	b	0	b
b	0	0	0	0
c	0	b	0	b

γ_2	0	a	b	c
0	0	0	0	0
a	0	a	0	0
b	0	0	b	0
c	0	0	0	c

Clearly $(M, +, \Gamma)$ is a Γ-nearring, and G is an MΓ-group. Now $H = \{0, c\}$ is a subgroup of G. But it is not an MΓ-subgroup. For this, $c\gamma_1 c = b \notin \{0, c\} = H$. Therefore $M\Gamma H \nsubseteq H$. Hence H is not an MΓ-subgroup of G.

3. Prime ideals of MΓ-groups.

Definition 2. *Let P be a proper ideal of G such that $M\Gamma G \nsubseteq P$. Then P is called prime if $A\Gamma B \subseteq P \implies A\Gamma G \subseteq P$ or $B \subseteq P$, for all ideals A of M, B of G.*

Definition 3. *An MΓ- group G is said to be 0-prime MΓ- group if M$\Gamma G \neq (0)$ and (0) is a prime ideal of G.*

Example 3. *Consider $M = \{0, a, b, c\}$, $\Gamma = \{\gamma_1, \gamma_2\}$, $G = M$ and the following binary operations.*

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

γ_1	0	a	b	c
0	0	0	0	0
a	0	a	0	0
b	0	0	0	0
c	0	b	0	0

γ_2	0	a	b	c
0	0	0	0	0
a	0	a	0	0
b	0	0	0	a
c	0	0	0	a

Then M is a Γ-nearring, and G is an MΓ-group. Since there are no ideals A, B of G such that $A\Gamma B = \{0\}$ we have that (0) is prime ideal of G.

Definition 4. *(Satyanarayana, MBV Rao, Syam Prasad [14]): A left ideal P of a nearring N is said to be a prime left ideal if A and B are left ideals of N such that $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.*

Example 4. *Let N be a nearring and P be a prime left ideal of N. Write $M = N$ and consider M as the gamma nearring with $\Gamma = \{\cdot\}$ (here '$\cdot$' deremarks the multiplication in N). Write $G = N$. Clearly G is an MΓ-group. Then P becomes a prime ideal of the MΓ-group G.*

Verification: Let A be an ideal of M and B be an ideal of G such that $A\Gamma B \subseteq P$. This implies $AB \subseteq P$ (since $\Gamma = \{\cdot\}$). Since A, B are left ideals in N, we have that $A \subseteq P$ or $B \subseteq P$. If $B \nsubseteq P$ then $A \subseteq P$. Since A is two sided ideal in N, we have $AN \subseteq A$. In the case $A \subseteq P$, we have that $A\Gamma G = AG = AN \subseteq A \subseteq P$.

Proposition 2. *Let G be an MΓ-group. Suppose $M\Gamma G \neq (0)$. If (0) is a prime ideal of G then the following two conditions are equivalent.*

(1) $B \neq (0)$ *(where B is an ideal of G), and*
(2) $A\Gamma B = (0) \iff A \subseteq (0 : \Gamma G)$.

Proof. $(1) \implies (2)$: Suppose (1) holds. That is $B \neq (0)$. To show $A\Gamma B = (0) \iff A \subseteq (0 : \Gamma G)$, suppose $A\Gamma B = (0)$. Since (0) is prime and $B \neq (0)$, we have $A\Gamma G = (0)$. This implies $A \subseteq (0 : \Gamma G)$. Conversely suppose that $A \subseteq (0 : \Gamma G)$. This means $A\Gamma G = \{0\}$. Now $A\Gamma B \subseteq A\Gamma G \subseteq \{0\}$. This implies $A\Gamma B = (0)$.
$(2) \implies (1)$: Suppose that $A\Gamma B = (0) \iff A \subseteq (0 : \Gamma G)$ holds. In a contrary way suppose that $B = (0)$. Then $M\Gamma B = (0) \implies M \subseteq (0 : \Gamma G)$ (by converse hypothesis) $\implies M\Gamma G = (0)$, a contradiction. □

Proposition 3. *Let G be an MΓ-group such that $(P : \Gamma G) \neq M$. If P is a prime ideal of G then the following two conditions are equivalent.*

(1) *B is an ideal of G and* $B \nsubseteq P$*, and*
(2) $A\Gamma B \subseteq P \iff A \subseteq (P : \Gamma G)$.

Proof. $(1) \implies (2)$: Suppose B is an ideal of G and $B \nsubseteq P$. To show $A\Gamma B \subseteq P \iff A \subseteq (P : \Gamma G)$, suppose $A\Gamma B \subseteq P$. Since P is prime and $B \nsubseteq P$, we have $A\Gamma G \subseteq P$. This implies $A \subseteq (P : \Gamma G)$. Conversely suppose that $A \subseteq (P : \Gamma G)$. This means $A\Gamma G \subseteq P$. Now $A\Gamma B \subseteq A\Gamma G \subseteq P$. This implies $A\Gamma B \subseteq P$.
$(2) \implies (1)$ Suppose that $A\Gamma B \subseteq P \iff A \subseteq (P : \Gamma G)$ holds. In a contrary way suppose that $B \subseteq P$. Then $M\Gamma B \subseteq P \implies M \subseteq (P : \Gamma G)$ (by converse hypothesis) $\implies M\Gamma G \subseteq P$, a contradiction. □

Theorem 4. *Let G be an* $M\Gamma$*- group, P be an ideal of G, A and B be ideals of M then the following conditions (1) and (2) are equivalent.*

(1) *P is 0-prime.*
(2) $< a > \Gamma < b > \subseteq P$ *implies that* $< a > \Gamma G \subseteq P$ *or* $b \in P$.
Moreover, if M is a zero symmetric Γ*-nearring then conditions (1) to (4) are equivalent.*
(3) *If M is zero symmetric,* $a\Gamma < b > \subseteq P$ *implies that* $a\Gamma G \subseteq P$ *or* $b \in P$.
(4) $a\Gamma B \subseteq P$ *implies that* $a\Gamma G \subseteq P$ *or* $B \subseteq P$.

Proof. $(1) \implies (2)$:
Suppose $< a > \Gamma < b > \subseteq P$. Write $A =< a >$ and $B =< b >$. Then $A\Gamma B \subseteq P$. This implies $A\Gamma G \subseteq P$ or $B \subseteq P$ (by (1)) $< a > \Gamma G \subseteq P$ or $< b > \subseteq P \implies < a > \Gamma G \subseteq P$ or $b \in P$. Hence (2).
$(2) \implies (1)$: Suppose (2).
In contrary way suppose that (1) is not true.
Then there exists an ideal A of M, an ideal B of G such that $A\Gamma B \subseteq P$ but $A\Gamma G \subsetneq P$ and $B \subsetneq P$. This implies $a\gamma g \notin P$ for some $a \in A$, $\gamma \in \Gamma$, $g \in G$ and $b \in B \backslash P$.
Now $< a > \Gamma < b > \subseteq A\Gamma B \subseteq P$. By (2) we have that $< a > \Gamma G \subseteq P$ or $b \in P$. Since $b \notin P$ we have $< a > \Gamma G \subseteq P$.
Now $a\gamma g \in < a > \Gamma G \subseteq P$ implies $a\gamma g \in P$, a contradiction.
$(2) \implies (3)$: Suppose $a\Gamma < b > \subseteq P$. This implies $a \in (P : \Gamma < b >) \implies < a > \subseteq (P : \Gamma < b >)$ (since $(P : \Gamma < b >)$ is an ideal and M is zero symmetric) $\implies < a > \Gamma < b > \subseteq P \implies < a > \Gamma G \subseteq P$ or $b \in P$ (by (2))

$\implies a\Gamma G \subseteq < a > \Gamma G \subseteq P$ or $b \in P$. This proves (3).
$(3) \implies (2)$: Suppose $< a > \Gamma < b > \subseteq P$. Then $a\Gamma < b > \subseteq < a > \Gamma < b > \subseteq P$. This implies $a\Gamma G \subseteq P$ or $b \in P$ (by (3)) $\implies a \in (P : \Gamma G)$ or $b \in P$
$\implies < a > \subseteq (P : \Gamma G)$ or $b \in P \implies < a > \Gamma G \subseteq P$.
$(3) \implies (4)$: Suppose (3). In contrary way, suppose (4) is not true. Then $a\Gamma B \subseteq P$, $a\Gamma G \nsubseteq P$ and $B \nsubseteq P$ for some $a \in A$. So there exists $\gamma \in \Gamma$, and $g \in G$ such that $a\gamma g \notin P$ and $b \in B\backslash P$.
Now $a\Gamma < b > \subseteq a\Gamma B \subseteq P \implies a\Gamma G \subseteq P$ or $b \in P$ (by (3)) $\implies a\Gamma G \subseteq P$ (since $b \notin B\setminus P$) $\implies a\gamma g \in a\Gamma G \subseteq P$.
$(4) \implies (3)$: Suppose $a\Gamma < b > \subseteq P$. Write $B =< b >$. Now $a\Gamma B \subseteq P \implies a\Gamma G \subseteq P$ or $B \subseteq P$ (by (4)) $\implies a\Gamma b \subseteq P$ or $b \in < b >= B \subseteq P$. □

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