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ON A SUBCLASS OF UNIVALENT HARMONIC MAPPINGS CONVEX IN THE IMAGINARY DIRECTION

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ABSTRACT. In the present article, we consider a class of univalent harmonic mappings, $\mathcal{C}_T = \left\{ T_c[f] = \frac{f+czf'}{1+c} + \overline{\frac{f-czf'}{1+c}};\ c > 0 \right\}$ and f is convex univalent in $\mathbb{D}$, whose functions map the open unit disk $\mathbb{D}$ onto a domain convex in the direction of the imaginary axis. We estimate coefficient, growth and distortion bounds for the functions of the same class.

1. INTRODUCTION

A domain $\Omega \subset \mathbb{C}$ is said to be convex if for any two points w_1 and w_2 in Ω, the line segment $tw_1 + (1-t)w_2$ $(0 \le t \le 1)$ lies entirely in Ω. A domain $\Omega \subset \mathbb{C}$ is said to be convex in a direction $\gamma \in [0, \pi)$, if for all $a \in \mathbb{C}$, the set $\{a + te^{i\gamma} : t \in \mathbb{R}\}$ has either empty or connected intersection with Ω. In particular, a domain is convex in the direction of the real (imaginary) axis if every line parallel to the real (imaginary) axis has either an empty or connected intersection with that domain. A function which maps the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ onto a convex domain or onto a domain convex in a direction γ, is said to be convex function or function convex in direction γ, respectively. In 2008, Muir [3] defined a transformation $T_c[f], c > 0$ as follows.

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For a univalent analytic function $f : \mathbb{D} \to \mathbb{C}$, with $f(0) = f'(0) - 1 = 0$,

$$T_c[f](z) = \frac{f(z) + czf'(z)}{1+c} + \overline{\frac{f(z) - czf'(z)}{1+c}}, \quad z \in \mathbb{D},\ c > 0. \tag{1.1}$$

Writing $I(z) = z/(1-z)$ we have,

$$\begin{aligned} T_1[I](z) &= \frac{I(z) + zI'(z)}{2} + \overline{\frac{I(z) - zI'(z)}{2}} \\ &= \frac{z - \frac{z^2}{2}}{(1-z)^2} - \overline{\frac{\frac{z^2}{2}}{(1-z)^2}} \quad , z \in \mathbb{D}, \end{aligned}$$

which is the 'Standard right half-plane' mapping which maps the open unit disk $\mathbb{D}$ onto the right half-plane $\{w \in \mathbb{C} : \operatorname{Re}(w) > \frac{-1}{2}\}$ and was earlier constructed by Clunie and Sheil-Small [1] using 'Shear Construction' technique stated in the following lemma.

Lemma 1.1. [1] *A locally univalent and sense-preserving harmonic function $f = h + \overline{g}$ on $\mathbb{D}$ is univalent and maps $\mathbb{D}$ onto a domain convex in the direction of ϕ if and only if the analytic mapping $h - e^{2i\phi} g$ is univalent and maps $\mathbb{D}$ onto a domain convex in the direction of ϕ.*

In [4], Muir proved that $T_c[f]$ defined by (1.1) is convex in the direction of the imaginary axis if and only if f is convex. In the present article, we consider the class

$$\mathcal{C}_T = \left\{ T_c[f] = \frac{f + czf'}{1+c} + \overline{\frac{f - czf'}{1+c}} : \ c > 0 \right\}$$

where c is any real number and f is convex univalent in $\mathbb{D}$. We shall study this class and estimate coefficient, growth and distortion bounds for the functions of the same class.

2. Background

Let $\mathcal{A}$ be the set of analytic functions defined on $\mathbb{D}$ that fix zero, and let $S \subset \mathcal{A}$ be the set of univalent functions with added normalization $f'(0) = 1$. Denote by $\mathcal{H}$ the class of all complex valued harmonic functions f in the open unit disk $\mathbb{D}$ normalized by $f(0) = 0 = f_z(0) - 1$. A function $f \in \mathcal{H}$ can be uniquely decomposed as $f = h + \overline{g}$, where h and g are analytic in $\mathbb{D}$. The functions h and g, respectively, are called analytic and co-analytic parts of f. By a result of Lewy [5], a necessary and sufficient condition for f to be locally univalent and sense preserving in $\mathbb{D}$ is that the jacobian J_f of f, defined by $J_f(z) = |h'(z)|^2 - |g'(z)|^2$, satisfies the condition $J_f(z) > 0$ in $\mathbb{D}$, or equivalently, if $h'(z) \neq 0$ in $\mathbb{D}$, the function $w = g'(z)/h'(z)$ (called dilatation of f) has the property $|w(z)| < 1$ for all $z \in \mathbb{D}$. Let S_H be the subclass of $\mathcal{H}$ consisting of all those functions that are univalent and sense-preserving in $\mathbb{D}$. By S_H^0, we denote the class of mappings $f \in S_H$ with an added normalization $f_{\overline{z}}(0) = 0$. Clearly, we have the inclusion $S \subset S_H^0 \subset S_H$. A domain D is close-to-convex if its compliment can be written as union of non-intersecting half lines. Let C and C_H denote the respective subclasses of S and S_H for which $f(\mathbb{D})$ is close-to-convex. Let $K(\phi)$ and $K_H(\phi)$ be the respective subclasses of S and S_H for which $f(\mathbb{D})$ is convex in the direction of ϕ. Note that $K(\phi) \subset C$ and $K_H(\phi) \subset C_H$. A domain D is said to be starlike with respect to a point $w_0 \in \mathbb{D}$, provided for every $w \in \mathbb{D}$, the line segment $tw + (1-t)w_0,\ 0 \leq t \leq 1$, lies in $\mathbb{D}$. Let S^* and S_H^* denote the respective subclasses of S and S_H for which $f(\mathbb{D})$ is starlike with respect to the origin. Similarly, let K and K_H denote the respective subclasses of S and S_H for which $f(\mathbb{D})$ is a convex domain.

3. Main Results

In this section of the present article, we establish estimates for coefficients, growth and distortion for the functions of the class $\mathcal{C}_T$. We shall need following results on analytic functions [2], in order to prove our main results in this section.

Lemma 3.1. *Let f be analytic in $\mathbb{D}$, with $f(0) = 0$ and $f'(0) = 1$. Then $f \in C$ if and only if $zf'(z) \in S^*$.*

Lemma 3.2. (i) *The coefficient of each function $f \in S^*$ satisfy $|a_n| \leq n$ for $n = 2, 3 \cdots$ strict inequality holds for all n, unless f is a rotation of the Koebe function $k(z)$ defined by $k(z) = \dfrac{z}{(1-z)^2}$.*

(ii) *If $f \in \mathcal{C}$, then $|a_n| \leq 1$ for $n = 2, 3 \cdots$ strict inequality holds for all n unless f is a rotation of function $l(z)$ defined by $l(z) = \dfrac{z}{1-z}$.*

Lemma 3.3. (i) *For each $f \in S^*$,*

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}, \quad |z| = r < 1$$

for each $z \in \mathbb{D}$, $z \neq 0$, equality occurs if and only if 'f' is a suitable rotation of the Koebe function $k(z)$ defined by $k(z) = \dfrac{z}{(1-z)^2}$.

(ii) *If $f \in \mathcal{C}$, then*

$$\frac{1}{(1+r)^2} \leq |f'(z)| \leq \frac{1}{(1-r)^2}, \quad |z| = r < 1$$

for each $z \in \mathbb{D}$, $z \neq 0$, equality occurs if and only if f is a rotation of the function $l(z)$ defined by $l(z) = \dfrac{z}{1-z}$.

Lemma 3.4. (i) *For each $f \in S^*$,*

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}, \quad |z| = r < 1.$$

For each $z \in \mathbb{D}$, $z \neq 0$, equality occurs if and only if 'f' is a suitable rotation of the Koebe function $k(z)$ defined by $k(z) = \dfrac{z}{(1-z)^2}$.

(ii) *For each $f \in C$,*

$$\frac{r}{1+r} \leq |f(z)| \leq \frac{r}{1-r}, \quad |z| = r < 1.$$

For each $z \in \mathbb{D}$, $z \neq 0$, equality occurs if and only if f is a suitable rotation of the function $l(z)$ defined by $l(z) = \dfrac{z}{1-z}$.

Theorem 3.1. *If the mapping $T_c[f] = H(z) + \overline{G(z)} \in \mathcal{C}_T$, then for every $n = 2, 3 \cdots$*

$$|A_n|^2 + |B_n|^2 \leq \frac{2}{(1+c)^2}(n^2+1).$$

Here, A_n and B_n are the coefficients of $H(z)$ and $G(z)$ in their power series representation defined by (1.1). Equality holds for the function $T_c\left[\dfrac{z}{(1-z)}\right]$ and its suitable rotations.

Proof. We have

$$T_c[f](z) = H(z) + \overline{G(z)} = \frac{f(z) + czf'(z)}{1+c} + \overline{\frac{f(z) - czf'(z)}{1+c}}, \; z \in \mathbb{D}, \; c > 0\,,$$

be locally-univalent and convex in the direction of the imaginary axis i.e. f is convex univalent. We have

$$H(0) = 0 = H'(0) - 1, \;\; G(0) = 0, \; G'(0) = \frac{1-c}{1+c}.$$

So, $H(z)$ and $G(z)$ have the series representation of the following form:

$$H(z) = z + \sum_{n=2}^{\infty} A_n z^n \quad \text{and} \quad G(z) = \sum_{n=1}^{\infty} B_n z^n, \quad \text{where } B_1 = \frac{1-c}{1+c}.$$

Thus,

$$H(z) + G(z) = \frac{2}{1+c} f(z)$$

since $f(z)$ is convex so by using (i) of Lemma 3.2 for $f(z)$ we have

$$\left| \frac{1+c}{2} \left(A_n + B_n\right) \right| \leq 1,$$

$$|A_n + B_n|^2 \leq \frac{4}{(1+c)^2}. \tag{3.1}$$

Similarly, we have

$$H(z) - G(z) = \frac{2c}{1+c} z f'(z),$$

thus by using Lemma 3.1 for the function $f(z)$ and (ii) of Lemma 3.2 for the function $zf'(z)$, we get

$$|A_n - B_n|^2 \leq \frac{4n^2c^2}{(1+c)^2}. \tag{3.2}$$

and so, using (3.1) and (3.2)

$$\begin{aligned} |A_n|^2 + |B_n|^2 &= \frac{1}{2} \left(|A_n - B_n|^2 + |A_n + B_n|^2 \right) \\ &\leq \frac{2}{(1+c)^2} (c^2 n^2 + 1). \end{aligned}$$

□

Corollary 3.1. *If $T_c\,[f]\,(z) = H(z) + \overline{G(z)} \in \mathcal{C}_{\mathcal{T}}$, then for every $n = 2, 3 \cdots$*

$$|A_n| < \sqrt{2(n^2c^2+1)} \ \text{ and } \ |B_n| < \sqrt{2(n^2c^2+1)}\ .$$

Here, A_n and B_n are coefficients of $H(z)$ and $G(z)$ in their power series representation defined by (1.1). Equality holds for the function $T_c \left[\frac{z}{(1-z)} \right]$ and its suitable rotations.

Proof. From Theorem 3.1,

$$|A_n| \leq \sqrt{|A_n|^2 + |B_n|^2 - |B_n|^2}$$
$$\leq \frac{\sqrt{2}}{1+c}\sqrt{c^2n^2+1}$$

as $c > 0$, we have

$$|A_n| < \sqrt{2(n^2c^2+1)}$$

and similarly,

$$|B_n| < \sqrt{2(n^2c^2+1)}$$

for $n = 2, 3, \cdots$. $\square$

Theorem 3.2. *If $T_c[f](z) = H(z) + \overline{G(z)} \in \mathcal{C}_{\mathcal{T}}$, then for every $z \in \mathbb{D}$,*

$$\frac{2}{(1+c)^2(1+r)^4}\left(1 + c^2\left(\frac{1-r}{1+r}\right)^2\right) \leq |H'(z)|^2 + |G'(z)|^2$$
$$\leq \frac{2}{(1+c)^2(1-r)^4}\left(1 + c^2\left(\frac{1+r}{1-r}\right)^2\right),$$

where $r = |z| < 1$. Equality holds for the function $T_c\left[\frac{z}{(1-z)}\right]$ and its suitable rotations.

Proof. Using (ii) of Lemma 3.3 for the function $f(z)$, we get

$$\frac{4}{(1+c)^2(1+r)^4} \leq |H'(z) + G'(z)|^2 \leq \frac{4}{(1+c)^2(1-r)^4}. \tag{3.3}$$

Similarly, with the use of Lemma 3.1 for the function $f(z)$ and then (i) of Lemma 3.3 for the function $zf'(z)$, we have

$$\frac{4c^2(1-r)^2}{(1+c)^2(1+r)^6} \leq |H'(z) - G'(z)|^2 \leq \frac{4c^2(1+r)^2}{(1+c)^2(1-r)^6}. \tag{3.4}$$

Now, from equation (3.3) and (3.4), we have

$$\frac{2}{(1+c)^2(1+r)^4}\left(1 + c^2\left(\frac{1-r}{1+r}\right)^2\right) \leq |H'(z)|^2 + |G'(z)|^2$$
$$\leq \frac{2}{(1+c)^2(1-r)^4}\left(1 + c^2\left(\frac{1+r}{1-r}\right)^2\right),$$

where $r = |z|,\ c > 0$. $\square$

Corollary 3.2. *Let $T_c[f](z) = H(z) + \overline{G(z)} \in \mathcal{C}_{\mathcal{T}}$, then for every $z \in \mathbb{D}$,*

$$\frac{\sqrt{2}}{2(1+r)^2}\sqrt{1+c^2\left(\frac{1-r}{1+r}\right)^2} < |H'(z)| < \frac{\sqrt{2}}{(1+c)(1-r)^2}\,, \tag{3.5}$$

and

$$0 \leq |G'(z)| < \frac{\sqrt{2}}{(1+c)(1-r)^2}\,, \tag{3.6}$$

where $r = |z|,\ c > 0$. Equality holds for the function $T_c\left[\frac{z}{(1-z)}\right]$ and its suitable rotations.

Proof. Since $c > 0$, then from Theorem 3.2, we have

$$\frac{2}{(1+r)^4}\left(1+c^2\left(\frac{1-r}{1+r}\right)^2\right) < |H'(z)|^2 + |G'(z)|^2 < \frac{2}{(1+c)^2(1-r)^4}\,. \tag{3.7}$$

The local univalence of $T_c[f](z)$ gives that $|G'(z)| < |H'(z)|,\ z \in \mathbb{D}$. Combining the above inequality with (3.7) we obtain inequalities (3.5) and (3.6). $\square$

Corollary 3.3. *Let $T_1[f](z) = H(z) + \overline{G(z)} \in \mathcal{C}_{\mathcal{T}}$, then*

$$\frac{1}{\sqrt{2(1+r^2)}(1+r)^2}\sqrt{1+\left(\frac{1-r}{1+r}\right)^2} \leq |H'(z)| \leq \frac{1}{\sqrt{2}(1-r)^2}\sqrt{1+\left(\frac{1+r}{1-r}\right)^2} \tag{3.8}$$

and

$$0 \leq |G'(z)| \leq \frac{1}{\sqrt{2(1+r^2)}(1-r)^2}\sqrt{1+\left(\frac{1+r}{1-r}\right)^2}, \tag{3.9}$$

where $r = |z|$. Equality holds for the function $T_c[\frac{z}{(1-z)}]$ and its suitable rotations.

Proof. Take $c = 1$ in Theorem 3.2, we derive

$$\frac{1}{2(1+r)^4}\left(1+\left(\frac{1-r}{1+r}\right)^2\right) \leq |H'(z)|^2 + |G'(z)|^2 \leq \frac{1}{2(1-r)^4}\left(1+\left(\frac{1+r}{1-r}\right)^2\right). \tag{3.10}$$

Now, if $c = 1$ thus $b_1 = G'(0) = 0$ which gives that the dilatation function $w(z) = \dfrac{G'(z)}{H'(z)}$, satisfies Schwarz's lemma and we have following inequality

$$0 \leq |G'(z)| \leq |z||H'(z)|.$$

Combining above inequality with (3.10), we get desired inequalities (3.8) and (3.9). □

Theorem 3.3. *If $T_c[f] = H(z) + \overline{G(z)} \in \mathcal{C}_T$, then for every $z \in \mathbb{D}$,*

$$\begin{aligned}\frac{2r^2}{(1+c)^2(1+r)^2}\left(1+\frac{c^2}{(1+r)^2}\right) &\leq |H(z)|^2+|G(z)|^2\\ &\leq \frac{2r^2}{(1+c)^2(1-r)^2}\left(1+\frac{c^2}{(1-r)^2}\right),\end{aligned}$$

where $r=|z|,\ c>0$. Equality holds for the function $T_c\left[\dfrac{z}{(1-z)}\right]$ and its suitable rotations.

Proof. By applying (ii) of Lemma 3.4 for the function $f(z)$, we have

$$\frac{4r^2}{(1+r)^2(1+c)^2} \leq |H(z)+G(z)|^2 \leq \frac{4r^2}{(1+c)^2(1-r)^2} \tag{3.11}$$

Similarly, using Lemma 3.1 for the function $f(z)$ and the (i) of Lemma 3.4 for the function $zf'(z)$ gives

$$\frac{4r^2c^2}{(1+c)^2(1+r)^4} \leq |H(z)-G(z)|^2 \leq \frac{4r^2c^2}{(1+c)^2(1-r)^4} \tag{3.12}$$

From (3.11) and (3.12), we have

$$\begin{aligned}\frac{2r^2}{(1+c)^2(1+r)^2}\left(1+\frac{c^2}{(1+r)^2}\right) &\leq |H(z)|^2+|G(z)|^2\\ &\leq \frac{2r^2}{(1+c)^2(1-r)^2}\left(1+\frac{c^2}{(1-r)^2}\right),\end{aligned}$$

where $r=|z|,\ c>0$. □

Corollary 3.4. *If $T_c[f](z) = H(z) + \overline{G(z)} \in \mathcal{C}_T$, then we have for every $z \in \mathbb{D}$,*

$$\begin{aligned}0 &\leq |H(z)| < \sqrt{2}\frac{r}{(1-r)}\sqrt{1+\frac{c^2}{(1-r)^2}},\\ 0 &\leq |G(z)| < \sqrt{2}\frac{r}{(1-r)}\sqrt{1+\frac{c^2}{(1-r)^2}},\\ 0 &\leq |T_c[f](z)| < \frac{2r}{(1-r)}\sqrt{1+\frac{c^2}{(1-r)^2}},\end{aligned}$$

where $r = |z|$, $c > 0$. *Equality holds for the function* $T_c\left[\dfrac{z}{(1-z)}\right]$ *and its suitable rotations.*

Proof. From Theorem 3.3, we have

$$\begin{aligned}\frac{2r^2}{(1+c)^2(1+r)^2}\left(1+\frac{c^2}{(1+r)^2}\right) &\leq |H(z)|^2+|G(z)|^2 \\ &\leq \frac{2r^2}{(1+c)^2(1-r)^2}\left(1+\frac{c^2}{(1-r)^2}\right).\end{aligned}$$

$|H(z)| \geq 0$ and $|G(z)| \geq 0$ are trivial inequalities, since we have for $c > 0$:

$$\begin{aligned}|H(z)|^2+|G(z)|^2 &\leq \frac{2r^2}{(1+c)^2(1-r)^2}\left(1+\frac{c^2}{(1-r)^2}\right) \\ &< \frac{2r^2}{(1-r)^2}\left(1+\frac{c^2}{(1-r)^2}\right)\end{aligned}$$

So,

$$0 \leq |H(z)| < \sqrt{2}\frac{r}{(1-r)}\sqrt{1+\frac{c^2}{(1-r)^2}}\,.$$

Similarly,

$$0 \leq |G(z)| < \sqrt{2}\frac{r}{(1-r)}\sqrt{1+\frac{c^2}{(1-r)^2}}\,,.$$

Also, $|T_c[f](z)| \geq 0$ and

$$\begin{aligned}|T_c[f](z)| &= |H(z)+\overline{G(z)}| \\ &\leq |H(z)|+|G(z)| \\ &\leq \sqrt{2\left(|H(z)|^2+|G(z)|^2\right)} \\ &\leq \frac{2r}{(1-r)(1+c)^2}\sqrt{1+\frac{c^2}{(1-r)^2}} \\ &< \frac{2r}{(1-r)}\sqrt{1+\frac{c^2}{(1-r)^2}}\,.\end{aligned}$$

□

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