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A CLASS OF HARMONIC STARLIKE FUNCTIONS DEFINED BY MULTIPLIER TRANSFORMATION

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ABSTRACT. We define two new subclasses, $HS(k,\lambda,b,\alpha)$ and $\overline{HS}(k,\lambda,b,\alpha)$, of univalent harmonic mappings using multiplier transformation. We obtain a sufficient condition for harmonic univalent functions to be in $HS(k,\lambda,b,\alpha)$ and we prove that this condition is also necessary for the functions in the class $\overline{HS}(k,\lambda,b,\alpha)$. We also obtain extreme points, distortion bounds, convex combination, radius of convexity and Bernandi-Libera-Livingston integral for the functions in the class $\overline{HS}(k,\lambda,b,\alpha)$.

1. INTRODUCTION

A continuous function $f = u + iv$ is a complex valued harmonic function in $\mathbb{C}$ if both u and v are real valued harmonic in $\mathbb{C}$. In any simply connected domain $\mathcal{D}$, a complex valued harmonic function $f = u + iv$ can be written as $f = h+\overline{g}$, where h and g are analytic in $\mathcal{D}$. We call h the analytic part and g the co-analytic part of f. We denote by $\mathcal{H}$ the class of complex valued harmonic mappings $f = h + \overline{g}$ defined in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $h(0) = g(0) = h'(0) - 1 = 0$. Such mappings have the following power series representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}. \tag{1.1}$$

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A necessary and sufficient condition for $f \in \mathcal{H}$ to be locally univalent and sense-preserving in $\mathbb{D}$ is that $|h'(z)| > |g'(z)|$ for all $z \in \mathbb{D}$ (see[4]). Let S_H denote the subclass of $\mathcal{H}$ consisting of all sense-preserving univalent harmonic mappings f and S_H^0 is the subclass of S_H whose members satisfy additional condition $f_{\overline{z}}(0) = 0$, i.e. $g'(0) = 0$. Denote by K_H^0, S_H^{*0} and C_H^0 the subclasses of S_H^0 whose functions map $\mathbb{D}$ onto convex, starlike and close-to-convex domains, respectively. Further $\overline{S}_H$ denote the subclass of S_H consisting of functions of the type $f_k = h + \overline{g}_k$, where

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n \quad \text{and} \quad g_k(z) = (-1)^k \sum_{n=1}^{\infty} |b_n| z^n. \tag{1.2}$$

In 1984 Clunie and Sheil-Small in their landmark paper [2], investigated some geometric properties of the class S_H and its subclasses. Since then researchers defined many subclasses of the class S_H and studied their various properties e.g. see [5, 6, 8, 11 and 14] and references therein. In particular, Nasr and Aouf [7] introduced the subclass $S^*(b)$ consisting of functions which are analytic and starlike of complex order b, $(b \in \mathbb{C}/\{0\})$ and satisfying

$$Re\left[1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)\right] \geq 0, \ z \in \mathbb{D}.$$

Halim and Janteng [3] defined the subclass $HS^*(b, \alpha)$, $b \neq 0$ with $|b| \leq 1$, of the class S_H consisting of the functions of the form $f = h + \overline{g}$, where h and g are given by (1.1) and satisfy

$$Re\left[1 + \frac{1}{b}\left(\frac{zf'(z)}{z'f(z)} - 1\right)\right] \geq \alpha, \quad 0 \leq \alpha < 1$$

where $z' = \frac{\partial}{\partial\theta}(z = re^{i\theta})$ and $f'(z) = \frac{\partial}{\partial\theta}(f(z) = f(re^{i\theta}))$. Further, Yaşar and Yalçın [14] studied the subclass $\overline{S}_H(m, b), b \in \mathbb{C}/\{0\}$, of $\overline{S}_H$ consisting of functions $f_m = h + \overline{g}_m$ that satisfy the following condition:

$$Re\left[1 + \frac{1}{b}\left(\frac{D^{m+1}f_m(z)}{D^m f_m(z)} - 1\right)\right] > 0.$$

Where D^m is the modified Salagean differential operator defined for the functions $f = h + \overline{g}$ as

$$D^m f(z) = D^m h(z) + (-1)^m \overline{D^m g(z)}.$$

One can refer to [11] and [5] for more details about Salagean and modified Salagean operators, respectively. Motivated by above papers, we define a new subclass $HS(k,\lambda,b,\alpha)$, ($k \in \mathbb{N}\cup\{0\}, 0 \leq \lambda \leq 1$, $b \neq 0$ with $|b| \leq 1$) of S_H consisting of harmonic functions $f = h + \overline{g}$ of the form (1.1) and satisfying

$$Re\left[1+\frac{1}{b}\left(\frac{I(k+1,\lambda)f(z)}{I(k,\lambda)f(z)}-1\right)\right] \geq \alpha,\ 0 \leq \alpha < 1. \tag{1.3}$$

Here $I(k,\lambda)$ is the modified multiplier transformation defined as

$$I(k,\lambda)f(z) = I(k,\lambda)h(z) + (-1)^k\overline{I(k,\lambda)g(z)},$$

where

$$I(k,\lambda)h(z) = z + \sum_{n=2}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^k a_n z^n \quad\text{and}\quad I(k,\lambda)g(z) = \sum_{n=1}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^k b_n z^n.$$

The multiplier transformation $I(k,\gamma)$ for analytic functions f was introduced by Cho and Kim [1]. We denote by $\overline{HS}(k,\lambda,b,\alpha)$, the class of functions of the type $f_k = h + \overline{g}_k$, which satisfy the condition (1.3). For particular, values of k,λ,b and α we obtain the corresponding results for several known subclasses as special cases. For example, for $\gamma \in [0,1)$ and
(i) if $k=0,\lambda=0,b=1-\gamma$ and $\alpha=0$, then $\overline{HS}(k,\lambda,b,\alpha) = T_H^*,$ [10];
(ii) if $k=1,\lambda=0,b=1-\gamma$ and $\alpha=0$, then $\overline{HS}(k,\lambda,b,\alpha) = TK_H,$ [10];
(iii) if $k,\lambda,b=\frac{1}{2}-\frac{i}{2}\tan\frac{\gamma}{2}$ and α, then $\overline{HS}(k,\lambda,b,\alpha) = \overline{RS}_H(k,\lambda,\alpha),$ [6].
If the co-analytic part of f is identically zero i.e. $g \equiv 0$ and
(iv) if $k=0,\lambda=0,b=1-\gamma$ and $\alpha=0$, then $HS(k,\lambda,b,\alpha) = T^*(\gamma),$ [9];
(v) if $k=1,\lambda=0,b=1-\gamma$ and $\alpha=0$, then $HS(k,\lambda,b,\alpha) = C(\gamma),$ [9];
(vi) if $k=0,\lambda=0,b$ and $\alpha=0$, then $HS(k,\lambda,b,\alpha) = T_n^*(b),$ [12];
(vii) if $k=m,\lambda=0,b$ and $\alpha=0$, then $HS(k,\lambda,b,\alpha) = T_{n,m}^*(b),$ [13].
In Section 2, we first obtain a sufficient condition for harmonic univalent functions $f = h + \overline{g}$ given by (1.1) to be in $HS(k,\lambda,b,\alpha)$ and then we prove that this condition is also necessary for the functions in the class $\overline{HS}(k,\lambda,b,\alpha)$. We also obtain extreme points, distortion bounds, convex combination, radius of convexity and Bernandi-Libera-Livingston integral for the functions in the class $\overline{HS}(k,\lambda,b,\alpha)$.

2. Main Results

Theorem 2.1. *Let a function* $f = h + \overline{g} \in S_H$ *given by (1.1) and satisfies*

$$\sum_{n=1}^{\infty} \left(\frac{n+\lambda}{1+\lambda}\right)^k \left\{ \left(\frac{(n+\lambda)-(1+\lambda)(1-|b|+\alpha|b|)}{(1+\lambda)|b|}\right)|a_n| + \left(\frac{(n+\lambda)+(1+\lambda)(1-|b|+\alpha|b|)}{(1+\lambda)|b|}\right)|b_n| \right\} \leq 2(1-\alpha), \tag{2.1}$$

where $0 \leq \alpha < 1$, $0 \leq \lambda \leq 1$, $k \in \mathbb{N}\bigcup\{0\}$, $b \neq 0$ *with* $|b| \leq 1$, *then* $f \in HS(k,\lambda,b,\alpha)$.

Proof. First we shall prove that f is sense-preserving and univalent in $\mathbb{D}$. Since $|z| < 1$, therefore

$$\begin{aligned} |h'(z)| &\geq 1 - \sum_{n=2}^{\infty} n|a_n||z|^{n-1} > 1 - \sum_{n=2}^{\infty} n|a_n| \\ &\geq 1 - \sum_{n=2}^{\infty} \left(\frac{n+\lambda}{1+\lambda}\right)^k \left(\frac{(n+\lambda)-(1+\lambda)(1-|b|+\alpha|b|)}{(1+\lambda)|b|(1-\alpha)}\right)|a_n| \\ &\geq \sum_{n=1}^{\infty} \left(\frac{n+\lambda}{1+\lambda}\right)^k \left(\frac{(n+\lambda)+(1+\lambda)(1-|b|+\alpha|b|)}{(1+\lambda)|b|(1-\alpha)}\right)|b_n|; \text{ using } (2.1) \\ &> \sum_{n=1}^{\infty} n|b_n||z|^{n-1} \geq |g'(z)| \ . \end{aligned}$$

This shows that f is sense-preserving in $\mathbb{D}$. Next we prove the univalence of f in $\mathbb{D}$. For $z_1 \neq z_2 \in \mathbb{D}$,

$$\begin{aligned} \left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| &\geq 1 - \left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right| > 1 - \frac{\sum_{n=1}^{\infty} n|b_n|}{1-\sum_{n=2}^{\infty} n|a_n|} \\ &\geq 1 - \frac{\sum_{n=1}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^k\left(\frac{(n+\lambda)+(1+\lambda)(1-|b|+\alpha|b|)}{(1+\lambda)|b|(1-\alpha)}\right)|b_n|}{1-\sum_{n=2}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^k\left(\frac{(n+\lambda)-(1+\lambda)(1-|b|+\alpha|b|)}{(1+\lambda)|b|(1-\alpha)}\right)|a_n|} \geq 0. \end{aligned}$$

Hence in view of condition (2.1), f is univalent in $\mathbb{D}$. In order to prove that $f \in HS(k,\lambda,b,\alpha)$ we shall show that for $0 \leq \alpha < 1$ and $|b| \leq 1$, $Re\left(1 + \frac{1}{b}\left(\frac{I(k+1,\lambda)f(z)}{I(k,\lambda)f(z)} - 1\right)\right) \geq \alpha$. We know that $Re(w) \geq \alpha$ if and only if $|1-\alpha+w| \geq |1+\alpha-w|$. So, it suffices to show that

$$\begin{aligned} &|(1-\alpha)bI(k,\lambda)f(z) + bI(k,\lambda)f(z) + I(k+1,\lambda)f(z) - I(k,\lambda)f(z)| - \\ &\quad |(1+\alpha)bI(k,\lambda)f(z) - (bI(k,\lambda)f(z) + I(k+1,\lambda)f(z) - I(k,\lambda)f(z))| \geq 0. \end{aligned}$$

Now,

$$\begin{aligned}
&|(1-\alpha)bI(k,\lambda)f(z)+bI(k,\lambda)f(z)+I(k+1,\lambda)f(z)-I(k,\lambda)f(z)|-\\
&\qquad|(1+\alpha)bI(k,\lambda)f(z)-(bI(k,\lambda)f(z)+I(k+1,\lambda)f(z)-I(k,\lambda)f(z))|\\
&=|(b(2-\alpha)-1)\,I(k,\lambda)f(z)+I(k+1,\lambda)f(z)|-\\
&\qquad|(b\alpha+1)\,I(k,\lambda)f(z)-I(k+1,\lambda)f(z)|\\
&=\Big|(2-\alpha)bz+\textstyle\sum_{n=2}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^k\left(2b-b\alpha-1+\frac{n+\lambda}{1+\lambda}\right)a_nz^n+\\
&\qquad\textstyle\sum_{n=1}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^k\left(2b-b\alpha-1-\frac{n+\lambda}{1+\lambda}\right)\overline{b_n}\overline{z}^n\Big|-\\
&\qquad\Big|b\alpha z+\textstyle\sum_{n=2}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^k\left(b\alpha+1-\frac{n+\lambda}{1+\lambda}\right)a_nz^n+\\
&\qquad\textstyle\sum_{n=1}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^k\left(b\alpha+1+\frac{n+\lambda}{1+\lambda}\right)\overline{b_n}\overline{z}^n\Big|\\
&\geq(2-\alpha)|b||z|-\textstyle\sum_{n=2}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^k\left((2-\alpha)|b|-1+\frac{n+\lambda}{1+\lambda}\right)|a_n||z|^n+\\
&\qquad\textstyle\sum_{n=1}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^k\left((2-\alpha)|b|-1-\frac{n+\lambda}{1+\lambda}\right)|b_n||z|^n-\\
&\qquad|b|\alpha z+\textstyle\sum_{n=2}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^k\left(|b|\alpha+1-\frac{n+\lambda}{1+\lambda}\right)|a_n||z|^n-\\
&\qquad\textstyle\sum_{n=1}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^k\left(|b|\alpha+1+\frac{n+\lambda}{1+\lambda}\right)|b_n||z|^n\\
&=2(1-\alpha)|b||z|-2\textstyle\sum_{n=2}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^k\left((1-\alpha)|b|-1+\frac{n+\lambda}{1+\lambda}\right)|a_n||z|^n-\\
&\qquad2\textstyle\sum_{n=1}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^k\left((\alpha-1)|b|+1+\frac{n+\lambda}{1+\lambda}\right)|b_n||z|^n\\
&\geq2(1-\alpha)|b||z|\Big(1-\textstyle\sum_{n=2}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^k|a_n||z|^{n-1}\left(\frac{(n+\lambda)-(1+\lambda)(1-|b|+|b|\alpha)}{(1+\lambda)(1-\alpha)|b|}\right)-\\
&\qquad\textstyle\sum_{n=1}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^k|b_n||z|^{n-1}\left(\frac{(n+\lambda)+(1+\lambda)(1-|b|+|b|\alpha)}{(1+\lambda)(1-\alpha)|b|}\right)\Big)\\
&\geq0,\quad\text{in view of}(2.1).
\end{aligned}$$

This completes the proof. □

Theorem 2.2. *Let $f_k=h+\overline{g}_k$ be given by (1.2). Then $f_k\in\overline{HS}(k,\lambda,b,\alpha)$ if and only if f_k satisfy condition (2.1).*

Proof. Since $\overline{HS}^*(k,\lambda,b,\alpha)\subset HS(k,\lambda,b,\alpha)$ therefore, 'if' part of the Theorem 2.2 can be proved from Theorem 2.1, thus we prove 'only if' part of Theorem 2.2. Let $f_k=h+\overline{g_k}\in\overline{HS}(k,\lambda,b,\alpha)$, then

$$Re\left[1+\frac{1}{b}\left(\frac{I(k+1,\lambda)f_k(z)}{I(k,\lambda)f_k(z)}-1\right)-\alpha\right]\geq0,\ \ 0\leq\alpha<1,\ |b|\leq1.$$

After substituting $I(k+1,\lambda)f_k(z)$ and $I(k,\lambda)f_k(z)$ in the above inequality we get,

$$Re\left[\frac{b(1-z)-\sum_{n=2}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^k|a_n|z^n\left[\frac{n+\lambda}{1+\lambda}+b-1-\alpha b\right]}{bI(k,\lambda)f(z)}-\right.$$
$$\left.(-1)^{2k}\frac{\sum_{n=1}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^k|b_n|\overline{z}^n\left[\frac{n+\lambda}{1+\lambda}-b+1+\alpha b\right]}{bI(k,\lambda)f(z)}\right]\geq 0,$$

$$Re\left[\frac{(1-\alpha)z-\sum_{n=2}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^k|a_n|z^n\left[\frac{n+\lambda}{b(1+\lambda)}+1-\frac{1}{b}-\alpha\right]}{z-\sum_{n=2}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^k|a_n|z^n+(-1)^{2k}\sum_{n=1}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^k|b_n|\overline{z}^n}-\right.$$
$$\left.(-1)^{2k}\frac{\sum_{n=1}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^k|b_n|\overline{z}^n\left[\frac{n+\lambda}{b(1+\lambda)}-1+\frac{1}{b}+\alpha\right]}{z-\sum_{n=2}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^k|a_n|z^n+(-1)^{2k}\sum_{n=1}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^k|b_n|\overline{z}^n}\right]\geq 0,$$

$$Re\left[\frac{(1-\alpha)-\sum_{n=2}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^k|a_n|z^{n-1}\left[\frac{n+\lambda}{b(1+\lambda)}+1-\frac{1}{b}-\alpha\right]}{1-\sum_{n=2}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^k|a_n|z^{n-1}+(-1)^{2k}\frac{\overline{z}}{z}\sum_{n=1}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)|b_n|\overline{z}^{n-1}}-\right.$$
$$\left.(-1)^{2k}\frac{\frac{\overline{z}}{z}\sum_{n=1}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^k|b_n|\overline{z}^{n-1}\left[\frac{n+\lambda}{b(1+\lambda)}-1+\frac{1}{b}+\alpha\right]}{1-\sum_{n=2}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^k|a_n|z^{n-1}+(-1)^{2k}\frac{\overline{z}}{z}\sum_{n=1}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)|b_n|\overline{z}^{n-1}}\right]\geq 0.$$

The above condition holds for all values of z, $|z|=r<1$. Choosing the values of z on positive real axis, where $0\leq z=r<1$, we have

$$Re\left[\frac{(1-\alpha)-\sum_{n=2}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^k|a_n|r^{n-1}\left[\frac{n+\lambda}{b(1+\lambda)}+1-\frac{1}{b}-\alpha\right]-\sum_{n=1}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^k|b_n|r^{n-1}\left[\frac{n+\lambda}{b(1+\lambda)}-1+\frac{1}{b}+\alpha\right]}{1-\sum_{n=2}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^k|a_n|r^{n-1}+\sum_{n=1}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^k|b_n|r^{n-1}}\right]\geq 0\,.$$

Since $Re(b)\leq|b|$, the above inequality reduces to

$$\left[\frac{(1-\alpha)-\sum_{n=2}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^k|a_n|r^{n-1}\left[\frac{n+\lambda}{|b|(1+\lambda)}+1-\frac{1}{|b|}-\alpha\right]}{1-\sum_{n=2}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^k|a_n|r^{n-1}+\sum_{n=1}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^k|b_n|r^{n-1}}-\right.$$
$$\left.\frac{\sum_{n=1}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^k|b_n|r^{n-1}\left[\frac{n+\lambda}{|b|(1+\lambda)}-1+\frac{1}{|b|}+\alpha\right]}{1-\sum_{n=2}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^k|a_n|r^{n-1}+\sum_{n=1}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^k|b_n|r^{n-1}}\right]\geq 0. \tag{2.2}$$

In case the condition (2.1) doesn't holds then, the numerator of left hand side of (2.2) becomes negative for r sufficiently close to 1. This leads to contradiction. Hence it proves the result. □

Theorem 2.3. *Members of the class $\overline{HS}(k,\lambda,b,\alpha)$ map unit disk onto a starlike domain.*

Proof. A function $f=h+\overline{g}=z-\sum_{n=2}^{\infty}|a_n|z^n+\sum_{n=1}^{\infty}|b_n|\overline{z}^n\in\overline{HS}(k,\lambda,b,\alpha)$ maps the unit disk $\mathbb{D}$ onto a starlike domain if and only if for $z\in\mathbb{D}$,

$$Re\left\{\frac{zh'(z)-\overline{zg'(z)}}{h(z)+\overline{g(z)}}\right\}>0. \tag{2.3}$$

We know that $Re(w) > 0$ if and only if $|1+w| > |1-w|$. Therefore to prove (2.3), it suffices to show that $|h(z)+\overline{g(z)}+zh'(z)-\overline{zg'(z)}| - |h(z)+\overline{g(z)} - zh'(z)+\overline{zg'(z)}| > 0$. Since

$$\begin{aligned}
&|h(z)+\overline{g(z)}+zh'(z)-\overline{zg'(z)}| - |h(z)+\overline{g(z)}-zh'(z)+\overline{zg'(z)}| \\
&= \left| z - \sum_{n=2}^{\infty}|a_n|z^n + \sum_{n=1}^{\infty}|b_n|\overline{z}^n + z(1-\sum_{n=2}^{\infty}n|a_n|z^{n-1}) - \overline{z}\sum_{n=1}^{\infty}n|b_n|\overline{z}^{n-1}\right| - \\
&\quad \left| z - \sum_{n=2}^{\infty}|a_n|z^n + \sum_{n=1}^{\infty}|b_n|\overline{z}^n - z(1-\sum_{n=2}^{\infty}n|a_n|z^{n-1}) + \overline{z}\sum_{n=1}^{\infty}n|b_n|\overline{z}^{n-1}\right| \\
&= \left| 2z - \sum_{n=2}^{\infty}(n+1)|a_n|z^n - \sum_{n=1}^{\infty}(n-1)|b_n|\overline{z}^n\right| - \\
&\qquad\qquad \left| \sum_{n=2}^{\infty}(n-1)|a_n|z^n + \sum_{n=1}^{\infty}(n+1)|b_n|\overline{z}^n\right| \\
&\geq 2|z|\left[1-\sum_{n=2}^{\infty}n|a_n||z|^{n-1} - \sum_{n=1}^{\infty}n|b_n||z|^{n-1}\right] \\
&> 2|z|\left[1-\sum_{n=2}^{\infty}n|a_n| - \sum_{n=1}^{\infty}n|b_n|\right] \\
&\geq 2|z|\left[1-\sum_{n=2}^{\infty}\left(\tfrac{n+\lambda}{1+\lambda}\right)^k \tfrac{(n+\lambda)-(1+\lambda)(1-|b|+\alpha|b|)}{(1-\alpha)(1+\lambda)|b|}|a_n| - \right. \\
&\qquad\qquad \left. \sum_{n=1}^{\infty}\left(\tfrac{n+\lambda}{1+\lambda}\right)^k \tfrac{(n+\lambda)+(1+\lambda)(1-|b|+\alpha|b|)}{(1-\alpha)(1+\lambda)|b|}|b_n|\right] > 0,
\end{aligned}$$

in view of condition (2.1). Hence it completes the proof. □

Extreme Points

In the next result we obtain the extreme points for the class $\overline{HS}(k,\lambda,b,\alpha)$.

Theorem 2.4. *If $f_k = h+\overline{g_k}$, then $f_k \in \overline{HS}(k,\lambda,b,\alpha)$ if and only if $f_k(z) = \sum_{n=1}^{\infty}(X_n h_n(z) + Y_n g_{k_n}(z))$, where*

$$h_1(z) = z,$$

$$h_n(z) = z - \tfrac{(1-\alpha)(1+\lambda)^{k+1}|b|}{[(n+\lambda)-(1+\lambda)(1-|b|+\alpha|b|)](n+\lambda)^k} z^n, \quad n = 2,3,4\ldots,$$

$$g_{k_n}(z) = z + (-1)^k \tfrac{(1-\alpha)(1+\lambda)^{k+1}|b|}{[(n+\lambda)+(1+\lambda)(1-|b|+\alpha|b|)](n+\lambda)^k}\overline{z^n}, \quad n = 1,2,3,\ldots,$$

and $\sum_{n=1}^{\infty}(X_n+Y_n) = 1$, $X_n \geq 0$, $Y_n \geq 0$. In particular, the extreme points of $\overline{HS}(k,\lambda,b,\alpha)$ are $\{h_n\}$ and $\{g_{k_n}\}$.

Proof. Substituting the values of $h_n(z)$ and $g_{k_n}(z)$ in $f_k(z)$, we have

$$\begin{aligned} f_k(z) &= \sum_{n=1}^{\infty}(X_n h_n(z) + Y_n g_{k_n}(z)) \\ &= z - \sum_{n=2}^{\infty} \tfrac{(1-\alpha)(1+\lambda)^{k+1}|b|}{[(n+\lambda)-(1+\lambda)(1-|b|+\alpha|b|)](n+\lambda)^k} X_n z^n + \\ &\qquad (-1)^k \sum_{n=1}^{\infty} \tfrac{(1-\alpha)(1+\lambda)^{k+1}|b|}{[(n+\lambda)+(1+\lambda)(1-|b|+\alpha|b|)](n+\lambda)^k} Y_n \overline{z^n}. \end{aligned}$$

Compairing (2.4) with $f_k(z) = h(z) + g_k(z) = z - \sum_{n=2}^{\infty} |a_n| z^n + (-1)^k \sum_{n=1}^{\infty} |b_n| \overline{z^n}$, we obtain

$$|a_n| = \frac{(1-\alpha)(1+\lambda)^{k+1}|b|}{[(n+\lambda)-(1+\lambda)(1-|b|+\alpha|b|)](n+\lambda)^k} X_n$$

and

$$|b_n| = (-1)^k \frac{(1-\alpha)(1+\lambda)^{k+1}|b|}{[(n+\lambda)+(1+\lambda)(1-|b|+\alpha|b|)](n+\lambda)^k} Y_n.$$

Now,

$$\begin{aligned} &\sum_{n=2}^{\infty} \left(\tfrac{n+\lambda}{1+\lambda}\right)^k \left[\tfrac{(n+\lambda)-(1+\lambda)(1-|b|+\alpha|b|)}{(1-\alpha)(1+\lambda)|b|}\right] |a_n| + \\ &\qquad \sum_{n=1}^{\infty} \left(\tfrac{n+\lambda}{1+\lambda}\right)^k \left[\tfrac{(n+\lambda)+(1+\lambda)(1-|b|+\alpha|b|)}{(1-\alpha)(1+\lambda)|b|}\right] |b_n| \\ &= \sum_{n=2}^{\infty} X_n + \sum_{n=1}^{\infty} Y_n = 1 - X_1 \leq 1. \end{aligned}$$

Therefore, $f_k \in \overline{HS}(k, \lambda, b, \alpha)$. Conversely, let $f_k \in \overline{HS}(k, \lambda, b, \alpha)$. Set

$$X_n = \left(\frac{n+\lambda}{1+\lambda}\right)^k \left[\frac{(n+\lambda)-(1+\lambda)(1-|b|+\alpha|b|)}{(1-\alpha)(1+\lambda)|b|}\right] |a_n|, \qquad n = 2, 3, 4, \ldots$$

and

$$Y_n = \left(\frac{n+\lambda}{1+\lambda}\right)^k \left[\frac{(n+\lambda)+(1+\lambda)(1-|b|+\alpha|b|)}{(1-\alpha)(1+\lambda)|b|}\right] |b_n|, \qquad n = 1, 2, 3, \ldots$$

where $\sum_{n=1}^{\infty} (X_n + Y_n) = 1$. Since $f_k \in \overline{HS}(k, \lambda, b, \alpha)$, thus $f_k(z) = z - \sum_{n=2}^{\infty} |a_n| z^n + (-1)^k \sum_{n=1}^{\infty} |b_n| \overline{z}^n$. by substituting values of $|a_n|$ and $|b_n|$, we get $f_k(z) = \sum_{n=1}^{\infty}(X_n h_n(z) + Y_n g_{k_n}(z))$, where $h_n(z)$ and $g_{k_n}(z)$ are as mentioned above. Hence the proof is complete. □

Convolution Properties

In this section, we show that the class $\overline{HS}(k,\lambda,b,\alpha)$ is closed under convolution. For harmonic functions

$$f_k(z) = z - \sum_{n=2}^{\infty} |a_n| z^n + (-1)^k \sum_{n=1}^{\infty} |b_n| \overline{z}^n$$

and

$$F_k(z) = z - \sum_{n=2}^{\infty} |A_n| z^n + (-1)^k \sum_{n=1}^{\infty} |B_n| \overline{z}^n,$$

their convolution, $f_k * F_k$ is given by

$$(f_k * F_k)(z) = f_k(z) * F_k(z) = z - \sum_{n=2}^{\infty} |a_n A_n| z^n + (-1)^k \sum_{n=1}^{\infty} |b_n B_n| \overline{z}^n.$$

Theorem 2.5. *Let $f_k \in \overline{HS}(k,\lambda,b,\alpha_1)$ and $F_k \in \overline{HS}(k,\lambda,b,\alpha_2)$, where $0 \le \alpha_1 \le \alpha_2 < 1$. Then $f_k * F_k \in \overline{HS}(k,\lambda,b,\alpha_2) \subset \overline{HS}(k,\lambda,b,\alpha_1)$.*

Proof. We wish to show that $(f_k * F_k)$ satisfies the coefficient condition (2.1). For $F_k \in \overline{HS}(k,\lambda,b,\alpha_2)$, we note that $|A_n| \le 1$ and $|B_n| \le 1$. Now, for the coefficients of convolution function $(f_k * F_k)$, we can write

$$\begin{aligned}
&\sum_{n=1}^{\infty} \left(\tfrac{n+\lambda}{1+\lambda}\right)^k \left[\tfrac{(n+\lambda)-(1+\lambda)(1-|b|+\alpha|b|)}{(1-\alpha_2)(1+\lambda)|b|}\right] |a_n A_n| + \\
&\qquad\qquad \sum_{n=1}^{\infty} \left(\tfrac{n+\lambda}{1+\lambda}\right)^k \left[\tfrac{(n+\lambda)+(1+\lambda)(1-|b|+\alpha|b|)}{(1-\alpha_2)(1+\lambda)|b|}\right] |b_n B_n| \\
&\le \sum_{n=1}^{\infty} \left(\tfrac{n+\lambda}{1+\lambda}\right)^k \left[\tfrac{(n+\lambda)-(1+\lambda)(1-|b|+\alpha|b|)}{(1-\alpha_2)(1+\lambda)|b|}|a_n| + \tfrac{(n+\lambda)+(1+\lambda)(1-|b|+\alpha|b|)}{(1-\alpha_2)(1+\lambda)|b|}|b_n|\right] \\
&\le \sum_{n=1}^{\infty} \left(\tfrac{n+\lambda}{1+\lambda}\right)^k \left[\tfrac{(n+\lambda)-(1+\lambda)(1-|b|+\alpha|b|)}{(1-\alpha_1)(1+\lambda)|b|}|a_n| + \tfrac{(n+\lambda)+(1+\lambda)(1-|b|+\alpha|b|)}{(1-\alpha_1)(1+\lambda)|b|}|b_n|\right] \\
&\le 2.
\end{aligned}$$

Hence $(f_k * F_k) \in \overline{HS}(k,\lambda,b,\alpha_2) \subset \overline{HS}(k,\lambda,b,\alpha_1)$. □

In order to prove our next result, we use the following lemma of Clunie and Sheil-Small [2].

Lemma 2.1. *If $F \in K_H^\circ$ is given by $F(z) = z + \sum_{n=2}^{\infty} (A_n z^n + \overline{B_n} \overline{z}^n)$, then $|A_n| \le \frac{n+1}{2}$ and $|B_n| \le \frac{n-1}{2}$.*

Theorem 2.6. *If* $f \in \overline{HS}^0(k,\lambda,b,\alpha)$ *and* $F \in K_H^\circ$, *then* $f*F \in \overline{HS}^0(k-1,\lambda,b,\alpha)$ *for* $0 \le \lambda \le 1$.

Proof. Since $f \in \overline{HS}^0(k,\lambda,b,\alpha)$, therefore

$$\sum_{n=2}^{\infty} \left(\frac{n+\lambda}{1+\lambda}\right)^k \left[\frac{(n+\lambda)-(1+\lambda)(1-|b|+\alpha|b|)}{(1+\lambda)|b|}|a_n| + \frac{(n+\lambda)+(1+\lambda)(1-|b|+\alpha|b|)}{(1+\lambda)|b|}|b_n|\right] \le (1-\alpha).$$

We know that $(f*F)(z) = z + \sum\limits_{n=2}^{\infty}(a_nA_nz^n + \overline{b_nB_n}\overline{z}^n)$. Now,

$$\begin{aligned}
&\sum_{n=2}^{\infty} \left(\frac{n+\lambda}{1+\lambda}\right)^{k-1} \left[\frac{(n+\lambda)-(1+\lambda)(1-|b|+\alpha|b|)}{(1+\lambda)|b|}|a_nA_n| + \frac{(n+\lambda)+(1+\lambda)(1-|b|+\alpha|b|)}{(1+\lambda)|b|}|b_nB_n|\right] \\
&= \sum_{n=2}^{\infty} \left(\frac{n+\lambda}{1+\lambda}\right)^{k-1} \left[\frac{(n+\lambda)-(1+\lambda)(1-|b|+\alpha|b|)}{(1+\lambda)|b|}|a_n||A_n| + \frac{(n+\lambda)+(1+\lambda)(1-|b|+\alpha|b|)}{(1+\lambda)|b|}|b_n||B_n|\right] \\
&\le \sum_{n=2}^{\infty} \left(\frac{n+\lambda}{1+\lambda}\right)^{k-1} \left[\frac{(n+\lambda)-(1+\lambda)(1-|b|+\alpha|b|)}{(1+\lambda)|b|}\left(\frac{n+1}{2}\right)|a_n| + \right. \\
&\qquad\qquad \left.\left(\frac{(n+\lambda)+(1+\lambda)(1-|b|+\alpha|b|)}{(1+\lambda)|b|}\right)\left(\frac{n-1}{2}\right)|b_n|\right]; \quad \text{using } Lemma \ 2.1 \\
&\le \sum_{n=2}^{\infty} \left(\frac{n+\lambda}{1+\lambda}\right)^{k-1} \left[\frac{(n+\lambda)-(1+\lambda)(1-|b|+\alpha|b|)}{(1+\lambda)|b|}\left(\frac{n+\lambda}{1+\lambda}\right)|a_n| + \right. \\
&\qquad\qquad \left.\left(\frac{(n+\lambda)+(1+\lambda)(1-|b|+\alpha|b|)}{(1+\lambda)|b|}\right)\left(\frac{n+\lambda}{1+\lambda}\right)|b_n|\right] \\
&\le \sum_{n=2}^{\infty} \left(\frac{n+\lambda}{1+\lambda}\right)^{k} \left[\left(\frac{(n+\lambda)-(1+\lambda)(1-|b|+\alpha|b|)}{(1+\lambda)|b|}\right)|a_n| + \right. \\
&\qquad\qquad \left.\left(\frac{(n+\lambda)+(1+\lambda)(1-|b|+\alpha|b|)}{(1+\lambda)|b|}\right)|b_n|\right] \le (1-\alpha).
\end{aligned}$$

Hence, $f*F \in \overline{HS}^0(k-1,\lambda,b,\alpha)$. □

Convex Combinations

Here, we prove that the class $\overline{HS}(k,\lambda,b,\alpha)$ is closed under convex combinations of its members. Let the function $f_{k,i}(z)$ be defined for i=1,2,... by

$$f_{k,i}(z) = z - \sum_{n=2}^{\infty}|a_{n,i}|z^n + (-1)^k\sum_{n=1}^{\infty}|b_{n,i}|\overline{z}^n. \tag{2.4}$$

Theorem 2.7. *The class* $\overline{HS}(k,\lambda,b,\alpha)$ *is closed under convex combinations.*

Proof. For $i = 1, 2, 3, ...,$ let $f_{k,i} \in \overline{HS}(k, \lambda, b, \alpha)$, where $f_{k,i}$ is given by (2.4). Since each $f_{k,i}(z)$ is in $\overline{HS}(k, \lambda, b, \alpha)$ therefore by Theorem 2.2, we obtain

$$\sum_{n=1}^{\infty} \left(\tfrac{n+\lambda}{1+\lambda}\right)^k \left[\left(\tfrac{(n+\lambda)-(1+\lambda)(1-|b|+\alpha|b|)}{(1+\lambda)|b|}\right)|a_{n,i}| + \right.$$
$$\left. + \left(\tfrac{(n+\lambda)+(1+\lambda)(1-|b|+\alpha|b|)}{(1+\lambda)|b|}\right)|b_{n,i}|\right] \leq 2(1-\alpha).$$

For $\sum_{i=1}^{\infty} t_i = 1$, $0 \leq t \leq 1$, the convex combinations of $f_{k,i}$ may be written as

$$c_i(z) = \sum_{i=1}^{\infty} t_i f_{k,i}(z) = z - \sum_{n=2}^{\infty}\left(\sum_{i=1}^{\infty} t_i |a_{n,i}|\right) z^n + (-1)^k \sum_{n=1}^{\infty}\left(\sum_{i=1}^{\infty} t_i |b_{n,i}|\right)\overline{z}^n.$$

Now for $c_i(z)$, consider

$$\sum_{n=1}^{\infty} \left(\tfrac{n+\lambda}{1+\lambda}\right)^k \left[\left(\tfrac{(n+\lambda)-(1+\lambda)(1-|b|+\alpha|b|)}{(1+\lambda)|b|}\right) |\sum_{i=1}^{\infty} t_i a_{n,i}| + \right.$$
$$\left. \left(\tfrac{(n+\lambda)+(1+\lambda)(1-|b|+\alpha|b|)}{(1+\lambda)|b|}\right) |\sum_{n=1}^{\infty} t_i b_{n,i}|\right]$$
$$= \sum_{i=1}^{\infty} t_i \left[\sum_{n=1}^{\infty} \left(\tfrac{n+\lambda}{1+\lambda}\right)^k \left(\tfrac{(n+\lambda)-(1+\lambda)(1-|b|+\alpha|b|)}{(1+\lambda)|b|}|a_{n,i}| + \right.\right.$$
$$\left.\left. \tfrac{(n+\lambda)+(1+\lambda)(1-|b|+\alpha|b|)}{(1+\lambda)|b|}|b_{n,i}|\right)\right]$$
$$\leq 2(1-\alpha)\sum_{i=1}^{\infty} t_i = 2(1-\alpha).$$

Thus, we get $c_i(z) \in \overline{HS}(k, \lambda, b, \alpha)$. □

Covering Theorem

Our next theorem is on the distortion bounds for functions in $\overline{HS}(k, \lambda, b, \alpha)$, which yields a covering result for this family.

Theorem 2.8. *If $f \in \overline{HS}(k, \lambda, b, \alpha)$ and $|z| = r < 1$, then*

$$(1-|b_1|)r - M \leq |f(z)| \leq (1+|b_1|)r + M \tag{2.5}$$

where,

$$M = \tfrac{(1+\lambda)^{k+1}}{(2+\lambda)^k}\left[\tfrac{(1-\alpha)|b|}{(2+\lambda)-(1+\lambda)(1-|b|+\alpha|b|)} - \tfrac{(2-|b|+\alpha|b|)}{(2+\lambda)-(1+\lambda)(1-|b|+\alpha|b|)}|b_1|\right] r^2.$$

Proof. First, We shall prove the left hand side of the inequality (2.5). Let $f \in \overline{HS}(k, \lambda, b, \alpha)$, then by using Theorem 2.2, we obtain

$$\begin{aligned}
|f(z)| &\geq (1-|b_1|)\, r - \textstyle\sum_{n=2}^{\infty} (|a_n| + |b_n|)\, r^n \\
&\geq (1-|b_1|)\, r - \textstyle\sum_{n=2}^{\infty} (|a_n| + |b_n|)\, r^2 \\
&\geq (1-|b_1|)\, r - \tfrac{(1-\alpha)(1+\lambda)^{k+1}|b|}{((2+\lambda)-(1+\lambda)(1-|b|+\alpha|b|))(2+\lambda)^k} \\
&\qquad \times \sum_{n=2}^{\infty} \tfrac{((2+\lambda)-(1+\lambda)(1-|b|+\alpha|b|))(2+\lambda)^k}{(1-\alpha)(1+\lambda)^{k+1}|b|} (|a_n| + |b_n|)\, r^2 \\
&\geq (1-|b_1|)\, r - \tfrac{(1-\alpha)(1+\lambda)^{k+1}|b| r^2}{((2+\lambda)-(1+\lambda)(1-|b|+\alpha|b|))(2+\lambda)^k} \\
&\qquad \times \sum_{n=2}^{\infty} \left(\tfrac{n+\lambda}{1+\lambda}\right)^k \left(\left(\tfrac{(n+\lambda)-(1+\lambda)(1-|b|+\alpha|b|)}{(1-\alpha)(1+\lambda)|b|}\right)|a_n|\right. \\
&\qquad\qquad \left. + \left(\tfrac{(n+\lambda)+(1+\lambda)(1-|b|+\alpha|b|)}{(1-\alpha)(1+\lambda)|b|}\right)|b_n|\right) \\
&\geq (1-|b_1|)\, r - \tfrac{(1-\alpha)(1+\lambda)^{k+1}|b| r^2}{((2+\lambda)-(1+\lambda)(1-|b|+\alpha|b|))(2+\lambda)^k} \times \left(1 - \tfrac{(2-|b|+\alpha|b|)}{(1-\alpha)|b|}|b_1|\right) \\
&= (1-|b_1|)\, r - M.
\end{aligned}$$

Following on the similar arguments we can easily prove the right hand side of the inequality (2.5). Hence proof of the theorem is complete. □

Radius of Convexity

Theorem 2.9. *If $f_k \in \overline{HS}(k, \lambda, b, \alpha)$ then f_k is convex in the disk*

$$|z| \leq \min_{q \in \mathbb{N}} \left(\frac{(1-\alpha)|b|(1-|b_1|)}{q\left((1-\alpha)|b| - (2-|b|+\alpha|b|)|b_1|\right)} \right)^{1/(q-1)}.$$

Proof. Let $f_k \in \overline{HS}(k, \lambda, b, \alpha)$ and let r, $0 < r < 1$ be fixed. Then $r^{-1} f_k(rz) \in \overline{HS}(k, \lambda, b, \alpha)$ and we get,

$$\begin{aligned}
\sum_{n=2}^{\infty} n^2 (|a_n| + |b_n|)\, r^{n-1} &= \sum_{n=2}^{\infty} n (|a_n| + |b_n|)\, n r^{n-1} \\
&\leq \sum_{n=2}^{\infty} \left(\tfrac{n+\lambda}{1+\lambda}\right)^k \left(\left(\tfrac{(n+\lambda)-(1+\lambda)(1-|b|+\alpha|b|)}{(1-\alpha)(1+\lambda)|b|}\right)|a_n|\right. \\
&\qquad \left. + \left(\tfrac{(n+\lambda)+(1+\lambda)(1-|b|+\alpha|b|)}{(1-\alpha)(1+\lambda)|b|}\right)|b_n|\right) n r^{n-1} \leq 1 - |b_1|,
\end{aligned}$$

provided, $n r^{n-1} \leq \frac{(1-|b_1|)}{1 - \frac{(2-|b|+\alpha|b|)}{(1-\alpha)|b|}|b_1|}$, or

$$r \leq \min_{q \in \mathbb{N}} \left(\frac{(1-\alpha)|b|(1-|b_1|)}{q\left((1-\alpha)|b| - (2-|b|+\alpha|b|)|b_1|\right)} \right)^{1/(q-1)}.$$

□

Bernandi-Libera-Livingston Integral of $\overline{HS}(k,\lambda,b,\alpha)$

In this last section, we study the closure property of the class $\overline{HS}(k,\lambda,b,\alpha)$ under the generalized Bernardi-Libera-Livingston integral operator $\mathfrak{L}_c$. For an analytic function f its generalized Bernardi-Libera-Livingston integral operator is defined as,

$$\mathfrak{L}_c[f(z)] = \frac{c+1}{z^c}\int_0^z \xi^{c-1} f(\xi)d\xi, \quad (c>-1),$$

whereas for harmonic functions $f = h + \overline{g}$ it is defined as,

$$\mathfrak{L}_c[f(z)] = \frac{c+1}{z^c}\int_0^z \xi^{c-1} h(\xi)d\xi + \overline{\frac{c+1}{z^c}\int_0^z \xi^{c-1} g(\xi)d\xi}, \quad (c>-1).$$

Theorem 2.10. *If $f_k \in \overline{HS}(k,\lambda,b,\alpha)$, then $\mathfrak{L}_c[f_k(z)] \in \overline{HS}(k,\lambda,b,\alpha)$.*

Proof. Let $f_k = h + \overline{g_k} \in \overline{HS}(k,\lambda,b,\alpha)$. Then,

$$\begin{aligned}\mathfrak{L}_c[f(z)] &= \tfrac{c+1}{z^c}\textstyle\int_0^z \xi^{c-1}h(\xi)d\xi + \overline{\tfrac{c+1}{z^c}\textstyle\int_0^z \xi^{c-1}g_k(\xi)d\xi} \\ &= \tfrac{c+1}{z^c}\textstyle\int_0^z \xi^{c-1}\left(\xi - \sum_{n=2}^{\infty}|a_n|\xi^n\right)d\xi + \\ &\qquad \overline{\frac{c+1}{z^c}\int_0^z \xi^{c-1}\left((-1)^k\sum_{n=1}^{\infty}|b_n|\xi^n\right)d\xi} \\ &= z - \textstyle\sum_{n=2}^{\infty}|A_n|z^n + (-1)^k\sum_{n=1}^{\infty}|B_n|\overline{z^n},\end{aligned}$$

where

$$|A_n| = \left(\frac{c+1}{c+n}\right)|a_n| \ \text{ and } \ |B_n| = \left(\frac{c+1}{c+n}\right)|b_n|.$$

Hence

$$\begin{aligned}&\textstyle\sum_{n=1}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^k\left[\left(\frac{(n+\lambda)-(1+\lambda)(1-|b|+\alpha|b|)}{(1+\lambda)|b|}\right)|A_n| + \left(\frac{(n+\lambda)+(1+\lambda)(1-|b|+\alpha|b|)}{(1+\lambda)|b|}\right)|B_n|\right] \\ &= \textstyle\sum_{n=1}^{\infty}\left[\left(\frac{n+\lambda}{1+\lambda}\right)^k\left(\frac{(n+\lambda)-(1+\lambda)(1-|b|+\alpha|b|)}{(1+\lambda)|b|}\right)\left(\frac{c+1}{c+n}\right)|a_n| + \right. \\ &\qquad\qquad \left.\left(\frac{(n+\lambda)+(1+\lambda)(1-|b|+|b|\alpha)}{(1+\lambda)|b|}\right)\left(\frac{c+1}{c+n}\right)|b_n|\right] \\ &\le \textstyle\sum_{n=1}^{\infty}\left[\left(\frac{n+\lambda}{1+\lambda}\right)^k\left(\frac{(n+\lambda)-(1+\lambda)(1-|b|+\alpha|b|)}{(1+\lambda)|b|}\right)|a_n| + \left(\frac{(n+\lambda)+(1+\lambda)(1-|b|+|b|\alpha)}{(1+\lambda)|b|}\right)|b_n|\right] \\ &\le 2(1-\alpha).\end{aligned}$$

Thus $\mathfrak{L}_c[f_k(z)] \in \overline{HS}(k,\lambda,b,\alpha)$. □

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