

EVALUATION FOR ESTIMATING OF THE PDF AND THE CDF OF GENERALIZED INVERTED EXPONENTIAL DISTRIBUTION WITH APPLICATION IN INDUSTRY

PARISA TORKAMAN

ABSTRACT. The generalized inverted exponential distribution is introduced as a lifetime model with good statistical properties. This paper, the estimation of the probability density function and the cumulative distribution function of with five different estimation methods: uniformly minimum variance unbiased(UMVU), maximum likelihood(ML), least squares(LS), weighted least squares (WLS) and percentile(PC) estimators are considered. The performance of these estimation procedures, based on the mean squared error (MSE) by numerical simulations are compared. Simulation studies express that the UMVU estimator performs better than others and when the sample size is large enough the ML and UMVU estimators are almost equivalent and efficient than LS, WLS and PC. Finally, the result using a real data set are analyzed.

1. INTRODUCTION

Most of the generalizations of the exponential distribution have the constant, non-increasing, non-decreasing and bathtub hazard rates, but in the applications, there may be data that have the inverted bathtub hazard rate. For example, in the investigate of breast and bladder cancer data, we found that the mortality increases initially after some time reaches a peak then slowly decreases. The generalized inverted exponential distribution (GIED), for such

2010 *Mathematics Subject Classification.* 62N02, 62E15.

Key words and phrases. Generalized inverted exponential, least square, estimator weighted Least Square estimator, percentile estimator.

types of data, can be a good model. The generalized inverted exponential distribution was introduced by Abouammoh and Alshinigiti [1] as a generalization of inverted exponential distribution (IED). The GIED has a uni-modal and right-skewed density function. Abouammoh and Alshinigiti [1] have shown that the hazard rate of GIED can be increasing or decreasing but not constant and depending on the value of the shape parameters. They have obtained the statistical and reliability properties of the GIED. For estimation of the unknown parameters have proposed the maximum likelihood and least square procedures. They also showed that the GIED in many situations can perform better than the IED, gamma, weibull, generalized exponential distribution.

Because of the convenient mathematical form of this model, GIED has been considered in many application fields, such as in queue theory, accelerated life testing, horse racing, modeling wind speeds and supermarkets queues and many more (Nadarajah and Kotz [17]). CDF and PDF of GIED are

$$F_X(x) = 1 - (1 - e^{-\lambda/x})^\alpha, \quad x \geq 0, \alpha > 0, \lambda > 0,$$

and

$$f_X(x) = \frac{\alpha\lambda}{x^2} e^{-\lambda/x} (1 - e^{-\lambda/x})^{\alpha-1}, \quad x \geq 0, \alpha > 0, \lambda > 0,$$

where α is the shape parameter and λ is the scale parameter.

Nadarajah and Kotz [17] have studied some properties of the GIED. Krishna and Kumar [12] discussed the estimation of the two parameters and reliability properties based on progressively type-II censored sample. Singh et al [19] obtained the maximum and Bayesian estimations for parameters of the GIED in case of the progressive type II censoring scheme with binomial removals. Dey and Pradhan [7] studied the estimation problem of GIED under the hybrid censoring scheme. Many authors have analyzed this distribution for example, see Dey and Dey [4], Dey et al [5], Dey et al [6], Panahi [18].

Because of the many applications of this distribution, it is important to analyze the efficient estimation of its PDF and CDF. For the other distribution similar estimation methods in various situations have been done. For example: Jabbari Nooghabi and Jabbari Nooghabi [17], Bagheri et al [3], Alizadeh et al [2] and Maleki et al [15]. Several estimation procedures in this paper are considered: maximum likelihood estimation (MLE), uniformly minimum variance unbiased estimation (UMVUE), least squares estimation (LS), weighted least squares estimation (WLS) and percentile estimation (PCE).

In this paper, it is assumed that the shape parameter is unknown but the scale parameter is not. The rest of this article is organized as follows. The MLE and the UMVUE of the PDF and the CDF and their MSEs are obtained in Section 2 and 3, other estimation methods, such as LS, WLS and PC are given in Section 4 and 5, respectively. Simulation studies and a real life data set is performed in Section 6 and 7. Finally, some discussion and conclusion are in Section 8.

2. MAXIMIM LIKELIHOOD ESTIMATORS OF THE PDF AND THE CDF

Suppose $X_1, X_2, \dots, X_n$ be a random sample of size n from the GIED. The MLE of α , say $\hat{\alpha}$ (when λ is known) is

$$\hat{\alpha} = \frac{-n}{\sum_{i=1}^n \log(1 - e^{-\lambda/x_i})}.$$

By using the invariance property of MLEs, we have:

$$\hat{f}_X(x) = \frac{\hat{\alpha}\lambda}{x^2} e^{-\lambda/x} (1 - e^{-\lambda/x})^{\hat{\alpha}-1}, \quad x \geq 0, \lambda > 0,$$

and

$$\hat{F}_X(x) = 1 - (1 - e^{-\lambda/x})^{\hat{\alpha}}, \quad x \geq 0, \lambda > 0,$$

respectively. We consider $T = -\sum_{i=1}^n \log(1 - e^{-\lambda/x_i})$ and it can easily be shown that $T \sim \text{Gamma}(\alpha, \lambda)$

$$T = \frac{-1}{\alpha} \sum_{i=1}^n \log(1 - e^{-\lambda/x_i}) = \frac{-1}{\alpha} \sum_{i=1}^n \log(1 - F(x_i)) = \sum_{i=1}^n z_i,$$

where denote equality in distribution, $1 - F(x_i)$ are independent uniform $[0,1]$ random variables and Z_i are independent exponential random variables with scale parameter α .

Therefore,

$$f_T(t) = \frac{\alpha^n}{\Gamma(n)} t^{n-1} e^{-\alpha t}, \quad t > 0, \alpha > 0.$$

By using the transformance $\hat{\alpha} = U = \frac{n}{T}$ and the PDF of $\hat{\alpha}$, we obtain

$$f_U(u) = \frac{n^n \cdot \alpha^n}{\Gamma(n)} \frac{e^{-n\alpha/u}}{u^{n+1}}, \quad u > 0, \alpha > 0.$$

The next theorem calculate $E(\hat{f}(x))^r$ and $E(\hat{F}(x))^r$.

Theorem 2.1. : *The r 'th moments of $\hat{f}(x)$ and $\hat{F}(x)$ are*

$$E(\hat{f}(x))^r = \frac{2c^r}{\Gamma(n)} (n\alpha)^{\frac{r+n}{2}} \left(\frac{-1}{r \log(1-k)} \right)^{\frac{r-n}{2}} K_{r-n}(2\sqrt{-rn \log(1-k)}),$$

and

$$E(\hat{F}(x))^r = 1 + 2 \frac{(n\alpha)^{\frac{n}{2}}}{\Gamma(n)} \sum_{i=1}^n (-1)^i \binom{r}{i} \left(\frac{-1}{i \log(1-k)} \right)^{\frac{-n}{2}} K_{-n}(2\sqrt{-in\alpha \log(1-k)}),$$

respectively, where $c = \frac{\lambda}{x^2} \frac{e^{-\lambda/x}}{1 - e^{-\lambda/x}}$, $k = e^{-\lambda/x}$, $\hat{\alpha} = u$ and $k_v(\cdot)$ is the modified Bessel function of the second kind of order v .

Proof. We have $\hat{f}(x) = c(1-k)^{u-1}$, $\hat{F}(x) = 1 - (1-k)^u$ for $w > 0$. therefore, we can write:

$$\begin{aligned} E(\hat{f}(x))^r &= \int_0^{+\infty} (c(1-k)^u)^r \frac{n^n \alpha^n}{\Gamma(n)} \frac{e^{-\frac{n\alpha}{u}}}{u^{n+1}} du \\ &= \frac{c^r n^n \alpha^n}{\Gamma(n)} \int_0^{+\infty} u^{r-n-1} e^{r u \log(1-k)} e^{-\frac{n\alpha}{u}} du \\ &= \frac{2c^r}{\Gamma(n)} (n\alpha)^{\frac{r+n}{2}} \left(\frac{-1}{r \log(1-k)} \right)^{\frac{r-n}{2}} K_{r-n}(2\sqrt{-rn \log(1-k)}). \end{aligned}$$

So that the last relationship follow via Eq.(3.479.1) in Gradshteyn and Ryzhik [8]. Also,

$$\begin{aligned} E(\hat{F}(x))^r &= E(\hat{F}(x) - 1 + 1)^r = \int_0^{+\infty} (-(1-k)^u + 1)^r \frac{(n\alpha)^{\frac{n}{2}}}{\Gamma(n)} \frac{e^{-\frac{n\alpha}{u}}}{u^{n+1}} du \\ &= \int_0^{+\infty} \sum_{i=0}^n (-1)^i \binom{r}{i} (1-k)^{ui} \frac{(n\alpha)^{\frac{n}{2}}}{\Gamma(n)} \frac{e^{-\frac{n\alpha}{u}}}{u^{n+1}} du \\ &= 1 + \int_0^{+\infty} \sum_{i=1}^n (-1)^i \binom{r}{i} (1-k)^{ui} \frac{(n\alpha)^{\frac{n}{2}}}{\Gamma(n)} e^{-\frac{n\alpha}{u}} u^{-n-1} du \\ &= 1 + \frac{(n\alpha)^{\frac{n}{2}}}{\Gamma(n)} \sum_{i=1}^n (-1)^i \binom{r}{i} \int_0^{+\infty} e^{ui \log(1-k)} e^{-\frac{n\alpha}{u}} u^{-n-1} du \\ &= 1 + 2 \frac{(n\alpha)^{\frac{n}{2}}}{\Gamma(n)} \sum_{i=1}^n (-1)^i \binom{r}{i} \left(\frac{-1}{i \log(1-k)} \right)^{\frac{-n}{2}} K_{-n}(2\sqrt{-in\alpha \log(1-k)}). \end{aligned}$$

□

Note that, $\hat{f}(x)$ and $\hat{F}(x)$ are biased estimators for $f(x)$ and $F(x)$, respectively. The MSEs of $\hat{f}(x)$ and $\hat{F}(x)$ are obtained based on Theorem 2.1 in the next Theorem.

Theorem 2.2. *The MSEs of $\hat{f}(x)$ and $\hat{F}(x)$ are*

$$MSE(\hat{f}(x)) = \frac{2(n\alpha)^{\frac{2+n}{n}}}{\Gamma(n)} \left(\frac{\lambda}{x^2} \frac{e^{-\frac{\lambda}{x}}}{1 - e^{-\frac{\lambda}{x}}} \right)^2 \left(\frac{-1}{2\log(1 - e^{-\frac{\lambda}{x}})} \right)^{\frac{2-n}{2}} K_{2-n}(2\sqrt{-2n\alpha\log(1 - e^{-\frac{\lambda}{x}})})$$

$$- 4f(x) \frac{(n\alpha)^{\frac{1+n}{n}}}{\Gamma(n)} \left(\frac{\lambda}{x^2} \frac{e^{-\frac{\lambda}{x}}}{1 - e^{-\frac{\lambda}{x}}} \right) \left(\frac{-1}{\log(1 - e^{-\frac{\lambda}{x}})} \right)^{\frac{1-n}{2}} K_{1-n}(2\sqrt{-n\alpha\log(1 - e^{-\frac{\lambda}{x}})}) + f^2(x),$$

and

$$MSE(\hat{F}(x)) = 1 + 2 \frac{(n\alpha)^{\frac{n}{2}}}{\Gamma(n)} \sum_{i=1}^n (-1)^i \binom{2}{i} \left(\frac{-1}{i \log(1 - e^{-\frac{\lambda}{x}})} \right)^{\frac{-n}{2}} K_{-n}(2\sqrt{-ni\alpha\log(1 - e^{-\frac{\lambda}{x}})})$$

$$- 2F(x) \left[1 - 2 \frac{(n\alpha)^{\frac{n}{2}}}{\Gamma(n)} \left(\frac{-1}{\log(1 - e^{-\frac{\lambda}{x}})} \right)^{\frac{-n}{2}} K_{-n}(2\sqrt{-n\alpha\log(1 - e^{-\frac{\lambda}{x}})}) \right] + F^2(x),$$

respectively.

Proof. We obtained $E(\hat{f}(X))^2$ and $E(\hat{f}(x))$ by putting $r = 2$ and $r = 1$ in the previous Theorem (2.1), we will have

$$MSE(\hat{f}(x)) = E(\hat{f}(X))^2 - 2f(x)E(\hat{f}(x)) + f^2(x).$$

The proof for $MSE(\hat{F}(x))$ is similar. □

3. UMVUES OF THE PDF AND THE CDF

We derive the UMVU estimators of the PDF and the CDF of the GIED and their MSEs. Suppose $X_1, X_2, \dots, X_n$ be a random variable from the GIED. So on, $T = \sum_{i=1}^n \log(1 - e^{-\lambda/x_i})$ is a complete sufficient statistics for that is an unknown parameter (λ is known). By considering the distribution of T , in the previous section and using the Lehmann Scheffe Theorem, the conditional probability function of given T ($f_{X_1|T}(x_1|T = t) = f^*(t)$) is the UMVUE of $f(x)$ because

$$E(f^*(t)) = \int f^*(t) f_T(t) dt = \int f_{X_1|T}(x_1|T = t) f_T(t) dt = \int f_{X_1,T}(x_1, t) dt = f_{X_1}(x_1),$$

where $f_{X_1,T}(x_1, t)$ denote the joint PDF of X_1 and T .

Theorem 3.1. *The joint PDF of X_1 and T is given by*

$$f_{X_1, T}(x_1, t) = \frac{\alpha\lambda}{\Gamma(n-1)x^2} \frac{e^{\lambda/x}}{1 - e^{\lambda/x}} (t + \log(1 - e^{\lambda/x}))^{n-2} e^{\alpha(t + \log(1 - e^{\lambda/x}))}.$$

Proof. We can write:

$$\begin{aligned} f_{X_1, T}(x_1, t) &= P(x_1 \leq X_1 \leq x_1 + dx_1, t \leq T \leq t + dt) \\ &= P(x_1 \leq X_1 \leq x_1 + dx_1, t \leq -\sum_{i=1}^n \log(1 - e^{-\lambda/x}) \leq t + dt) = \\ &= P(x_1 \leq X_1 \leq x_1 + dx_1, t + \log(1 - e^{-\lambda/x}) \leq -\sum_{i=2}^n \log(1 - e^{-\lambda/x_i}) \leq \\ &\leq t + \log(1 - e^{-\lambda/x}) + dt) \\ &= P(x_1 \leq X_1 \leq x_1 + dx_1) P(t + \log(1 - e^{-\lambda/x}) \leq -\sum_{i=2}^n \log(1 - e^{-\lambda/x_i}) \leq \\ &\leq t + \log(1 - e^{-\lambda/x}) + dt) L \\ &= \frac{\alpha\lambda}{x^2} e^{-\lambda/x} (1 - e^{-\lambda/x})^{\alpha-1} \cdot \frac{\alpha^{n-1}}{\Gamma(n-1)} (t + \log(1 - e^{\lambda/x}))^{n-2} e^{\alpha(t + \log(1 - e^{\lambda/x}))}. \end{aligned}$$

The proof is complete. □

Theorem 3.2. *By considering $T = t$ then*

$$\tilde{f}(x) = \frac{(n-1)\lambda}{x^2} \frac{e^{\lambda/x}}{1 - e^{\lambda/x}} \frac{(t + \log(1 - e^{\lambda/x}))^{n-2}}{t^{n-1}}, \quad -\log(1 - e^{\lambda/x}) < t < +\infty,$$

and

$$\tilde{F}(x) = 1 - \left(\frac{t + \log(1 - e^{\lambda/x})}{t} \right)^{n-1}, \quad -\log(1 - e^{\lambda/x}) < t < +\infty,$$

are the UMVU estimators for $f_X(x)$ and $F_X(x)$, respectively.

Proof. By using distribution of $T = t$ and based on Theorem 2.1, $\tilde{f}(x)$ is the UMVUE of $f(x)$. By the derivative of $\tilde{F}(x)$ with respect to x , $\tilde{f}(x)$ is calculated

and show that $\tilde{F}(x)$ is the UMVU estimator of $F(x)$.

$$\begin{aligned}
 \frac{d}{dx}\tilde{F}(x) &= -\frac{d}{dx}\left[1 + \frac{\log(1 - e^{\lambda/x})}{t}\right]^{n-1} \\
 &= -(n-1)\left[1 + \frac{\log(1 - e^{\lambda/x})}{t}\right]^{n-2} \frac{d}{dx} \frac{\log(1 - e^{\lambda/x})}{t} \\
 &= (n-1)\left[1 + \frac{\log(1 - e^{\lambda/x})}{t}\right]^{n-2} \cdot \frac{1}{t} \cdot \frac{\frac{\lambda}{x^2} e^{\lambda/x}}{1 - e^{\lambda/x}} \\
 &= \frac{(n-1)\lambda}{x^2} \frac{e^{\lambda/x}}{1 - e^{\lambda/x}} \frac{(t + \log(1 - e^{\lambda/x}))^{n-2}}{t^{n-1}} = \tilde{f}(x).
 \end{aligned}$$

□

Note that it can easily be shown that $\hat{f}(x)$ is the unbiased estimator of $f(x)$.

$$\begin{aligned}
 E(\tilde{f}(X)) &= \int_{-\log(1-e^{\lambda/x})}^{+\infty} \frac{(n-1)\lambda}{x^2} \frac{e^{\lambda/x}}{1 - e^{\lambda/x}} \frac{(t + \log(1 - e^{\lambda/x}))^{n-2}}{t^{n-1}} \frac{\alpha^n}{\Gamma(n)} t^{n-1} e^{-\alpha t} dt \\
 &= \frac{(n-1)\lambda}{x^2} \frac{e^{\lambda/x}}{1 - e^{\lambda/x}} \frac{\alpha^n}{\Gamma(n)} e^{\alpha(\log(1-e^{\lambda/x}))} \int_{-\log(1-e^{\lambda/x})}^{+\infty} (t + \log(1 - e^{\lambda/x}))^{n-2} e^{-\alpha(t + (\log(1-e^{\lambda/x})))} dt \\
 &= \frac{\alpha\lambda}{x^2} e^{-\lambda/x} (1 - e^{-\lambda/x})^{\alpha-1} = f(x).
 \end{aligned}$$

Theorem 3.3. *The mean square errors of $f(x)$ and $F(x)$ are computed by:*

$$MSE(\tilde{f}(x)) = \frac{c^2 \alpha^n}{\Gamma(n)} \sum_{i=1}^{2(n-2)} (\log(1 - e^{-\lambda/x}))^i \binom{2(n-2)}{i} \Gamma(n-2-i, -\log(1-k)\alpha) - f^2(x).$$

and

$$MSE(\tilde{F}(x)) = \frac{\alpha^n}{\Gamma(n)} \sum_{i=1}^2 (-1)^i \binom{2}{i} \sum_{j=0}^{(n-1)i} \binom{(n-1)i}{j} (\log(1 - e^{-\lambda/x}))^j \Gamma(n-j, -\log(1-k)\alpha) - F^2(x)$$

so that, $k = e^{-\lambda/x}$, $c = \frac{(n-1)\lambda}{x^2} \frac{e^{-\lambda/x}}{1 - e^{-\lambda/x}}$ and $\Gamma(u, x) = \int_x^{+\infty} t^{u-1} e^{-t} dt$ is the incomplete gamma.

Proof. We can write

$$\begin{aligned}
E(\tilde{f}(x))^r &= \int_{-\log(1-k)}^{+\infty} c^r \left(\frac{(t + \log(1 - e^{-\lambda/x}))^{n-2}}{t^{n-1}} \right)^r \frac{\alpha^n}{\Gamma(n)} t^{n-1} e^{-\alpha t} dt \\
&= c^r \frac{\alpha^n}{\Gamma(n)} \int_{-\log(1-k)}^{+\infty} \left(1 + \frac{\log(1 - e^{-\lambda/x})}{t} \right)^{(n-2)r} t^{n-r-1} e^{-\alpha t} dt \\
&= c^r \frac{\alpha^n}{\Gamma(n)} \int_{-\log(1-k)}^{+\infty} \sum_{i=0}^{(n-2)r} \binom{(n-2)r}{i} \left(\frac{\log(1 - e^{-\lambda/x})}{t} \right)^i t^{n-r-1} e^{-\alpha t} dt \\
&= c^r \frac{\alpha^n}{\Gamma(n)} \sum_{i=0}^{(n-2)r} \log(1 - e^{-\lambda/x})^i \binom{(n-2)r}{i} \int_{-\log(1-k)}^{+\infty} t^{n-r-i-1} e^{-\alpha t} dt \\
&= c^r \frac{\alpha^n}{\Gamma(n)} \sum_{i=0}^{(n-2)r} \binom{(n-2)r}{i} \Gamma(n - r - i, -\log(1 - k)\alpha).
\end{aligned}$$

also

$$\begin{aligned}
E(\tilde{F}(x))^r &= \int_{-\log(1-k)}^{+\infty} \left(1 - \left(\frac{t + \log(1 - e^{-\lambda/x})}{t} \right)^{n-1} \right)^r \frac{\alpha^n}{\Gamma(n)} t^{n-1} e^{-\alpha t} dt \\
&= \frac{\alpha^n}{\Gamma(n)} \int_{-\log(1-k)}^{+\infty} \sum_{i=0}^2 (-1)^i \binom{2}{i} \sum_{j=0}^{(n-1)i} \binom{(n-2)r}{j} (\log(1 - e^{-\lambda/x}))^j t^{n-j-1} e^{-\alpha t} dt \\
&= \frac{\alpha^n}{\Gamma(n)} \sum_{i=0}^2 (-1)^i \binom{2}{i} \sum_{j=0}^{(n-1)i} \binom{(n-2)r}{j} (\log(1 - e^{-\lambda/x}))^j \int_{-\log(1-k)}^{+\infty} t^{n-j-1} e^{-\alpha t} dt \\
&= \frac{\alpha^n}{\Gamma(n)} \sum_{i=0}^2 (-1)^i \binom{2}{i} \sum_{j=0}^{(n-1)i} \binom{(n-2)i}{j} (\log(1 - e^{-\lambda/x}))^j \Gamma(n - j, -\log(1 - k)\alpha).
\end{aligned}$$

By setting $r = 2$ in previous equations and based on following two equations, the proof is complete.

$$\begin{aligned}
MSE(\tilde{f}(x)) &= E(\tilde{f}(x))^2 - f^2(x), \\
MSE(\tilde{F}(x)) &= E(\tilde{F}(x))^2 - F^2(x).
\end{aligned}$$

□

Note that the UMVU estimator of α is

$$\tilde{\alpha} = \frac{-(n-1)}{\sum_{i=1}^n \log(1 - e^{-\lambda/x_i})}.$$

4. LEAST SQUARES AND WEIGHED LEAST SQUARES ESTIMATORS

In this part, we use the method of Swain to achieve estimators of the probability density function and the cumulative distribution function of the GIED. Swain obtained a procedure for estimating Beta distribution parameters with the name least estimators. Let $X_1, X_2, \dots, X_n$ are i.i.d with GIED and $X_{(i)}$, $i = 1, 2, \dots, n$ be the order statistic. To use these methods, we need the expectation and the variance of, see Johnson et al [9]. So, we have:

$$\begin{aligned} E(F(X_{(i)})) &= \frac{i}{n+1}, \\ \text{Var}(F(X_{(i)})) &= \frac{i(n-i+1)}{(n+1)^2(n+2)}. \end{aligned}$$

Procedure 1:

We minimize the following relationship with respect to $\hat{\lambda}$, thus Least squares estimators obtain as follows:

$$\sum_{i=1}^n \left(F(X_{(i)}) - \frac{i}{n+1} \right)^2.$$

The LS estimation of α (λ is known) for the GIED, say $\tilde{\alpha}_{LS}$, can be achieved by minimizing

$$\sum_{i=1}^n \left(1 - (1 - e^{-\lambda/x_{(i)}})^{\alpha} - \frac{i}{n+1} \right)^2.$$

Estimation of the PDF and the CDF are according to the method described as follow:

$$\begin{aligned} \tilde{f}_{LS}(x) &= \frac{\tilde{\alpha}_{LS}\lambda}{x^2} e^{-\lambda/x} (1 - e^{-\lambda/x})^{\tilde{\alpha}_{LS}-1}, \\ \tilde{F}_{LS}(x) &= 1 - (1 - e^{-\lambda/x})^{\tilde{\alpha}_{LS}}. \end{aligned}$$

Procedure 2:

We minimize the following equation with respect to the unknown parameters α , thus weight least squares (WLS) estimation is given by

$$\sum_{i=1}^n w_j \left(F(X_{(i)}) - \frac{i}{n+1} \right)^2,$$

where

$$w_j = \frac{1}{\text{Var}(F(X_{(i)}))} = \frac{(n+1)^2(n+2)}{i(n-i+1)}.$$

Then, the WLS estimators of α (λ is known) for the GIED, say $\tilde{\alpha}_{WLS}$ is the value minimizing

$$\sum_{i=1}^n w_j \left(1 - (1 - e^{-\lambda/x_{(i)}})^{\tilde{\alpha}} - \frac{i}{n+1} \right)^2,$$

with respect to α . So, based on WLS method estimation of the PDF and the CDF are as following equations:

$$\begin{aligned} \tilde{f}_{WLS}(x) &= \frac{\tilde{\alpha}_{WLS} \lambda}{x^2} e^{-\lambda/x} (1 - e^{-\lambda/x})^{\tilde{\alpha}_{WLS}-1}, \\ \tilde{F}_{WLS}(x) &= 1 - (1 - e^{-\lambda/x})^{\tilde{\alpha}_{WLS}}. \end{aligned}$$

It is difficult to calculate the expectation and the MSE of LS and WLS estimators by mathematical methods, therefore we obtain them using simulation methods.

5. PERCENTILE ESTIMATOR

Kao [10, 11] introduced a new estimation method based on the percentile. To study most, see and Johnson et al [9] and Mann et al [16]. Percentile estimators are based on inverting the CDF. Because of the CDF has a closed form, in this section, we can be estimated parameters of GIED with this method. Suppose $X_1, X_2, \dots, X_n$ is a random sample with common $F(\cdot)$ and $X_{(i)}$, $i = 1, \dots, n$ be the order statistic. We will consider $p_i = \frac{i}{n+1}$. The percentile estimators of parameters are calculated by minimizing:

$$\sum_{i=1}^n \left(1 - (1 - p_i)^{(\frac{1}{\alpha})} - e^{-\lambda/x_{(i)}} \right)^2,$$

with respect to α for the GIED. The estimation of the PDF and the CDF are obtained with percentile procedure as follow:

$$\begin{aligned} \tilde{f}_X(x) &= \frac{\tilde{\alpha}_{PC} \lambda}{x^2} e^{-\lambda/x} (1 - e^{-\lambda/x})^{\tilde{\alpha}_{PC}-1}, \\ \tilde{F}_X(x) &= 1 - (1 - e^{-\lambda/x})^{\tilde{\alpha}_{PC}}. \end{aligned}$$

It is difficult to calculate the expectation and the MSE of LS and WLS estimators by mathematical methods, therefore we obtain them using simulation methods

6. SIMULATION STUDY

We study the performance of the described procedures including MLE, UMVUE, LSE, WLSE, PCE of the PDF and the CDF. The performance of these estimators is evaluated based on the Mean Square Errors (MSEs). Note that all computations are done using the R statistical program. MSEs are obtained by producing a thousand repetitions of the sample of size $n = 15, \dots, 120$ from the GIED with different of the parameters $(\alpha, \lambda) = (0.7, 0.7), (1.5, 2.7)$ and $(1.8, 1.2)$.

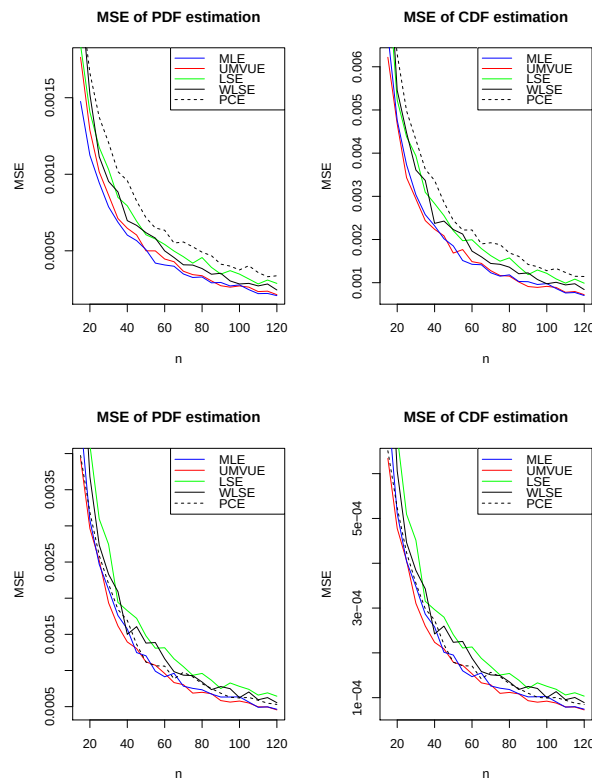


FIGURE 1. MSEs of MLE, UMVUE, LSE, WLSE and PCE for the PDF(right) and the CDF(left) for parameters of $(\alpha, \lambda) = (0.7, 0.7), (1.5, 2.7), (1.8, 1.2)$

In Figures 1 and 2, the MSEs is plotted against the sample size (n). To produce samples from the GIED is used the inversion method. We consider the estimators of the PDF and the CDF for $x = 1$ (see, Bagheri et al [3]). Figures illustrate that the UMVUE and MLE are equivalent and better than the other

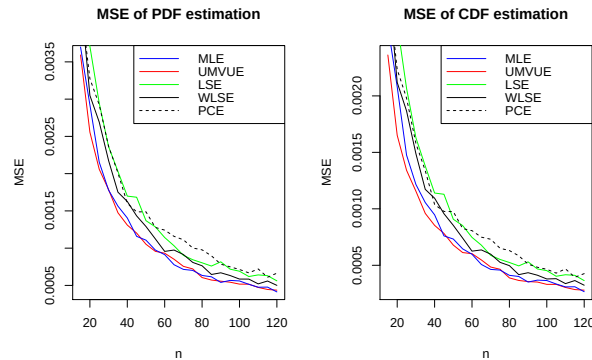


FIGURE 2. MSEs of MLE, UMVUE, LSE, WLSE and PCE for the PDF(right) and the CDF(left) for parameters of $(\alpha, \lambda) = (0.7, 0.7), (1.5, 2.7), (1.8, 1.2)$

estimators for the PDF and the CDF of GIED and have the smallest MSEs. It is worth to mention, that MSEs for each estimator are decreasing function of n .

7. DATA ANALYSIS

We examine a real dataset in the industry to compare the performance of MLE, UMVUE, LSE, WLSE and PCE for the PDF and the CDF of the GIE distribution. The data resulted from a test on the endurance of deep groove ball-bearing that presented in Lawless [13]. Data were analyzed using Lieblein and Zelen [14].

These data are as follows:

17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.40, 51.84, 51.96, 54.12, 55.56, 67.80,

68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40.

In this study, the GIED was fitted to these data by considering that is unknown and is known. We can place the parameters equal with because of is a scale parameter, so that s is the standard deviation of data. (See, Bagheri et al [4]). For five estimation methods, Figures 2 and 3 indicate the PDF vs. experimental PDF and the CDF vs. experimental CDF.

Based on these Figures, indicate that the UMVU and ML estimators are better fit than other procedures.

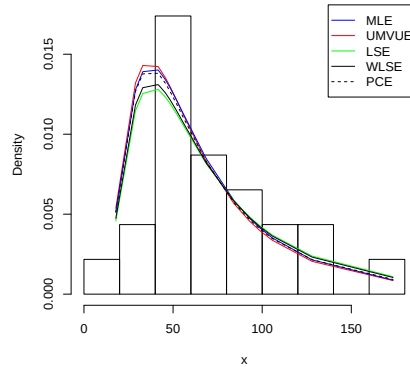


FIGURE 3. Fitted PDFs on the histogram of ball bearing dataset for five estimation methods.

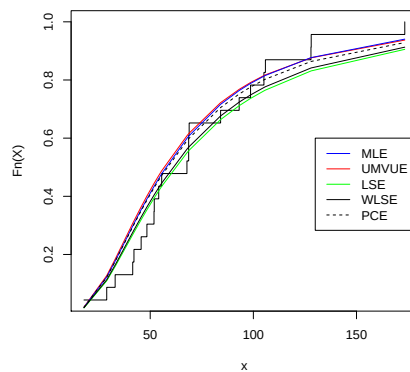


FIGURE 4. Fitted CDFs on the empirical of ball bearing dataset for five estimation methods.

We also considered means of model selection criteria for comparing estimation procedures.

These criteria are: maximum likelihood (ML), Akaike information criteria (AIC), corrected AIC(AICc), Hannan-Quinn criterion(HQC) and Bayes information criterion(BIC), and have by

$$ML = -2 \log L(\alpha)$$

$$AIC = -2 \log L(\alpha) + 2k$$

$$AIC_c = -2 \log L(\alpha) + 2k \left(\frac{n}{n - k - 1} \right)$$

$$BIC = -2 \log L(\alpha) + k \log n$$

$$HQC = -2 \log L(\alpha) + 2k \log(\log n)$$

So that, $\log \log L(\alpha)$ denotes the logarithm likelihood and n and k are the number of observations and parameters of the distribution, respectively. Table 1 gives the values of these models selection criteria for the fore different estimation methods. The smallest values of these criteria denote that it is fitted better compared to other methods. Based on the mentioned model selection criteria, the UMVU and ML estimators have better performance than others. Also the PC, WLS and LS estimators have the smallest values of the MSEs, respectively. (See the results of simulation study).

TABLE 1. Model selection criteria for the ball bearing data.

Methods of estimation	ML	AIC	AICc	BIC	HQC
MLE	99.7010	101.7010	101.8914	102.8365	101.9866
UMVUE	99.4779	101.4779	101.6683	102.6134	101.7635
LSE	99.9456	101.9456	102.1360	103.0811	102.2312
WLSE	99.8539	101.8539	102.0443	102.9894	102.1395
PCE	99.7226	101.7336	101.9131	102.8581	102.0082

TABLE 2. Estimates of α parameter when λ is known.

	Estimate of α
MLE	3.0894
UMVUE	2.9550
LS	2.6308
WLS	2.7227
PC	2.9477

Corollary 7.1. *In this paper, we consider five different procedures of estimation for the PDF and the CDF of the generalized inverted exponential distribution, including Maximum likelihood estimator (ML), uniformly minimum variance unbiased estimator (UMVU), least-square estimator (LS), weighted least square estimator (WLS) and percentile estimator (PC). Proposed procedures are compared based on their MSEs with together. MSEs help us to evaluate proposed procedures in comparison with each other. Simulations and real data set are used to compare*

the performance of the described estimation procedures. The results denote that in the set of unbiased estimators the UMVU estimators are better and by increasing the sample size in biased class the ML and the UMVU are equivalent and they outperform others.

REFERENCES

- [1] A. M. ABOUAMMOH, A. M. ALSHINIGITI: *Reliability estimation of generalized inverted exponential distribution*, Journal of Statistical Computation and Simulation, **79**(11) (2009), 1301-1315.
- [2] M. ALIZADEH, S. E. BAGHERI, E. BALOUI JAMAKHANEH, S. NADARAJAH: *Estimates of the PDF and the CDF of the exponentiated Weibull distribution*, Brazilian Journal of Probability and Statistics, **29** (2015), 695-716.
- [3] S. F. BAGHERI, M. ALIZADEH, E. BALOUI JAMKHAHEH, E, S. NADARAJAH : *Evaluation and comparision of estimations in the generalized exponential-Poisson distribution*, Journal of Statistical Comutation and Simulation, **45** (2014), 2345-2360.
- [4] P. CHEN, Z. YE, X. ZHAO : *Minimum Distance Estimation for the Generalized Pareto Distribution*, Technometrics, **59** (2017), 528-541.
- [5] S. DEY, T. DEY: *Generalized inverted exponential distribution: Different methods of estimation*, American Journal of Mathematical and Management Sciences, **33** (2014), 194-215.
- [6] S. DEY, S. SINGH, M. Y. TRIPATHI, A. ASGHARZADEH: *Estimation and prediction for a progressively censored generalized inverted exponential distribution*, Statistical Methodology. **32** (2014), 185-202.
- [7] S. DEY, T. DEY, D. J. LUCKETT: *Statistical inference for the generalized inverted exponential distribution based on upper record value* , Math. Comput. Simul., **120** (2016), 64-78.
- [8] S. DEY, B. PRADHAN: *Generalized inverted exponential distribution under hybrid censoring*, Stat. Methodol., **18** (2014), 101-114.
- [9] J. SWAIN, S. VENKATRAMAN, J. WILSON: *Least squares estimation of distribution function in Johnson translation system*, Journal of Statistical Computation and Simulation., **29** (1988), 271-297.
- [10] N. L. JOHNSON, S. KOTZ, N. BALAKRISHNAN: *Continuous Univariate Distributions*, New York:Wiley. MR1299979, 1994.
- [11] K. H. J. KAO: *Computer methods for estimating Weibull parameters in reliability studies*, Trannsaction of IRE-Reliability and Quality Control. **13** (1958), 15-22.
- [12] K.H. KAO: *A graphical estimation of mixed Weibull parameters in life testing electron tubes*, Trannsaction of IRE-Reliability and Quality Control., **13** (1958), 389-407.

- [13] H. KRISHNA, K. KUMAR: *Reliability estimation in generalized inverted exponential distribution with progressively type II censored sample*, Journal of Statistical Computation and Simulation. **83**(6) (2013), 1007-1019.
- [14] N. R. MANN, R. E. SCHAFER, N. D. SINGPURWALLA: *Methods for statistical Analysis of Reliability and Life Data*, New York: Wiley. MR0365976, 1974.
- [15] S. K. SINGH, U. SING, M. KUMAR: *Estimation of Parametres of Generalized Inverted Exponential Distribution for progressive Type II censored sample with Binomial Removals*, Journal of Probability and Statistics, (2013), 1-12.
- [16] U. SINGH, S. SINGH, R. K. SING : *Comparative Study of Traditinal Estimation Methods and Maxumum Product Spacing Method in Generalized Inverted Exponential Distribution*, Journal of Statistical Applications and Probability, **2**(2) (2014), 153-169.
- [17] N. M. JABBARI NOOGHABI, N. H. JABBARI NOOGHABI : *Efficient Estimation of PDF and CDF and r th moment for the Exponentiated Pareto Distribution in the presence of outliers*, Statistics A Journal of Theoretical and Applied statistics, **44** (2010), 1-20.
- [18] J. F. LAWLESS: *Statistical models and methods for lifetime data*, John Wiley and Sons. New York, 1982.
- [19] J. LIEBLEIN, M. ZELEN: *Statistical investigation of the fatigue life of deep groove ball bearing*, StRes. Natl. Bur Stand., **57** (1956), 273-316.

DEPARTMENT OF STATISTICS

FACULTY OF MATHEMATICAL SCIENCES AND STATISTICS

MALAYER UNIVERSITY, MALAYER, IRAN

E-mail address: p.torkaman@malayeru.ac.ir