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NUMERICAL BLOW-UP ON WHOLE DOMAIN FOR A QUASILINEAR PARABOLIC EQUATION WITH NONLINEAR BOUNDARY CONDITION

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ABSTRACT. This paper deals with numerical approximation of the following quasilinear parabolic equation $u_t = u^{1+\gamma}u_{xx},\ 0 < x < 1,\ t > 0$, with a nonlinear boundary condition $u_x(0,t) = -u^q(0,t),\ u_x(1,t) = 0,\ t > 0$. We show that the solution of the semidiscrete scheme, obtained by the finite differences method blows up in a finite time when $0 < q < 1$. Convergence of the numerical blow-up time to the theoretical one when the mesh size goes to zero is also established. Finally, we give some numerical results to illustrate certain point of our work.

1. INTRODUCTION

Consider the following parabolic quasilinear problem:

$$\begin{cases} u_t = u^{1+\gamma}u_{xx}, \quad (x,t) \in (0,1) \times (0,T), \\ u_x(0,t) = -u^q(0,t),\ u_x(1,t) = 0, \quad t \in (0,T), \\ u(x,0) = u_0(x), \quad x \in [0,1], \end{cases} \tag{1.1}$$

where $\gamma > 0$ and $q > 0$ are given constants, and u_0 is a positive bounded smooth function defined on $[0,1]$ such that $u_0'(0) = -u_0^q(0)$ and $u_0'(1) = 0$. The problem (1.1) arises in fluid dynamics, which is essentially the study of gases and liquids in motion, see [11] for more details. From the standard

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theory of parabolic equation, local existence and uniqueness of positive solution of the above problem follow. A solution of a evolution equation is said to blow-up in finite time if this solution become unbounded in that finite time. We call blow-up point, the point of the space where solution become unbounded. The set of all blow-up points is called the blow-up set.

We know from Jong-Shenq Guo [8] and from Kavitha S., Bhakya K. [11] that, if $q > 0$ and for every positive bounded smooth initial data $u_0,$ the solution u of (1.1) blows up in finite time T . Moreover, if $q > 1,\ u_0' \leq 0$ and $u_0'' \geq 0$ in $[0, 1],\ x = 0$ is the only blow-up point. But if $0 < q < 1,\ u_0' \leq 0$ and $u_0'' \geq 0$ in $[0, 1]$, blow-up occurs on the whole space $[0, 1]$, see [8].

The blow-up phenomenon has been the focus of many authors in recent years. Some were interested in the theoretical analysis [5, 8, 11], and others in the numerical one [1, 2, 4, 10, 6, 7, 12].

This work is concerned with the numerical approximations of (1.1) for the case $0 < q < 1$. The case $q > 1$ has been studied in [6] by Ganon, Taha, Touré. Our aim is to prove the blow-up of the numerical solution and the convergence of the numerical blow-up time without put strong assumption on initial data (we only use assumptions that guarantee the blow-up of solution of the continuous problem), which is not the case of some numerical methods (see Theorems 5-8 in [1] and relation (35) and Remark 3.1 in [3]).

This paper is organized as follows : in the next section, we present a semidiscrete scheme of the problem (1.1). In Section 3, we give some properties of this semidiscrete scheme. In Section 4, under suitable conditions, we prove that the solution of the semidiscrete scheme of (1.1) blows up in finite time and the numerical blow-up time converges to the theoretical one when the mesh size goes to zero. Finally, in the last section, we illustrate our analysis by giving some numerical results.

2. Semidiscrete problem

Let I be a positive integer and define the grid $x_i = ih,\ i = 0, \ldots, I,$ where $h = \dfrac{1}{I}$ is the mesh parameter. We approximate the solution u of the problem (1.1) by the solution $U_h(t) = (U_0(t), ..., U_I(t))^T$ of the following semidiscrete scheme

$$\frac{dU_i(t)}{dt} = U_i^{1+\gamma}(t)\delta^2 U_i(t), \quad i = 1, \ldots, I-1, \quad t \in (0, T_h), \tag{2.1}$$

$$\frac{dU_0(t)}{dt} = U_0^{1+\gamma}(t)\left(\delta^2 U_0(t) + \frac{2}{h}U_0^q(t)\right), \quad t \in (0, T_h), \tag{2.2}$$

$$\frac{dU_I(t)}{dt} = U_I^{1+\gamma}(t)\delta^2 U_I(t), \quad t \in (0, T_h), \tag{2.3}$$

$$U_i(0) = \varphi_i > 0, \quad i = 0, \ldots, I, \tag{2.4}$$

where for $t \in (0, T_h)$,

$$\delta^2 U_i(t) = \frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{h^2}, \quad i = 1, \ldots, I-1,$$

$$\delta^2 U_0(t) = \frac{2U_1(t) - 2U_0(t)}{h^2},$$

$$\delta^2 U_I(t) = \frac{2U_{I-1}(t) - 2U_I(t)}{h^2},$$

and $[0, T_h)$, the maximal time interval on which $\|U_h(t)\|_\infty$ is finite, with $\|U_h(t)\|_\infty = \max_{0 \le i \le I} |U_i(t)|$. When T_h is finite, we say that the solution $U_h(t)$ blows up in finite time and the time T_h is called the blow-up time of the solution $U_h(t)$.

Denote

$$\delta_*^2 U_i(t) = \begin{cases} \delta^2 U_i(t) & \text{if } i = 1, \ldots, I, \\ \delta^2 U_0(t) + \frac{2}{h}U_0^q(t) & \text{if } i = 0. \end{cases}$$

3. Properties of the semidiscrete scheme

In this section, we give some important results on the semidiscrete scheme that have been proved in [6], namely :

Let U_h be a solution of (2.1)-(2.4),

1) then $U_i(t) \ge 0, \quad i = 0, \ldots, I,\ t \in (0, T_h)$;
2) if the initial data at (2.4) verifies $\delta_*^2 \varphi_i \ge 0,\ i = 0, \ldots, I$, then $\frac{dU_i(t)}{dt} \ge 0$ and $U_i(t) > 0$ for $0 \le i \le I,\ t \in [0, T_h)$;
3) if the initial data at (2.4) verifies $\delta_*^2 \varphi_i \ge 0,\ i = 0, \ldots, I$ and $\varphi_i > \varphi_{i+1}$, then $U_i(t) > U_{i+1}(t), \quad 0 \le i \le I-1,\ t \in [0, T_h)$.

The following theorem, proved in [6] shows that under appropriate conditions, problem (2.1)-(2.4) has a unique solution that converges to the theoretical one when the mesh size goes to zero.

Theorem 3.1. *Assume that the problem* (1.1) *has a solution* $u \in C^{4,1}([0,1] \times [0,T_d])$ *and the initial condition at* (2.4) *verifies* $\|\varphi_h - u_h(0)\|_\infty = o(1)$ *as* $h \to 0$, *where* $u_h(t) = \big(u(x_0,t), \ldots, u(x_I,t)\big)^T$. *Then, for* h *small enough, the semidiscrete problem* (2.1)-(2.4) *has a unique solution* $U_h \in C^1([0,T_d], \mathbb{R}^{I+1})$ *such that*

$$\max_{0 \le t \le T_d} \|U_h(t) - u_h(t)\|_\infty = O\big(\|\varphi_h - u_h(0)\|_\infty + h^2\big) \text{ as } h \to 0.$$

4. Numerical blow-up

In this section, we prove that the solution U_h of the semidiscrete problem (2.1)-(2.4) blows up in finite time and its semidiscrete blow-up time converges to the real one when the mesh size goes to zero.
We set $(H): u_0 > 0,\ u_0' \le 0$ and $u_0'' \ge 0$ in $[0,1]$.

Theorem 4.1. *Let* $0 < q < 1$. *Assume that the problem* (1.1) *has a solution* u *which blows up in finite time* T *such that* $u \in C^{4,1}([0,1] \times [0,T))$ *and the initial condition at* (2.4) *verifies* $\|\varphi_h - u_h(0)\|_\infty = o(1)$ *as* $h \to 0$. *Under the assumption* (H), *the unique solution* U_h *of* (2.1)-(2.4) *blows up in finite time* T_h *for sufficiently small* h, *and we have the following relation :*

$$\lim_{h \to 0} T_h = T.$$

Proof. We prove Theorem 4.1 by using the Theorem 1.1 given in [13] by Ushijima. The proof consists in checking three conditions : conditions A0, A1 and A2 (see [13]).
Step 1 (Condition A0) The solution u of (1.1) blows up in finite time T (see [8, 11]).
Step 2 (Condition A1) From [8], we know that $u > 0$, $u_x < 0$, $u_t > 0$ and $u_{xx} > 0$. Let us define the functional J as follows :

$$J[u](t) = \int_0^1 u^{\frac{1-q}{\varepsilon(q)}}(x,t)dx, \quad t \in [0,T),$$

where $\varepsilon(q) > \dfrac{1-q}{q}$.
It is not hard to see that

$$\lim_{t\to T} J[u](t) = \infty$$

since u blows up in the whole interval $[0,1]$.
Denote $\alpha = u(1,0) = u_0(1) > 0$ and $\beta = \dfrac{1-q}{\varepsilon(q)} > 0$.

$$\begin{aligned} \frac{dJ(t)}{dt} &= \beta \int_0^1 u^{\beta-1}(x,t)u_t(x,t)dx \\ &\geq \beta\alpha^{\beta+\gamma} \int_0^1 u_{xx}(x,t)dx \\ &= \beta\alpha^{\beta+\gamma} u^q(0,t) \\ &\geq \beta\alpha^{\beta+\gamma} \int_0^1 u^q(x,t)dx. \end{aligned}$$

Using Jensen's inequality to the inequality above, we obtain

$$\frac{dJ(t)}{dt} \geq \beta\alpha^{\beta+\gamma}\big(J\big)^{\frac{q}{\beta}}.$$

Note that $\dfrac{q}{\beta} = \dfrac{q\varepsilon(q)}{1-q} > 1$ since $\varepsilon(q) > \dfrac{1-q}{q}$.
Now, we define J_h, the semidiscretization of J by

$$J_h(t) = h\sum_{i=0}^{I} U_i^\beta(t), \quad t \in [0, T_h).$$

By a straightforward computation, we get

$$\frac{dJ_h(t)}{dt} \geq \beta\alpha_h^{\beta+\gamma}\big(J_h\big)^{\frac{q}{\beta}}, \quad t \in [0, T_h),$$

where $\alpha_h = \varphi_I > 0$.
Putting $G(s) = \beta\alpha_h^{\beta+\gamma}(s)^{\frac{q}{\beta}}$, it is clear that

$$\frac{dJ_h(t)}{dt} \geq G(J_h),$$

and there exists $s_0 > 0$ such that

$$\begin{cases} G(s) > 0 \quad \text{for } \ s > s_0, \\ \displaystyle\int_{s_0}^{\infty} \frac{ds}{G(s)} < \infty \quad \text{since} \quad \frac{q}{\beta} > 1. \end{cases}$$

Step 3 (Condition A2) Using Theorem 3.1, we show that for any $\epsilon > 0$,

$$\lim_{h\to 0} \sup_{t\in[0,T-\epsilon]} |J[u](t) - J_h[U_h](t)| = 0.$$

Finally, conditions A0, A1 and A2 are satisfied. According to Theorem 1.1 of [13], we obtain the desired results. □

5. Numerical experiments

In this section, we estimate the numerical blow-up time of (2.1)-(2.4) by using the algorithm proposed by C. Hirota and K. Ozawa [9]. We first transform the semidiscrete scheme (2.1)-(2.4) into a tractable form by the arc length transformation technique like this :

$$\begin{cases} \dfrac{d}{d\ell}\begin{pmatrix} t \\ U_0 \\ \vdots \\ U_I \end{pmatrix} = \dfrac{1}{\sqrt{1+\sum_{i=0}^{I} f_i^2}} \begin{pmatrix} 1 \\ f_0 \\ \vdots \\ f_I \end{pmatrix}, \quad 0 < \ell < \infty, \\ t(0) = 0, \quad U_i(0) = \varphi_i > 0, \quad 0 \le i \le I, \end{cases} \tag{5.1}$$

where

$$\begin{aligned} f_0 &= \frac{2}{h^2} U_0^{1+\gamma}\Big(U_1 - U_0 + hU_0^q\Big), \\ f_i &= \frac{1}{h^2} U_i^{1+\gamma}\Big(U_{i+1} - 2U_i + U_{i-1}\Big), \quad 1 \le i \le I-1, \\ f_I &= \frac{2}{h^2} U_I^{1+\gamma}\Big(U_{I-1} - U_I\Big). \end{aligned}$$

"ℓ" is such that $d\ell^2 = dt^2 + \sum_{i=0}^{I} dU_i^2$ and is called the arc length.
The variables t and U_i are fonctions of ℓ, and C. Hirota and K. Ozawa[9] proved that

$$\lim_{\ell\to\infty} t(\ell) = T_h \quad \text{and} \quad \lim_{\ell\to\infty} \|U_h(\ell)\|_\infty = \infty.$$

Secondly, we introduce $\{v_j\}$ which is a sequence of the arc length and we apply an ODE solver (DOP54) to (5.1) for each value of $\{v_j\}$. We generate then a linearly convergent sequence to the blow-up time, which sequence is finally accelerated by the Aitken Δ^2 method. The three tolerances parameters, AbsTol, RelTol and InitialStep of the DOP54 (see [9, 7] for more details) are set as follows AbsTol = RelTol = 1.d–15, InitialStep = 0, the sequence of the arc length $v_j = 2^{10} \cdot 2^j$ $(j = 0, \ldots, 10)$ and the initial condition

$$\varphi_i = 0.5 * (i * h)^2 - i * h + 1, \quad 0 \le i \le I.$$

In the following Tables, T_h is the approximate blow-up time corresponding to meshes of $I = 16, 32, 64, 128$; n is the numbers of iterations and the order s of the method is computed from

$$s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}$$

Table 1 : For $\gamma = 0.2,\ q = 0.7$

I	T_h	n	s
16	1.3059245101	130827	–
32	1.3060855729	256853	–
64	1.3061258449	636465	2.00
128	1.3061359133	2042966	2.00

Table 2 : For $\gamma = 0.2,\ q = 0.9$

I	T_h	n	s
16	1.0205106012	60012	–
32	1.0206288011	113554	–
64	1.0206583598	220576	2.00
128	1.0206657500	500129	2.00

Table 3 : For $\gamma = 0.5,\ q = 0.9$

I	T_h	n	s
16	0.8992412182	62475	–
32	0.8994400927	118469	–
64	0.8994898198	230738	2.00
128	0.8995022521	528716	2.00

Remark 5.1. *The above tables assure the convergence of the numerical blow-up time to the continuous one, since the rate of convergence is 2, which is just the accuracy of the difference approximation in space.*
We also notice that the blow-up time diminishes when the parameter q or γ increases.

Others illustrations are given by some plots in the below figures.

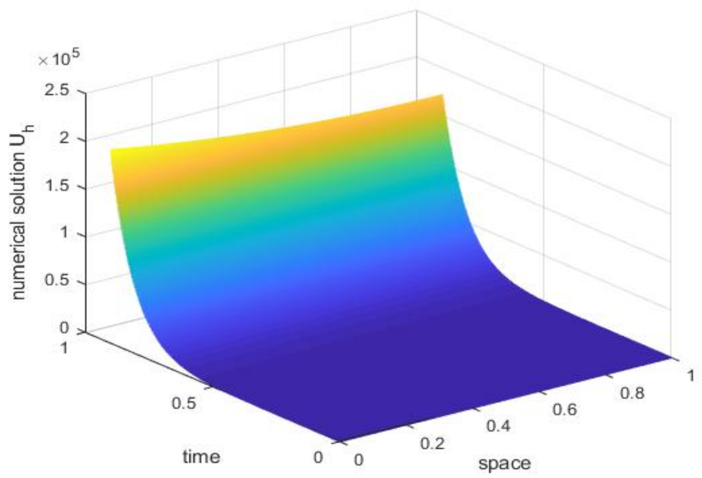

Figure 1. Evolution of the numerical solution
for $I = 32,\ \gamma = 0.5,\ q = 0.9$

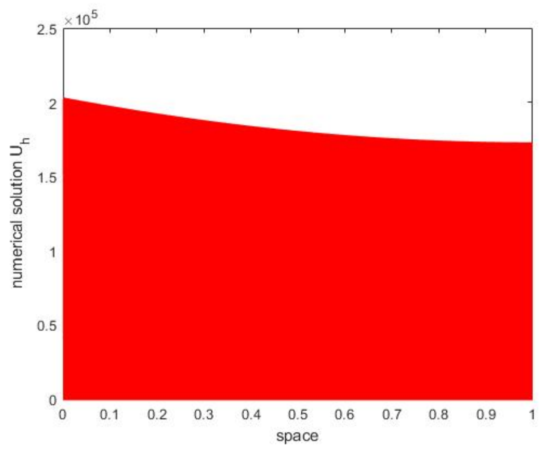

Figure 2. Evolution of U_h according to the space
for $I = 32,\ \gamma = 0.5,\ q = 0.9$

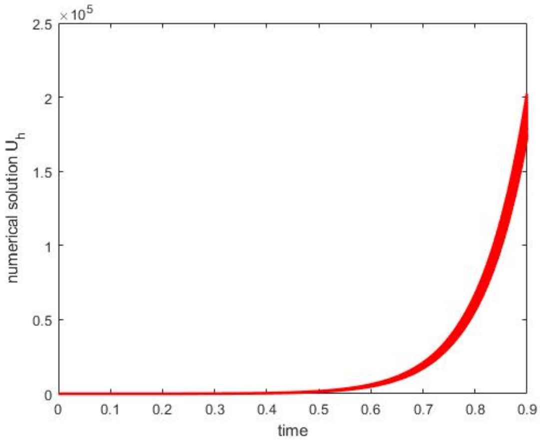

Figure 3. Evolution of U_h according to the time for $I = 32,\ \gamma = 0.5,\ q = 0.9$

Remark 5.2. *Figures 1, 2 and 3 show that the numerical solution blows up in finite time on the whole space for $\gamma > 0$ and $0 < q < 1$, which is in line with the theoretically established result (see* [8]*).*

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