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APPROXIMATE ANALYTICAL SOLUTION FOR NON-LINEAR REACTION DIFFUSION EQUATIONS IN MODELING OF A BACTERIAL AND FUNGAL BIOFILTER

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Abstract. The aim of the present work is to find the solution for concentration of pollutant in the biofilm phase using HPM with complex inversion method. The obtained results are validated and illustrate the efficiency of future technique from the figure. Furthermore, the reported analytical result is useful in future modification of mathematical modeling of the biofilter.

1. Introduction

In 1893, the Biofilters has been used to control to air pollution in chemical industrial. Netherland and Germany are the first countries constructed to control the waste air. Many authors were reported the biofilteration techniques provide an effective and inexpensive technology to remove air pollution and its biologically open system [1, 2, 3, 4, 5, 6]. In biofilters, Bacteria and fungi are certainly the two dominant microorganism groups [7] and there grow high and slow respectively. Toluene has been used in many industries and there are more capable than elimination capacities of bacteria and fungi in biofilter [8, 9, 10, 11, 12]. Recently, dynamic Biofilter model applied to control the toluene and it is more suitable for duration [13, 14, 15]. From

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the biofilteration model of mathematical model, develop the partial differential equation of mass balance equation taking into the account the chemical and physical phenomena. In Mathematical point of view, the linear type differential equation has been solved by exactly; it's represented the changes can be done in chemical and physical phenomena. At the time, the non-linear type differential model cannot be solved exactly. Some methods are available in approximation solution formed, such as a Homtopy perturbation method, Adomian decomposition method, and Variation iteration method and so on [16, 17, 18, 19, 20, 21, 22]. All the method produced the solution in different approximation, it is validated by the numerical solution for the approximately closed to exactness.
A.D.Dorado et.al., [23] represents the nonlinear partial differential equation in the modeling of bacteria and fungi biofilter. In this work, the analytical expressions have been derived from the concentration of the Pollution concentration in the biofilm phase using Homotopy perturbation method (HPM) [24, 25, 26, 27]. Also, the obtained analytical results are compared with numerical solution with the help of MATLAB software [28] and the error percentage is noted. The represented solutions are more reliable and easy to predict the dynamic behavior of the system by varying the parameters.

2. Problem Description

A.D. Dorado et.al.,[23] represent a mathematical modeling of a bacterial and fungal biofilter and it established mass balance equation for biofilm as follows [23]:

$$\frac{\partial C_b}{\partial t} = D\frac{\partial^2 C_b}{\partial t^2} + v_{max}\frac{C_b}{K_S + C_b} \tag{2.1}$$

Subject to the initial and boundary conditions are as follows [23]:

$$t = 0\,;\ \ 0 \leq x \leq \delta\,;\ \ C_b = 0 \tag{2.2}$$

$$t < 0\,;\ \ x = 0\,;\ \ C_b = \frac{C_g}{H} \tag{2.3}$$

$$t < 0\,;\ \ x = \delta\,;\ \ \frac{\partial C_b}{\partial t} = 0 \tag{2.4}$$

where the function C_b denote the pollutant concentration in the biofilm phase, x represent the position in the biofilm from the surface, D is the diffusion coefficient, v_{max} is the volumetric maximum growth rate, K_S is the semi - saturation or affinity constant, C_g is the concentration of toluene in the gas phase, H is the gas - liquid distribution coefficient given by Henry's law and δ is the biofilm thickness.
From the above equation (2.1) with the initial and boundary conditions equations (2.2) – (2.4) which has to be solved in this case written in dimensionless form as follow:

$$\frac{\partial C_b'(x',t')}{\partial t'} = \frac{\partial^2 C_b'(x',t')}{\partial x'^2} + \mu \frac{C_b'(x',t')}{K + C_b'(x',t')} \,. \tag{2.5}$$

Subject to the initial and boundary conditions are as follows [23]:

$$t' = 0\,;\;\; 0 \le x' \le \delta\,;\;\; C_b'(0,t') = 0\,, \tag{2.6}$$

$$t' < 0\,;\;\; x' = 0\,;\;\; C_b'(x',t') = a\,,$$

$$t' < 0\,;\;\; x' = 1\,;\;\; \frac{\partial C_b}{\partial t} = 0\,. \tag{2.7}$$

3. Mathematical Procedures

Nonlinear phenomena appear in many fields like physics, chemical engineering and applied mathematics etc. In these phenomena, many scientists formulated the nonlinear partial differential equations and not produced solutions. For finding solutions for exact analytical solutions for these nonlinear equations is difficult. In 1992, Liao et.al., [16] develop the Homotopic technique to solve the nonlinear differential equations. These method yields a very rapid convergence of the solution series in most cases and few iterations leading to very accurate solutions.

3.1. **Basic concept of Homotopy perturbation method (HPM).**

To explain this method, consider the nonlinear differential equation in the form:

$$D(u) - f(r) = 0; \qquad r \in \Omega \tag{3.1}$$

with the boundary conditions

$$D(u, \frac{\partial u}{\partial t}) = 0; \qquad r \in \varGamma$$

where D is a general differential operator, B is a boundary operator, $f(r)$ is a known analytical function and Γ is the boundary of the domain Ω .
In general, the operator D can be divided into a linear part L and a non-linear part N. Eqn.(3.1) can therefore be written as

$$L(u) + N(u) - f(r) = 0\,.$$

By the Homotopy technique, we construct a Homotopy $v(r, p)$: $\Omega \times [0, 1] \longrightarrow R$ that satisfies

$$H(v, p) = (1 - p)[L(v) - L(u_0)] + p[D(v) - f(r)] = 0\,, \tag{3.2}$$

or

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0\,, \tag{3.3}$$

where $p \in [0, 1]$ is an embedding parameter, and u_0 is an inital approximation of Eqn.(3.1) that satisfies the boundary conditions. From Eqn.(3.2) or Eqn.(3.3), we have:

$$\begin{aligned} H(v, 0) &= L(v) - L(u_0) = 0\,, \\ H(v, 1) &= D(v) - f(r) = 0\,. \end{aligned}$$

When p=0, Eqn.(3.2) or Eqn.(3.3) become linear equations. When p =1, they become non-linear equations. The process of changing p from zero to unity is that of $L(v)$ – $L(u_0)$=0 to $D(v)$–$f(r)$=0.
We first use the embedding parameter p as a "small parameter" and assume that the solutions of Eqn.(3.2) or Eqn.(3.3) can be written as a power series in p :

$$v = v_0 + pv_1 + p^2 v_2 + \cdots$$

Setting p = 1 results in the approximate solution of Eqn.(3.1):

$$u = \lim_{n \longrightarrow \infty} v = v_0 + pv_1 + p^2 v_2 + \cdots$$

4. Application of Described Manners in the Issue

4.1. **Basic concept of Homotopy Perturbation Method (HPM).** In this section, the above nonlinear partial differential equations Eqn.(2.5) – Eqn.(2.7) have been solved analytically by using Homotopy perturbation method [3] –[7]:

To use the Homotopy perturbation method (HPM), we can first construct a Homotopy as follows:

$$(1-p)\left[\frac{\partial^2 C_b'(x',t')}{\partial x'^2} - \frac{\partial C_b'(x',t')}{\partial t'} + \mu\frac{C_b'(x',t')}{K+1}\right] + p\left[\frac{\partial^2 C_b'(x',t')}{\partial x'^2} - \frac{\partial C_b'(x',t')}{\partial t'} + \mu\frac{C_b'(x',t')}{K+C_b'(x',t')}\right] = 0$$

where $p \in [0,1]$is an embedding parameter. We can obtain the following approximation solution:

$$C_b'(x',t') \approx \cos\left(\sqrt{\frac{\mu}{K+1}}x\right) + \tan\left(\sqrt{\frac{\mu}{K+1}}\right)\sin\left(\sqrt{\frac{\mu}{K+1}}x\right) + \pi\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)\sin\left(\frac{(2n+1)\pi x}{2}\right)}{\frac{(2n+1)^2\pi^2}{4}-\frac{\mu}{K+1}}e^{-\left(\frac{(2n+1)^2\pi^2}{4}-\frac{\mu}{K+1}\right)t}.$$

4.2. **Limiting case.**

4.2.1. *Unsaturated (first order) case:* When $C'(x',t') \ll$ K, the Eqn.(2.5) reduce to

$$\frac{\partial C_b'(x',t')}{\partial t'} = \frac{\partial^2 C_b'(x',t')}{\partial x'^2} + \mu\frac{C_b'(x',t')}{K}. \tag{4.1}$$

Eqn.(4.1) is linear mass balance equation which is exactly solvable. By solving them, the concentration of pollutant in the biofilm phase can be obtained as follow:

$$C_b'(x',t') \approx \cos\left(\sqrt{\frac{\mu}{K}}x\right) + \tan\left(\sqrt{\frac{\mu}{K}}\right)\sin\left(\sqrt{\frac{\mu}{K}}x\right) + \pi\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)\sin\left(\frac{(2n+1)\pi x}{2}\right)}{\frac{(2n+1)^2\pi^2}{4}-\frac{\mu}{K}}e^{-\left(\frac{(2n+1)^2\pi^2}{4}-\frac{\mu}{K}\right)t}.$$

4.2.2. *Saturated (zero order) case:* When $C'(x',t') \gg$ K, the Eqn.(2.5) reduce to

$$\frac{\partial C_b'(x',t')}{\partial t'} = \frac{\partial^2 C_b'(x',t')}{\partial x'^2} + \mu\,. \tag{4.2}$$

By solving Eqn.(4.2), we can obtain the solution of the pollutant concentration in the biofilm phase as follow:

$$C_b'(x',t') \approx 1 + 2(\mu-1)\sum_{n=0}^{\infty}(-1)^n \frac{\cos\left(\frac{(2n+1)\pi}{2}(x-1)\right)}{(2n+1)\pi} e^{-\frac{(2n+1)^2\pi^2}{4}t}\,.$$

4.3. **Numerical results and discussion.**

In this work, we can find the solution for the concentration of pollutant in the biofilm phase (Eqn.(2.5)) for the boundary conditions (Eqn.(2.6) – Eqn.(2.7)) using HPM.

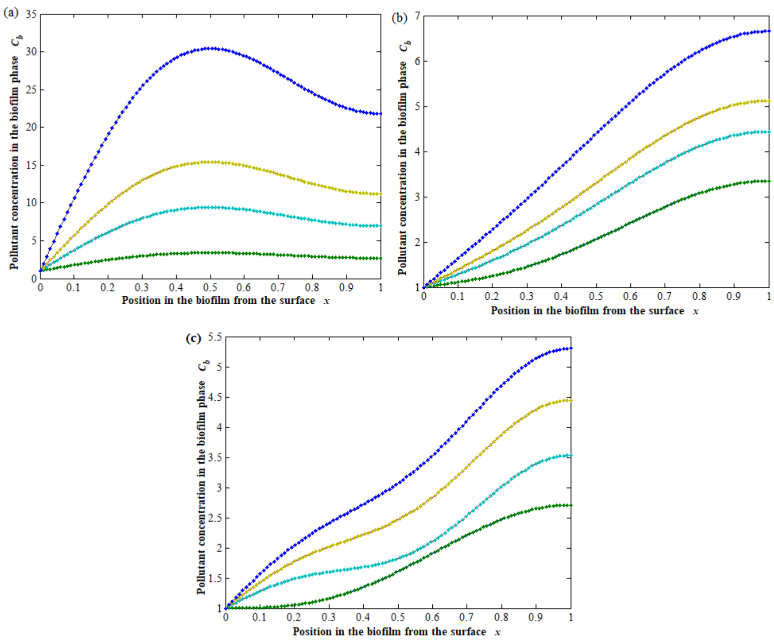

FIGURE 1. Nature for the obtained solution $C_b'(x',t')$ with μ=0.1, 5, 10, 15; K=10 (a) Saturated Case (b) Unsaturated Case(First order)(c) Unsaturated Case(Zero order)

In Fig.1, represent the comparison between the analytical result and numerical result. Form these, we observed that there is no deviation between numerical and obtained solution in both cases. Upon comparison, it gives a satisfactory agreement for all values of the parameters μ and K.

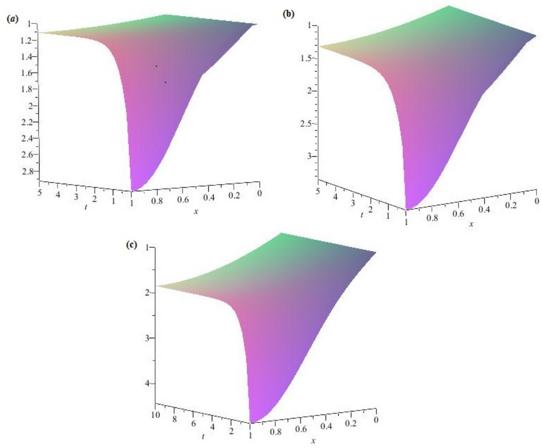

FIGURE 2. Surface of $C_b'(x', t')$ pollutant in the biofilm phase(Saturated Case) at the parameters (a) μ=0.1 and K=0.5; (b) μ=5 and K=10; (c) μ=10 and K=10.

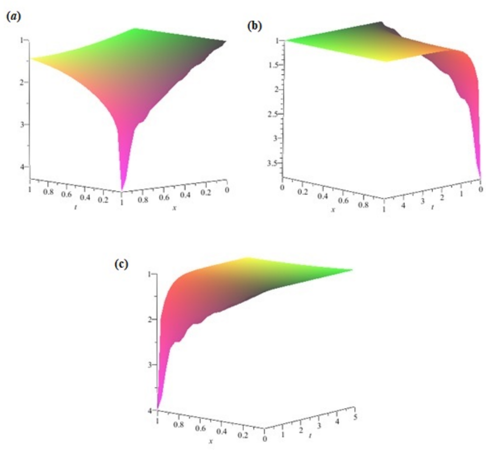

FIGURE 3. Surface of $C_b'(x', t')$ pollutant in the biofilm phase(Saturated Case) at the parameters (a) μ=0.1 and K=0.5; (b) μ=5 and K=10; (c) μ=10 and K=10.

In addition, the surfaces of obtained solution (Saturated and Unsaturated case) have been represented in Fig.2 - 4.

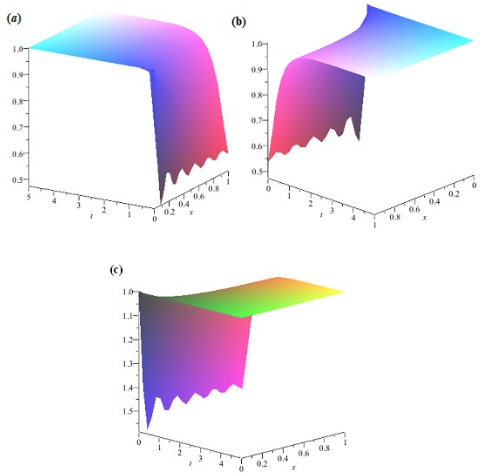

FIGURE 4. Surface of $C_b'(x',t')$ pollutant in the biofilm phase(Unsaturated Case - Zero order) at the parameters (a) μ=0.1 and K=0.5; (b) μ=5 and K=10; (c) μ=10 and K=10.

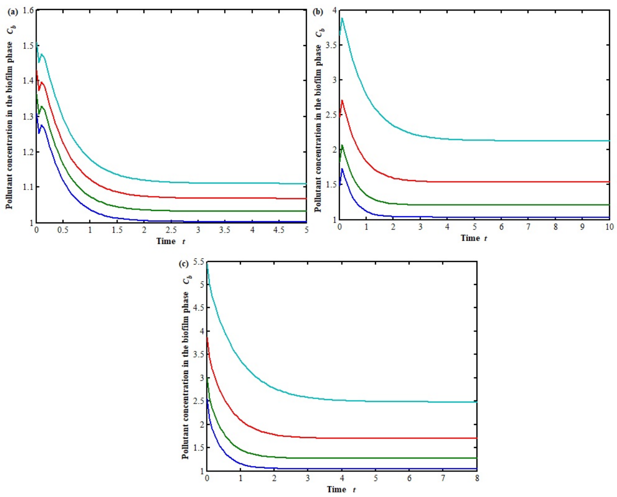

FIGURE 5. Nature for the obtained solution $C_b'(x',t')$ (Unsaturated Case - First order) (a) μ=0.1 and K=0.5; (b) μ=5 and K=10; (c) μ=10 and K=10.

Fig. 2 – 4 describe the surfaces of the concentration of pollutant in the biofilm phase at the unsaturated and saturated case. From these figures, we can conclude that the physical behavior of the considered nonlinear equation and it helps the researchers.

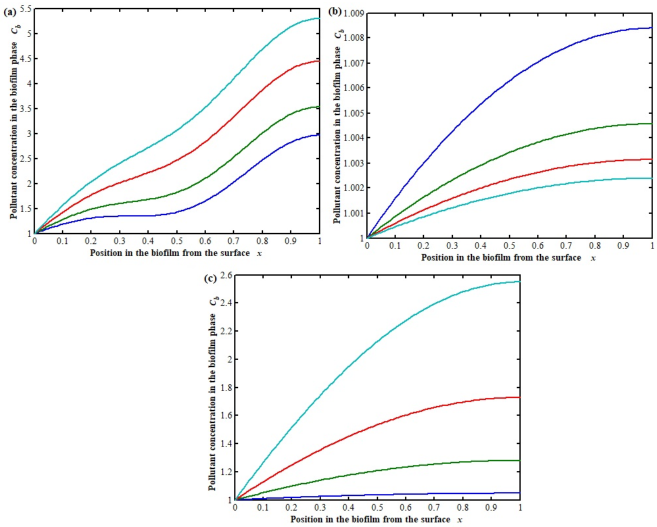

FIGURE 6. Nature for the obtained solution $C_b'(x', t')$ (Saturated Case) with (a) μ=0.1, 5, 10, 13 and K=10; (b) μ=0.1 and K=5, 10, 15, 20; (c) μ=1, 5, 10, 15 and K=10.

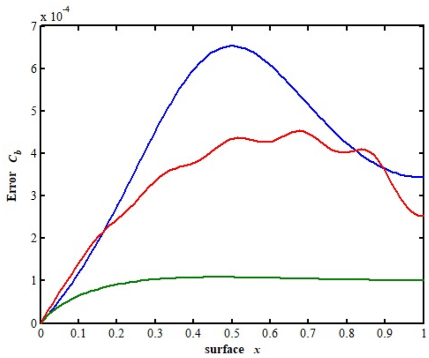

FIGURE 7. Error graph of the solution.

Natures of the obtained solution with parameters μ and K have captured in Fig. 5 and 6.

The table.1 represents the error calculation for our results and numerical simulation results. 0.0004 % is the maximum relative errors and these is in Fig.7.

TABLE 1. Error calculation $C_b'(x', t')$ with numerical and analytical results for various values of parameters μ and K.

Saturated case (Zero Order): μ= 0.1 and K =10				Unsaturated case (1st Order): μ= 15 and K =10				Saturated case: μ= 0.1 and K =1			
x	Num.	ADM	% of	x	Num.	ADM	% of	x	Num.	ADM	% of
0.0	1.0000	1.0000	0.0000	0.0	1.0000	1.0000	0.0000	0.0	1.0000	1.0000	0.0000
0.2	0.6689	0.6686	0.0004	0.2	2.2786	2.2783	0.0001	0.2	1.2248	1.2248	0.0000
0.4	0.4817	0.4814	0.0006	0.4	3.6647	3.6643	0.0001	0.4	1.4695	1.4695	0.0000
0.6	0.4774	0.4771	0.0006	0.6	5.0903	5.0897	0.0001	0.6	1.7693	1.7693	0.0000
0.8	0.5673	0.5670	0.0005	0.8	6.2187	6.2180	0.0001	0.8	2.2263	2.2263	0.0000
1.0	0.6180	0.6178	0.0003	1.0	6.6523	6.6516	0.0001	1.0	3.7874	3.7874	0.0000
Average Error %			0.0004	Average Error %			0.0000	Average Error %			0.0000

5. CONCLUSIONS

In this paper, HPM is used to solving the nonlinear differential equation of the pollutant in the biofilm phase. Also, we can represent the solution of the first order and zero order case for the concentration of pollutant in the biofilm phase. Furthermore, according to achieved results, these works are useful to understand the behavior of the system.

Finally, it will be obvious that HPM is an excellent analytical method due to its accuracy, efficiency and convergence it could be applicable for solving strongly nonlinear differential equations.

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