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BRK-GROUPS IN TOPOLOGICAL SPACES

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ABSTRACT. In our paper [10], we introduced topological structures on BRK-Algebras. In this paper, we introduce the notion of BRK-groups and discuss some basics topological concepts on it.

1. INTRODUCTION AND PRELIMINARIES

Imai and Iseki [2] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras in the year of 1996. These algebras have been extensively studied since their introduction. In 1983, Hu and Li [1] introduced the notion of a BCH-algebra which is a generalization of the notion of BCK and BCI-algebras and studied a few properties of these algebras. In 2001, Neggers et al. [7] introduced a new notion, called a Q-algebra and generalized some theorems discussed in BCI/BCK-algebras. In 2002, Neggers and Kim [8] introduced a new notion, called a B-algebra, and obtained several results. In 2007, Walendziak [11] introduced a new notion, called a BF-algebra, which is a generalization of B-algebra. In [5], C. B. Kim and H. S. Kim introduced BG-algebra as a generalization of B-algebra. We introduce a new notion, called a BRK-algebra, which is a generalization of $BCK/BCI/BCH/Q/QS/BM$-algebras. In 2012, Ravi Kumar Bandaru [9] introduced the notion of BRK-algebra. Motivated by this, we discuss the topological concepts in BRK-algebra in [10]. Groups recur throughout mathematics and the methods of

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group theory have influenced may parts of algebra. Almost all structures of abstract algebra are special case of groups. In this paper, we introduce the notion of topological BRK-groups and discuss some basic interesting results.

First, we recall certain definitions from [1,4,6–8,11] that are required in the paper.

Definition 1.1. *A BRK-algebra (briefly, BRK Alg) [9] $(I, \star, 0)$ is a non-empty set I with a constant 0 and a binary operation $\star$ satisfying the following axioms:*

$$(i)\ x \star 0 = x,$$

$$(ii)\ (x \star y) \star x = 0 \star y\,,$$

for any $x, y \in$ I. In I, we can define a binary relation " $\leq$ " by

$$x \leq y\ iff\ x \star y = 0\,.$$

Definition 1.2. *[10] Let I be a BRK Alg and M, N are any non-empty subsets of I. We define a subset $M \star N$ of I as follows, $M \star N = \{m \star n | m \in M, n \in N\}$.*

Definition 1.3. *Let I be a BRK-algebra, and $i \in I$. Define a left map [9] $L_i : I \to I$ (resp. right map $R_i : I \to I$) by, $L_a(i) = a \star i$ (resp. $R_a(i) = i \star a$), for all $i \in I$.*

The set of all left (resp. right) maps on I is defined as, $L(I)$ (resp. $R(I)$).

Definition 1.4. *[10] Let $(I, \star, 0)$ be a BRK Alg. Let τ be the collection of subsets of I, τ is said to be a topology on I, if $I, \phi \in \tau$; arbitrary union of members of τ is in τ; finite intersection of members of τ is in τ.*

Definition 1.5. *[10] Let $(I, \star, 0)$ be a BRK Alg and τ a topology on I. Then $I = (I, \star, 0, \tau)$ is called a topological BRK Alg (briefly, τBRK Alg), if the operation " $\star$ " is continuous or equivalently, for any $m, n \in I$ and for every open set O of $m \star n$, there exist two open sets M and N respectively, such that $M \star N$ is a subset of O.*

Remark 1.1. *[10] The members of τ are called an open set in I. The complement of $M \in \tau$, that is $I \backslash M$ is called a closed set in I, denoted by $\tilde{M}$. The open set of an element $x \in I$, is a member of τ containing I.*

Definition 1.6. *[10] Let $(I, \star, 0)$ be a BRK Alg and M be any subset of I is said to BRK-subalgebra (briefly, BRK $sAlg$), if for every $m, n \in I$, $m \star n \in M$.*

Definition 1.7. *[3] A topological group G is a group that is also a topological space, such that the map of $G \times G$ into G sending $m \times n$ into $m \star n$, and the map G into G sending m into m^{-1} are continuous.*

2. TOPOLOGICAL BRK-GROUPS

Theorem 2.1. *In a $BRK\ Alg$ $(I, \star, 0)$ if $x \star 0 = x$, for all $x \in I$ then $(I, \star, 0)$ forms an abelian group and every element has an involution.*

Proof.

1. Commutative: $x \star y = (x \star y) \star 0$
   $= (x \star y) \star (x \star x)$ since $x \star x = 0$
   $= (x \star (x \star x)) \star y$
   $= (x \star 0) \star y = x \star y.$ (By Theorem 4.5 in [10].)
2. Closure: Since $\star$ is a binary operation, $x \star y \in I$.
3. Associative : $(x \star y) \star z = x \star (y \star z)$ (By Theorem 4.5 in [10]).
4. Identity: Putting $x = y = z$ in the Associative law $(x \star y) \star z = x \star (y \star z)$ and using (1) & (3), we obtain $0 \star x = x \star 0 = x$. This means that 0 is an identity element of I.
5. Inverse: By (3), x of I has its inverse the element x itself. If there exist another element $y \in I$, such that $x \star y = y \star x = 0$; then $x = y$. That is every element in I has involution.

□

Notation : The abelian group $(I, \star, 0)$ is called a BRK-group (briefly, $BRK\ grp$).

Example 1. *Let $(I = \{0, x, y, z\}, \star, 0)$ be a $BRK\ Alg$ in which $\star$ is defined by*

$\star$	0	x	y	z
0	0	0	0	0
x	x	0	z	y
y	y	z	0	x
z	z	y	x	0

and a topology $\tau = \{\phi, I, \{0, x\}, \{y, z\}\}$. The set $(I, \star, 0)$ forms a $BRK\ grp$ with the binary operation $0 \star a = a$, for all $a \in I$.

Remark 2.1. *In general, every $BRK\ grp$ is a group. But a group is not a $BRK\ grp$.*

Example 2. *Let $\mathbb{R}$, the set of all real numbers with the binary operation '$+$', $\mathbb{R}$ is a group but not a $BRK\ grp$ under '$+$'.*

Theorem 2.2. *Let I be a $BRK\ grp$ and M, any subset of I. Then M is closed under the operation $\star$ iff M is a BRK-subgroup (briefly, $BRK\ sgrp$).*

Proof. Let I be a $BRK\ grp$ and M a subset of I be closed under the operation $\star$. Since for every $x \in M$, $x \star x = 0$, $0 \in M$. Clearly, $0 \star x = x \star 0 = x \Rightarrow M$ has an identity element. In I, each element has involution $\Rightarrow$ each element in M also has an involution. Therefore there exist a unique inverse for all $m \in M$. Associative and commutative properties holds in I. This implies that the associative condition also holds in M. Therefore M is a $BRK\ grp$ under the same operation in I. Conversely, let M be a $BRK\ sgrp$ of I. Trivially, M is closed under binary operation $\star$. $\square$

Definition 2.1. *A topological BRK-group (briefly, $\tau BRK\ grp$) G is a $BRK\ grp$ that is also a topological space, such that the map of $G \times G$ into G sending $x \times y$ into $x \star y$ and the map G into G sending x into x^{-1} are continuous.*

Example 3. *In Example 1, clearly the maps $x \times y$ into $x \star y$ and x into x^{-1} are continuous on I. Therefore, $(I, \star, 0, \tau)$ is a $\tau BRK\ grp$.*

Theorem 2.3. *Let $(I, \star, \tau)$ be a $\tau BRK\ Alg$ with $0 \star x = x$, for all $x \in I$. Then $(I, \star, \tau)$ is a $\tau BRK\ grp$.*

Proof. Let I be a $\tau BRK\ Alg$ with $0 \star x = x$, for all $x \in I$. From Theorem 2.1, I is a $BRK\ grp$. Now define a map $f : I \times I \to I$ by,

$$(x, y) \to x \star y.$$

Since I is a $\tau BRK\ Alg$, f is continuous. Again define the map $g : I \to I$ by, $x \to x^{-1}$. Since $x^{-1} = x$, for all x implies g is a constant function $\Rightarrow g$ is continuous. $\square$

Remark 2.2. *In general, every element in a $BRK\ grp$ is involutive and so the map x into x^{-1} in the Defintion 2.1 is always continuous in any $BRK\ grp$. Therefore it is sufficient to show that the map $x \times y$ into $x \star y$ is continuous, which provides a $\tau BRK\ grp$.*

Theorem 2.4. *Every subalgebra in a $\tau BRK\ grp$, under subspace topology, forms a $\tau BRK\ grp$.*

Proof. Let I be a $\tau BRK\ grp$, and M be any subalgebra of I. Since every $BRK\ sAlg$ of I is a $BRK\ grp$, we want to prove in the map f from $M \star M$ into M given by $f(x, y) = x \star y$ is continuous. For any $U \in \tau$ and $a, b \in M$ and $U \cap M$ is an arbitary open set of $a \star b$ in M, then there exist an open set V and W of a and b in I such that $V \star W \subseteq U$. Thus $V \cap M$ and $W \cap M$ are the open sets of a and b in M such that $(V \cap M) \star (W \cap M) \subseteq (U \cap M)$. Hence f is continuous. □

Definition 2.2. *Let I be a $BRK\ grp$ and M be any subset of I. The left coset of M is defined as, $a \star M = \{a \star h/h \in M\}$, $a \in I$.*

Theorem 2.5. *Let X be a $\tau BRK\ grp$, and M be a $BRK\ sgrp$ of I. If any left coset of M is open, then M is open.*

Proof. Let I be a $\tau BRK\ grp$, and M be a any $BRK\ sgrp$ of I. Given $x \star M$ is open for any $x \in I$. We want to show that, M is open. Let $m \in M$, $\Rightarrow$ $x \star m \in x \star M$. Since I is a $\tau BRK\ grp$, for any open set W of $x \star m$, there exist a two open sets U and V of x and m respectively such that $U \star V \subseteq W$. Since $x \star M$ is open, $W \subseteq x \star M$. This implies that, $U \star V \subseteq x \star M$. Claim: $V \subseteq M$. Suppose $V \not\subseteq M$. Then there exist an atleast one element $n \in V$ but $n \notin M$. $\Rightarrow$ $x \star n \in U \star V \subseteq x \star M = \{x \star m/m \in M\} \Rightarrow n \in M$, which is a contradiction to our assumption. $\Rightarrow V \subseteq M$. Hence, M is open. □

Theorem 2.6. *Let I be a $\tau BRK\ grp$ and $M,\ N$ be any two $BRK\ sgrp$'s of I. If $M \star N$ is open then both M and N are open.*

Proof. Let I be a $\tau BRK\ grp$ and $M,\ N$ be $BRK\ sgrp$'s of I. Let $m \in M$ and $n \in N$, so that $m \star n \in M \star N$. Consider an open set W of $m \star n$. Since $M \star N$ is open, we can prove that there exist open sets U and V of m and n respectively such that $U \star V \subseteq W$. Since $M \star N$ is open, $W \subseteq M \star N$. $\Rightarrow U \star V \subseteq M \star N$. Claim: $U \subseteq M$. Suppose not. That is assume that $U \not\subseteq M$. Then there exist an element $c \in U$ and $c \notin M \Rightarrow c \star V \subseteq U \star V \subseteq M \star N \Rightarrow c \in M$. which contradicts the fact that $c \in M$. Therefore, $U \subseteq M$. Similarly, we can prove that $V \subseteq N$. Hence, both M and N are open. □

Theorem 2.7. *In a $\tau BRK\ grp$, every open $BRK\ sgrp$ is closed.*

Proof. Let I be a $\tau BRK\ grp$ and M be an open $BRK\ sgrp$. Let $m \in I \backslash M$, Since $x \star x = 0$ and M is open, there exist an open set U of I such that $U \star U \subseteq M$. Claim : $U \subseteq M$. Suppose $U \not\subset I$. Then $U \subseteq I \backslash M$, there exist an element $y \in U \cap M \Rightarrow z \star y \in U \star U \subseteq M$, for all $z \in U$. Since M is a $BRK\ sgrp$, $z = x \star (x \star z) \in M \Rightarrow U \subseteq M$, contradicting $U \not\subset M$. Hence, M is closed. □

Conclusion

In this paper, we have introduced the concept of $\tau BRK\ grp$ and studied their properties. In addition, we extend this concept we can get interesting results on it. In our future work, we introduce the concept of fuzzy $\tau BRK\ grp$, interval-valued fuzzy $\tau BRK\ grp$, intuitionistic fuzzy structure of $\tau BRK\ grp$, intuitionistic fuzzy ideals of $\tau BRK\ grp$. I hope this work would serve as a foundation for further studies on the structure of $\tau BRK\ Alg$'s.

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