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NON-EXISTENCE OF SKOLEM MEAN LABELING FOR FOUR STAR GRAPHS

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ABSTRACT. In this paper, we prove if $r > s < t$, the four star $G = k_{1,r} \cup k_{1,r} \cup k_{1,s} \cup k_{1,t}$ is not a skolem mean graph if $|s-t| > 4+2r$ for $r = 2,3\cdots$; $s = 1,2\cdots$ and $t \geq 2r+s+5$.

1. INTRODUCTION

In [1], V. Balaji and etl proved that the three star graph $K_{1,\ell} \cup K_{1,m} \cup K_{1,n}$, $\ell \leq m < n$ is skolem mean graph if $|m-n| \leq \ell+4$. In [2], they have proved that the four star graph $K_{1,\ell} \cup K_{1,\ell} \cup K_{1,m} \cup K_{1,n}$, $\ell \leq m < n$ is skolem mean graph if $|m-n| \leq 2\ell+4$. In [3], V. Balaji and etl proved that the three star graph $K_{1,p} \cup K_{1,q} \cup K_{1,r}$, $p > q < r$ is skolem mean graph if and only if $|p-q| \leq r+4$.

Definition 1.1. *A graph G with p nodes and q links is said to be a skolem mean graph if there exists a function f from the node set of G to $\{1,2,\cdots,p\}$ such that the induced map f^* from the link set of G to $\{2,3,\cdots,p\}$ defined by*

$$f^*(e=uv) = \begin{cases} \frac{f(u)+f(v)}{2} & \text{if } \ f(u)+f(v) \text{ is even;} \\ \frac{f(u)+f(v)+1}{2} & \text{if } \ f(u)+f(v) \text{ is odd,} \end{cases}$$

then the resulting links get distinct labels from the set $\{2,3,\cdots,p\}$.

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2. Main result

Theorem 2.1. *If $r > s < t$, the four star $K_{1,r} \cup K_{1,r} \cup K_{1,s} \cup K_{1,t}$ is not a skolem mean graph if $|s-t| > 4+2r$ with $r-s=1$, for $r = 2, 3 \cdots$ and $s = 1, 2 \cdots$ and $t \geq 2r+s+5$.*

Proof. Let $G = K_{1,r} \cup K_{1,r} \cup K_{1,s} \cup K_{1,t}$ where,

$$V(G) = \{v_{a,b} : 1 \leq a \leq 2, 0 \leq b \leq 3\} \cup \{v_{3,b} : 0 \leq b \leq 2\} \cup \{v_{4,b} : 0 \leq b \leq 13\}$$
$$E(G) = \{v_{a,0}v_{a,b} : 1 \leq a \leq 2, 1 \leq b \leq 3\} \cup \{v_{3,0}v_{3,b} : 1 \leq b \leq 2\} \cup \{v_{4,0}v_{4,b} : 1 \leq b \leq 13\}.$$

Then, $p = 25$ and $q = 21$.

Suppose G is a skolem mean graph, then there exists a function f from the node set of G to $1, 2 \cdots$ such that the induced map f^* from the link set of G to $2, 3 \cdots$ defined by

$$f^*(e = uv) = \begin{cases} \frac{f(u)+f(v)}{2} & \text{if } f(u)+f(v) \text{ is even;} \\ \frac{f(u)+f(v)+1}{2} & \text{if } f(u)+f(v) \text{ is odd} \end{cases}.$$

Then, the resulting links get distinct labels from the set $\{2, 3 \cdots p\}$. Let $x_{a,b}$ be the label given to the node $v_{a,b}$ for $1 \leq a \leq 2$, $0 \leq b \leq 3$, $v_{3,b}$ for $0 \leq b \leq 2$ and $v_{4,b}$ for $0 \leq b \leq 13$.

Let $y_{a,b}$ be the respective link label of the link $v_{a,o}v_{a,b}$ for $1 \leq a \leq 2$, $0 \leq b \leq 3$, $v_{3,0}v_{3,b}$ for $1 \leq b \leq 2$ and $v_{4,0}v_{4,b}$ for $0 \leq b \leq 13$.

Let us first consider the case that $x_{4,0} = 24$. If $v_{4,b} = 2t-1$ and $v_{4,c} = 2t$ for some n and for some b and c, then,

$$f^*(v_{4,0}v_{4,b}) = \frac{24+2t}{2} = \frac{24+2t-1}{2} = 12+t = f^*(v_{4,0}v_{4,b})$$

This is not possible as f^* is a bijection. Therefore, the 13 nodes $x_{4,b}$ for $1 \leq b \leq 13$ are among the 13 numbers $(1/2)$, $(3/4)$, $(5/6)$, $(7/8)$, $(9/10)$, $(11/12)$, $(13/14)$, $(15/16)$, $(17/18)$, $(19/20)$, $(21/22)$, 23 and 25. Since $x_{4,0} = 24$, first let us consider all the biggest link labels possible for $K_{1,13}$. That is, for 13 nodes $x_{4,b}$ for $1 \leq b \leq 13$. Consider the 13 choices that may induce the larger link values.

Therefore, the 13 choices are $(1/2)$, $(3/4)$, $(5/6)$, $(7/8)$, $(9/10)$, $(11/12)$, $(13/14)$, $(15/16)$, $(17/18)$, $(19/20)$, $(21/22)$, 23 and 25. The respective link

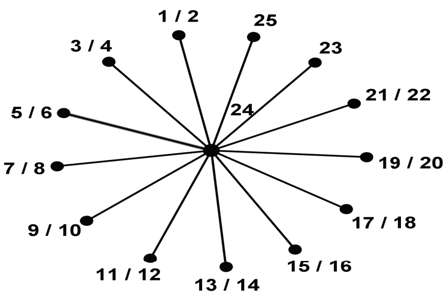

FIGURE 1. $K_{1,13}$

labels are 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24 and 25. Then, the set $\{y_{4,b} : 1 \le b \le 13\} = \{13, 14 \cdots 25\}$.

Case(A): $x_{4,3} = 21$ (we have $x_{4,0} = 24, x_{4,1} = 25, x_{4,2} = 23$ and $x_{1,0} = 22$). Now, 22 is a label of either $x_{a,0}$ for $1 \le a \le 2$ or $x_{a,b}$ for $1 \le a \le 2$; $1 \le b \le 3$. That is, 22 is a label of pendent or non-pendent node in $K_{1,2}$ or $K_{1,3}$ component of G. Without loss of generality, let us assume that $x_{1,0} = 22$.

Case(A$_1$): $x_{1,0} = 22$ (we have $x_{4,0} = 24$, $x_{4,1} = 25$, $x_{4,2} = 23$)
If $x_{1,0} = 22$, then, $x_{1,1}$ take the values one among 1,2 [As $x_{1,1} \ge 3$ would imply that $y_{1,1} \ge 13$. This is not possible]. Let $x_{1,1} = 1$ and $x_{4,13} = 2$, then, respective link labels are $y_{1,1} = 12$ and $y_{4,3} = 23$. Next, $t_{4,4}$ is either 20 or 19.

Case(A$_2$): $x_{4,4} = 19$ *or* 20
If $x_{4,4} = 19$, then, let $x_{1,2} = 20$, then, $y_{1,2} = 21$, but, $y_{4,5} = 21$ is already alloted. Hence, $x_{1,2} = 20$ is not possible.

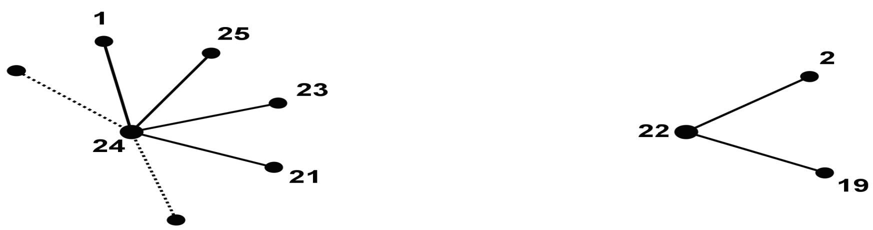

FIGURE 2

Case(B): $x_{4,13} = 2$ *or* 1 we have $x_{4,0} = 24, x_{4,1} = 25, x_{4,2} = 23$ and $x_{1,0} = 1$. Now, 2 is a label of either $x_{a,0}$ for $1 \le a \le 2$ or $x_{a,b}$ for $1 \le a \le 2$; $1 \le b \le 3$. That is, 2 is a label of pendent or non-pendent node in $K_{1,2}$ or $K_{1,3}$ component of G. Without loss of generality, let us assume that $x_{1,0} = 2$. Then, 2 is a label of non-pendent node in $K_{1,2}$ component of G.

Case(B$_1$): $x_{4,3} = 22$ *or* 21
Let $x_{4,3} = 21$. That is, 22 is a label of pendent node in a $K_{1,2}$ component of G. Let us assume that $x_{1,1} = 22$. Let $x_{4,0} = 24$, $x_{4,1} = 25$, $x_{4,2} = 23$, $x_{4,3} = 21$, $x_{4,13} = 1$, $x_{1,0} = 2$, $x_{1,1} = 22$. Then, $y_{4,13} = 13$, $y_{4,1} = 25$, $y_{4,2} = 24$, $y_{4,3} = 23$ and $y_{1,1} = 12$.

Case(B$_2$): $x_{4,4} = 19$ *or* 20
Let $t_{4,4} = 19$, then, 20 should be a label of another pendent or adjacent node in $K_{1,2}$ component of G. Then $x_{1,2} = 20$. Let $x_{4,0} = 24$, $x_{4,1} = 25$, $x_{4,2} = 23$, $x_{4,3} = 21$, $x_{4,13} = 1$ and $x_{4,4} = 19$, $x_{1,0} = 2$, $x_{1,1} = 22$, $x_{1,2} = 20$. Then, $y_{4,13} = 13$, $y_{4,1} = 25$, $y_{4,2} = 24$, $y_{4,3} = 23$, $y_{1,1} = 12$ and $y_{1,2} = 11$.

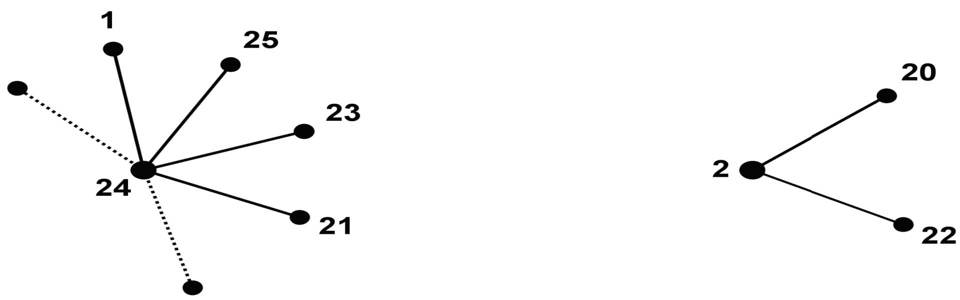

FIGURE 3

Case(C): $x_{4,5} = 17$ *or* 18
Let $x_{4,5} = 18$, then, 17 should be a label of pendent or non-pendent node in $K_{1,3}$ component of G. Without loss of generality. Let $x_{2,0} = 17$, then 17 should be label of non-pendent node in $K_{1,3}$ component of G.

Case(C$_1$): $x_{4,12} = 3$ *or* 4
If $x_{4,12} = 4$ and $x_{1,2} \geq 4$, then, 3 should be a label of pendent node of $K_{1,3}$ component of G, then, $x_{2,1} \geq 11$. This is not possible.

Case(C$_2$): $x_{4,12} = 3$ *or* 4
Now, let $x_{4,5} = 18$, so 17 should be a label of unlabeled node. To avoid the complication, let us allot 17 to a pendent node. Without loss of generality, let it be $x_{2,1} = 17$, i.e $x_{2,1} = 17$, $y_{2,b}$, $1 \leq b \leq 3$.

Case(D): $x_{4,12} = 3 \ or \ 4$
Let $x_{4,12} = 4$, then, 3 should be a label of non-pendent node in $K_{1,3}$ component of G. Then, $t_{2,0} = 3$.

Case(D$_1$): $t_{4,6} = 15 \ or \ 16$
If $t_{4,6} = 16$, so 15 should be a label of a pendent node in $K_{1,3}$ of G. Then, $x_{2,2} = 15$.

Case(D$_2$): $x_{4,7} = 13 \ or \ 14$
Let $x_{4,7} = 14$, then, 13 should a be label of another pendent node in $K_{1,3}$ component of G.

Without loss of generality , let $x_{2,3} = 13$. Let $x_{4,0} = 24$, $x_{4,1} = 25$, $x_{4,2} = 23$, $x_{4,3} = 21$, $x_{4,13} = 1$, $x_{4,4} = 19$,, $x_{4,5} = 18$, $x_{4,6} = 16$, $x_{4,7} = 14$, $x_{4,12} = 4$, $x_{1,0} = 2$, $x_{1,1} = 22$, $x_{1,2} = 20$, $x_{2,0} = 3$, $x_{2,1} = 17$, $x_{2,2} = 15$ and $x_{2,3} = 13$. Then $y_{4,13} = 13$, $y_{4,1} = 25$, $y_{4,2} = 24$, $y_{4,3} = 23$, $y_{1,1} = 12$, $y_{1,2} = 11$, $y_{2,1} = 10$, $y_{2,2} = 9$ and $y_{2,3} = 8$.

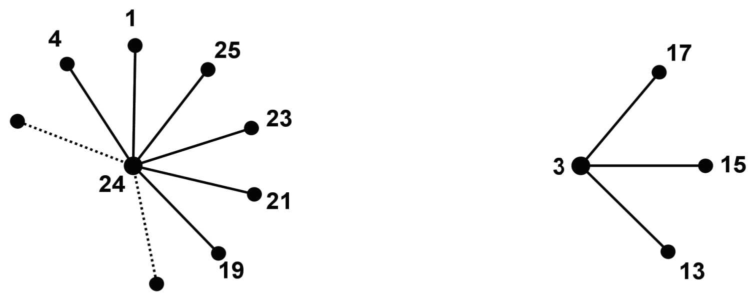

FIGURE 4

Case(E): $x_{4,8} = 11 \ or \ 12$ (we have $x_{4,0} = 24$, $x_{4,1} = 25$, $x_{4,2} = 23$ and $x_{1,0} = 1$)
Now, 12 is a label of $t_{b,0}$ for $1 \leq b \leq 3$. That is, 12 is a label of pendent or non-pendent node in another $K_{1,3}$ component of G. Without loss of generality, let us assume that $t_{3,0} = 12$. Then 12 is a label of non-pendent node in $K_{1,3}$ component of G.

Case(E$_1$): $x_{4,11} = 5 \ or \ 6$
If $x_{4,11} = 6$, so 5 should be a label of a pendent node in $K_{1,3}$ component of G. Then, $x_{3,1} = 5$. Let $x_{3,0} = 11$, $x_{3,1} = 5$, $x_{4,8} = 12$, $x_{4,11} = 6$. Then, $y_{4,8} = 18$, $y_{4,11} = 15$, $y_{3,1} = 8$. But , the link value 8 is already allotted $y_{2,3} = 8$. Hence,

$x_{3,0} = 11$ is not a non-pendent node in second $K_{1,3}$ component of G.Hence, $y_{3,1} = 8$ is not possible. Hence, $x_{3,0} \neq 11$. Similarly $x_{3,0} \neq 12$.

Case(F): $x_{4,11} = 5$ *or* 6

If $x_{4,11} = 6$, so 5 should be a label of a pendent or non-pendent node in another $K_{1,3}$ component of G. Without loss of generality, let $x_{3,0} = 5$, then, 5 should be a label of a non-pendent node in $K_{1,3}$ component of G.

Case(F$_1$): $x_{4,8} = 11$ *or* 12

If $x_{4,8} = 12$, so 11 should be a label of a pendent node in second $K_{1,3}$ component of G. Let us assume that $x_{3,1} = 11$. Then, $y_{3,1} = 8$, but, the link value 8 is already allotted $y_{2,3} = 8$.Then, $x_{3,1} \neq 11$ and similarly $x_{3,1} \neq 12$.

Case(F$_2$): $x_{4,9} = 9$ *or* 10

If $x_{4,9} = 10$, then, 9 should be a label of a pendent node in second $K_{1,3}$ component of G. Let us assume that the node is $x_{3,1} = 9$, then we get the link value $y_{3,1} = 7$.

Case(F$_3$): $x_{4,10} = 7$ *or* 8

If $x_{4,9} = 8$, then, 7 should be a label of a pendent node in second $K_{1,3}$ component of G. Let us assume that $x_{3,2} = 7$. Then, $x_{3,1} = 6$. But, $x_{3,2} = 8$ is not possible. Suppose $x_{3,2} = 8$, then, we get the link value $y_{3,2} = 7$, but the link value 7 is already allotted $y_{3,1} = 7$. Hence, $x_{3,2} \neq 8$.

Case(G): $x_{4,8} = 11$ *or* 12

If $x_{4,8} = 12$, then, 11 should be a label of a pendent node in second $K_{1,3}$ component of G. Let us assume that $x_{3,3} = 11$. Then, $y_{3,3} = 8$, but, the link value 8 is already allotted $y_{2,3} = 8$.Then, $x_{3,3} \neq 11$ and similarly $x_{3,3} \neq 12$.

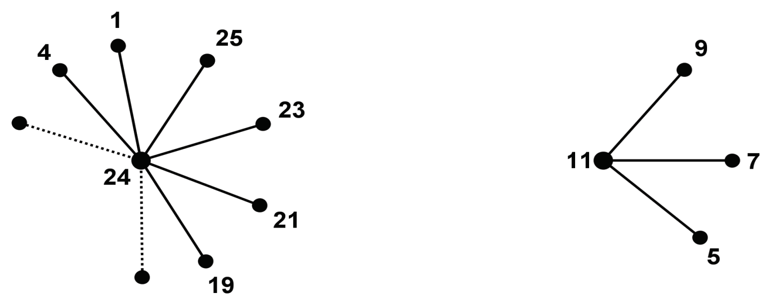

FIGURE 5

Let $x_{4,0} = 24$, $x_{4,1} = 25$, $x_{4,2} = 23$, $x_{4,3} = 21$, $x_{4,13} = 1$, $x_{4,4} = 19$, $x_{4,5} = 18$, $x_{4,6} = 16$, $x_{4,7} = 14$, $x_{4,12} = 4$, $x_{4,8} = 12$, $x_{4,9} = 10$, $x_{4,10} = 8$, $x_{4,11} = 6$, $x_{1,0} = 2$, $x_{1,1} = 22$, $x_{1,2} = 20$, $x_{2,0} = 3$, $x_{2,1} = 17$, $x_{2,2} = 15$, $x_{2,3} = 13$, $x_{3,0} = 5$, $x_{3,1} = 9$, $x_{3,2} = 7$, $x_{3,3} = 11$.

Then, $y_{4,13} = 13$, $y_{4,1} = 25$, $y_{4,2} = 24$, $y_{4,3} = 23$, $y_{1,1} = 12$, $y_{1,2} = 11$, $y_{2,1} = 10$, $y_{2,2} = 9$, $y_{2,3} = 8$,$y_{3,0} = 7$,$y_{3,2} = 6$,$y_{3,3} = 8$.

Suppose that $x_{4,8} = 11$ and one of the unlabeled node should be 12, we know that all the node labels smaller than 5 are already allotted to the nodes. So, giving label greater than 5 to the adjacent node of the unknown node, labeled 12 will induce a link label 9, but, 9 is already the link label of $y_{2,2}$, which fails the bijection of the labeling defined. Obviously, $G = 2K_{1,3} \cup K_{1,2} \cup K_{1,13}$ is not a skolem mean graph for $x_{4,0} = 24$.

A similar argument can prove that G is not a skolem mean graph, when, $x_{4,0}$ takes other values as such the edges $y_{4,j}$ get the higher values.

Hence, we failed to generate a skolem mean labeling for $G = 2K_{1,3} \cup K_{1,2} \cup K_{1,13}$, even, when the $K_{1,13}$ component of G takes the smaller of the values. Hence, $G = 2K_{1,3} \cup K_{1,2} \cup K_{1,13}$ is not a skolem mean graph, when, G assumes smaller as well as greater values. Hence, $G = 2K_{1,3} \cup K_{1,2} \cup K_{1,13}$ is not a skolem mean graph. That is, G is not a skolem mean graph, when, $|s - t| = 5 + 2r$. In a similar way, we shall prove that $G = 2K_{1,4} \cup K_{1,2} \cup K_{1,15}$ is also not a skolem mean graph. Argumently, we may assert that graph with bigger difference between s and t will never make a skolem mean graph.

Hence, the four star $G = K_{1,r} \cup K_{1,r} \cup K_{1,s} \cup K_{1,t}$ is not a skolem mean graph, if $|s - t| > 4 + 2r$ for $r = 2, 3 \cdots$; $s = 1, 2 \cdots$. □

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