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INTEGRAL INEQUALITIES OF HADAMARD TYPE FOR SUB E-FUNCTIONS

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ABSTRACT. In this paper, we show that the power function of sub E-function $f^n(x)$ is sub E-function. Furthermore, we establish some new integral inequalities of Hadamard type involving sub E-functions and concave E-functions.

1. INTRODUCTION

Let $f : I \to \mathbb{R}$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. There are many generalizations of the notion of convex functions see [3,4,7,10]. One way to generalize the notion of convex function is to replace linear functions by another family of functions in the sense of Beckenbach [3]. In this paper, we deal with a family $\{E(x)\}$ of exponintial functions

$$E(x) = A \exp Bx,$$

where A, B arbitrary constants.

The Hermite-Hadamard integral inequality for convex functions $f : [a, b] \to \mathbb{R}$

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}\,, \tag{1.1}$$

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is well known in the literature and has many applications for special means, see for example [2, 6, 9]. The Hermite-Hadamard integral inequality (1.1) was established for sub E-functions in [1] as

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq L(f(a), f(b)),$$

where, $L(f(a), f(b)) := \dfrac{f(b)-f(a)}{\ln f(b) - \ln f(a)}, \quad f(a), f(b) \geq 0, \quad f(a) \neq f(b).$

In this work, we proved that the higher powers of sub E-function is sub E-function in addition to establish some new integral inequalities of Hadamard type involving sub E-functions and concave E-functions.

2. Definitions and Preliminary Results

In this section, we introduce the basic definitions and results which will be used later. For more informations see [1], [5], [8].

Definition 2.1. *A positive function $f : I \to (0, \infty)$ is called sub E-function on I, if for any $a, b \in I$ with $a < b$ the graph of $f(x)$ for $a < x < b$ lies on or under the graph of a function*

$$E(x) = Ae^{Bx},$$

where A and B are taken so that $E(a) = f(a)$, and $E(b) = f(b)$.
Equivalenty, for all $x \in [a, b]$

$$\begin{aligned} f(x) &\leq E(x) \\ &= exp\left[\frac{(b-x)\ln f(a) + (x-a)\ln f(b)}{b-a}\right]. \end{aligned} \tag{2.1}$$

If the inequality (2.1) holds with " $\geq$ ", then the function will be called concave E-function on I.

Note the following: There is more than one formula for the function $E(x)$ other than that stated in (2.1); for example,

$$E(x) = f(a)e^{B(x-a)};\ B = \frac{\ln f(b) - \ln f(a)}{b-a},$$

or in a multiplicative form

$$E(x) = [f(a)]^{\frac{b-x}{b-a}}.[f(b)]^{\frac{x-a}{b-a}}.$$

Remark 2.1. *The sub E-functions possess a number of properties analogous to those of convex functions. For example: If $f : I \to (0, \infty)$ is sub E-function, then for any $a, b \in I$, the inequality $f(x) \geq E(x)$ holds outside the interval $[a, b]$.*

Definition 2.2. *Let a function $f : I \to (0, \infty)$ be sub E-function. A function*

$$T_u(x) = Ae^{Bx},$$

is said to be supporting function for $f(x)$ at the point $u \in (a, b)$ if

(1) $T_u(u) = f(u)$,
(2) $T_u(x) \leq f(x) \;\; \forall x \in I$.

That is, if $f(x)$ and $T_u(x)$ agree at $x = u$, the graph of $f(x)$ lies on or above the support curve.

Proposition 2.1. *If $f : I \to \mathbb{R}$ is a differentiable sub E-function, then the supporting function for $f(x)$ at the point $u \in I$ has the form*

$$T_u(x) = f(u) \exp\left[(x - u)\frac{f'(u)}{f(u)}\right].$$

Remark 2.2. *For a sub E-function $f : I \to (0, \infty)$, we write the supporting function at $u \in I$ in the following form*

$$T_u(x) = f(u) \exp\left[(x - u)\frac{M_{u,f}}{f(u)}\right].$$

The constant $M_{u,f}$ is equal to $f'(u)$ if f is differentiable at the point $u \in I$; otherwise $f'_-(u) \leq M_{u,f} \leq f'_+(u)$.

Theorem 2.1. *Let $f : I \to (0, \infty)$ be a two times continuously differentiable function. The function f is sub E-function on I if and only if $f(x)f''(x) - (f'(x))^2 \geq 0$ for all x in I.*

Theorem 2.2. *A function $f : I \to (0, \infty)$ is sub E-function on I if and only if there exist a supporting function for $f(x)$ at each point $x \in I$.*

Theorem 2.3. *If a function $f : [a, b] \to \mathbb{R}$ is continuous and g is an integrable function that does not change sign on $[a, b]$, then there exists c in (a, b) such that*

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx.$$

3. Main Results

Theorem 3.1. *Let $f : I \to (0,\infty)$ be sub E-function and two times continuously differentiable then the higher powers of $f(x)$ is sub E-function.*

Proof. Since $f(x)$ is non-negative and sub E-function then,

$$f(x) \geq 0,\ f(x)f''(x) - (f'(x))^2 \geq 0\ \forall x \in I\,. \tag{3.1}$$

$$\begin{aligned}(f^n(x))' &= nf^{n-1}(x)f'(x)\,,\\ (f^n(x))'' &= n(n-1)f^{n-2}(x)(f'(x))^2 + nf^{n-1}(x)f''(x)\,.\end{aligned}$$

Hence,

$$\begin{aligned}f^n(x)(f^n(x))'' - ((f^n(x))')^2 &= f^n(x)[n(n-1)(f^{n-2}(x))(f'(x))^2 + nf^{n-1}(x)f''(x)]\\ &\quad -n^2f^{2n-2}(x)(f'(x))^2\\ &= n(n-1)f^{2n-2}(x)(f'(x))^2 + nf^{2n-1}(x)f''(x)\\ &\quad -n^2f^{2n-2}(x)(f'(x))^2\\ &= nf^{2n-1}(x)f''(x) - nf^{2n-2}(x)(f'(x))^2\\ &= nf^{2n-2}[f(x)f''(x) - (f'(x))^2].\end{aligned}$$

Now using (3.1), we conclude that:

$$f^n(x)(f^n(x))'' - ((f^n(x))')^2 \geq 0\,.$$

Hence, $f^n(x)$ is sub E-function. □

Theorem 3.2. *Let $f : I \to (0,\infty)$ be sub E-function, $n \in \mathbb{N}$ and $a, b \in I$ with $a < b$, then*

$$\frac{f^n(a)}{nB}\left[e^{nB(b-a)}-1\right] \leq \int_a^b f^n(x)dx \leq \frac{f^{n+1}(u)}{nf'(u)}\left[\exp\left[n(b-u)\frac{f'(u)}{f(u)}\right]-\exp\left[n(a-u)\frac{f'(u)}{f(u)}\right]\right],$$

where, $B = \dfrac{\ln f(b) - \ln f(a)}{b-a}$.

Proof. Let u an arbitrary point in (a,b). As $f(x)$ is a sub E-function, then from Definition 2.1 we observe that the graph of $f(x)$ lies nowhere above the function

$$E(x) = f(a)e^{B(x-a)};\ B = \frac{\ln f(b) - \ln f(a)}{b-a}\,, \tag{3.2}$$

and nowhere below any supporting function.

$$T_u(x) = f(u)\exp\left[(x-u)\frac{f'(u)}{f(u)}\right] \tag{3.3}$$

at the point $u \in (a,b)$. Thus,

$$T_u(x) \le f(x) \le E(x),\ x \in [a,b].$$

As $f(x)$, $T_u(x)$ are non-negative functions, then

$$T_u^n(x) \le f^n(x) \le E^n(x) \ \ \forall n \in \mathbb{N}.$$

$$\int_a^b T_u^n(x)dx \le \int_a^b f^n(x)dx \le \int_a^b E^n(x)dx. \tag{3.4}$$

Using (3.2), one has

$$\begin{aligned}
\int_a^b f^n(x)dx &\le \int_a^b E^n(x)dx \\
&= \int_a^b f^n(a)e^{nB(x-a)}dx \\
&= \frac{f^n(a)}{nB}e^{nB(x-a)}\Big|_a^b \\
&= \frac{f^n(a)}{nB}\left[e^{nB(b-a)} - 1\right]. \qquad (3.5)
\end{aligned}$$

Using (3.3), (3.4), one obtains:

$$\begin{aligned}
\int_a^b f^n(x)dx &\ge \int_a^b T_u^n(x)dx \\
&= \int_a^b f^n(u)\exp\left[n(x-u)\frac{f'(u)}{f(u)}\right]dx \\
&= f^n(u)\frac{f(u)}{nf'(u)}\left[\exp\left[n(x-u)\frac{f'(u)}{f(u)}\right]\right]\Big|_a^b \\
&= \frac{f^{n+1}(u)}{nf'(u)}\left[\exp\left[n(b-u)\frac{f'(u)}{f(u)}\right] - \exp\left[n(a-u)\frac{f'(u)}{f(u)}\right]\right]. \qquad (3.6)
\end{aligned}$$

Hence, from (3.4), (3.5), (3.6) we get the required inequality. □

Theorem 3.3. *If $f : I \to (0,\infty)$ is sub E-function on I then,*

$$f\left(\frac{x+y}{2}\right) \le \sqrt{f(x)f(y)},\ \forall x,y \in I.$$

Proof. For all $a, b \in I$ with $a < b$, from Definition 2.1, let $x = \dfrac{a+b}{2}$

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \exp\left[\frac{\frac{b-a}{2}\ln f(a) + \frac{b-a}{2}\ln f(b)}{b-a}\right] \\ &= \exp\left[\frac{\ln f(a) + \ln f(b)}{2}\right] \\ &= \exp\left[\frac{\ln f(a)f(b)}{2}\right] \\ &= \sqrt{f(a)f(b)}. \end{aligned}$$

□

Theorem 3.4. *Let $f,\ g : I \to (0,\infty)$ be continuous, sub E-functions on I, $a,\ b \in I$ with $a < b$, $c_1, c_2 \in (a,b)$ and $\alpha,\ \beta > 0$ with $\alpha + \beta = 1$. Then the following inequality holds*

$$\int_a^b f(x)g(x)dx \leq \alpha f(a)g(c_1)\left[\frac{1}{B_1}(e^{B_1(b-a)} - 1)\right] + \beta g(a)f(c_2)\left[\frac{1}{B_2}(e^{B_2(b-a)} - 1)\right].$$

Proof. Since f, g are sub E-functions, we have

$$f(x) \leq f(a)e^{B_1(x-a)},\ B_1 = \frac{\ln f(b) - \ln f(a)}{b-a}, \tag{3.7}$$

$$g(x) \leq g(a)e^{B_2(x-a)},\ B_2 = \frac{\ln g(b) - \ln g(a)}{b-a}, \tag{3.8}$$

multiplying both sides of (3.7) and (3.8) by $\alpha g(x)$ and $\beta f(x)$ respectively and adding the resulting inequalities we get

$$f(x)g(x) \leq \alpha f(a)g(x)e^{B_1(x-a)} + \beta g(a)f(x)e^{B_2(x-a)}. \tag{3.9}$$

Integrating both sides of (3.9) with respect to x from a to b, we get

$$\int_a^b f(x)g(x)dx \leq \alpha f(a)\int_a^b g(x)e^{B_1(x-a)}dx + \beta g(a)\int_a^b f(x)e^{B_2(x-a)}dx.$$

Let $c_1, c_2 \in (a,b)$, by using integral form of mean value theorem, we get

$$\begin{aligned} \int_a^b f(x)g(x)dx &\leq \alpha f(a)g(c_1)\int_a^b e^{B_1(x-a)}dx + \beta g(a)f(c_2)\int_a^b e^{B_2(x-a)}dx, \\ &= \alpha f(a)g(c_1)\left[\frac{1}{B_1}(e^{B_1(b-a)} - 1)\right] + \beta g(a)f(c_2)\left[\frac{1}{B_2}(e^{B_2(b-a)} - 1)\right]. \end{aligned}$$

Hence, the theorem follows. □

Theorem 3.5. *Let $f,\ g : I \to (0,\infty)$ be continuous, sub E-functions on I, $a,\ b \in I$ with $a < b$ and $\alpha,\ \beta > 0$ with $\alpha+\beta = 1$. Then the following inequality holds:*

$$\begin{aligned}\int_a^b f(x)g(x)dx \ &\geq\ \frac{\alpha}{M_{a,f}} f^2(a)g(c_1)\Big[\exp\frac{M_{a,f}(b-a)}{f(a)} - 1\Big]\\ &+\ \frac{\beta}{M_{a,g}} g^2(a)f(c_2)\Big[\exp\frac{M_{a,g}(b-a)}{g(a)} - 1\Big].\end{aligned}$$

Proof. Since f, g are sub E-functions on I, from Definition 2.2, we have that $\forall x, y \in I$

$$f(x) \ \geq\ f(y)\exp\left[(x-y)\frac{M_{y,f}}{f(y)}\right], \tag{3.10}$$

$$g(x) \ \geq\ g(y)\exp\left[(x-y)\frac{M_{y,g}}{g(y)}\right], \tag{3.11}$$

where $M_{y,f}$ is a fixed real number depending on y, f. Multiplying both sides of (3.10) and (3.11) by $\alpha g(x)$ and $\beta f(x)$ respectively and adding the resulting inequalities, we get

$$f(x)g(x) \geq \alpha g(x)f(y)\exp\left[(x-y)\frac{M_{y,f}}{f(y)}\right] + \beta f(x)g(y)\exp\left[(x-y)\frac{M_{y,g}}{g(y)}\right], \tag{3.12}$$

by taking $y = a$ in (3.12), we get

$$f(x)g(x) \geq \alpha g(x)f(a)\exp\left[(x-a)\frac{M_{a,f}}{f(a)}\right] + \beta f(x)g(a)\exp\left[(x-a)\frac{M_{a,g}}{g(a)}\right]. \tag{3.13}$$

Integrating both sides of (3.13) with respect to x from a to b, we get

$$\int_a^b f(x)g(x)dx \geq \alpha f(a)\int_a^b g(x)\exp\left[(x-a)\frac{M_{a,f}}{f(a)}\right]dx + \beta g(a)\int_a^b f(x)\exp\left[(x-a)\frac{M_{a,g}}{g(a)}\right]dx,$$

Let $c_1, c_2 \in (a,b)$, by using integral form of mean value theorem, we get

$$\begin{aligned}&\int_a^b f(x)g(x)dx \geq \alpha f(a)g(c_1)\int_a^b \exp\left[(x-a)\frac{M_{a,f}}{f(a)}\right]dx + \beta g(a)f(c_2)\int_a^b \exp\left[(x-a)\frac{M_{a,g}}{g(a)}\right]dx,\\ &= \frac{\alpha}{M_{a,f}} f^2(a)g(c_1)\Big[\exp\frac{M_{a,f}(b-a)}{f(a)} - 1\Big] + \frac{\beta}{M_{a,g}} g^2(a)f(c_2)\Big[\exp\frac{M_{a,g}(b-a)}{g(a)} - 1\Big].\end{aligned}$$

Hence, the theorem follows. □

Theorem 3.6. *Let $f : I \to (0,\infty)$ be sub E-function on I, $g : I \to (0,\infty)$ be concave E-function on I, $a, b \in I$ with $a < b$ and $\alpha > 1$ with $\alpha+\beta = 1$. Then the* *following inequality holds*

$$\begin{aligned}\int_a^b f(x)g(x)dx &\geq \alpha f(\frac{a+b}{2})\int_a^b g(x)\exp\left[(x-\frac{a+b}{2})\frac{M(\frac{a+b}{2},f)}{f(\frac{a+b}{2})}\right]dx\\ &+ \beta g(\frac{a+b}{2})\int_a^b f(x)\exp\left[(x-\frac{a+b}{2})\frac{M(\frac{a+b}{2},g)}{g(\frac{a+b}{2})}\right]dx.\end{aligned}$$

Proof. Since f is sub E-function on I and g is concave E-function on I, we have that $\forall x, y \in I$

$$f(x) \geq f(y)\exp\left[(x-y)\frac{M_{y,f}}{f(y)}\right], \tag{3.14}$$

$$g(x) \leq g(y)\exp\left[(x-y)\frac{M_{y,g}}{g(y)}\right], \tag{3.15}$$

where $M_{y,f}$ is a fixed real number depending on y, f. Multiplying both sides of (3.14) and (3.15) by $\alpha g(x)$ and $\beta f(x)$ respectively and adding the resulting inequalities, we get

$$f(x)g(x) \geq \alpha g(x)f(y)\exp\left[(x-y)\frac{M_{y,f}}{f(y)}\right]+\beta f(x)g(y)\exp\left[(x-y)\frac{M_{y,g}}{g(y)}\right]. \tag{3.16}$$

By taking $y = \dfrac{a+b}{2}$ in (3.16), hence

$$\begin{aligned}f(x)g(x) &\geq \alpha g(x)f(\frac{a+b}{2})\exp\left[(x-\frac{a+b}{2})\frac{M(\frac{a+b}{2},f)}{f(\frac{a+b}{2})}\right]\\ &+ \beta f(x)g(\frac{a+b}{2})\exp\left[(x-\frac{a+b}{2})\frac{M(\frac{a+b}{2},g)}{g(\frac{a+b}{2})}\right],\end{aligned} \tag{3.17}$$

integrating both sides of (3.17) with respect to x from a to b, we get the desired inequality

$$\begin{aligned}\int_a^b f(x)g(x)dx &\geq \alpha f(\frac{a+b}{2})\int_a^b g(x)\exp\left[(x-\frac{a+b}{2})\frac{M(\frac{a+b}{2},f)}{f(\frac{a+b}{2})}\right]dx\\ &+ \beta g(\frac{a+b}{2})\int_a^b f(x)\exp\left[(x-\frac{a+b}{2})\frac{M(\frac{a+b}{2},g)}{g(\frac{a+b}{2})}\right]dx,\end{aligned}$$

□

Theorem 3.7. *Let* $f, g : I \to (0, \infty)$ *be sub* E*-functions on* I*,* $a, b \in I$ *with* $a < b$ *and* $\alpha, \beta > 0$ *with* $\alpha + \beta = 1$*. Then the following inequality holds*

$$\begin{aligned}\int_a^b f(x)g(x)dx \;&\geq\; \alpha f(\frac{a+b}{2}) \int_a^b g(x) \exp\left[(x - \frac{a+b}{2})\frac{M(\frac{a+b}{2},f)}{f(\frac{a+b}{2})}\right] dx \\ &+\; \beta g(\frac{a+b}{2}) \int_a^b f(x) \exp\left[(x - \frac{a+b}{2})\frac{M(\frac{a+b}{2},g)}{g(\frac{a+b}{2})}\right] dx.\end{aligned}$$

Proof. Since $f,\ g$ are sub E-functions on $I,$ from Definition 2.2, we have that $\forall x, y \in I$

$$f(x) \;\geq\; f(y) \exp\left[(x - y)\frac{M_{y,f}}{f(y)}\right], \tag{3.18}$$

$$g(x) \;\geq\; g(y) \exp\left[(x - y)\frac{M_{y,g}}{g(y)}\right], \tag{3.19}$$

where $M_{y,f}$ is a fixed real number depending on y, f. Multiplying both sides of (3.18) and (3.19) by $\alpha g(x)$ and $\beta f(x)$ respectively and adding the resulting inequalities, we get

$$f(x)g(x) \geq \alpha g(x)f(y) \exp\left[(x-y)\frac{M_{y,f}}{f(y)}\right] + \beta f(x)g(y) \exp\left[(x-y)\frac{M_{y,g}}{g(y)}\right]. \tag{3.20}$$

By taking $y = \frac{a+b}{2}$ in (3.20), hence

$$\begin{aligned}f(x)g(x) \;&\geq\; \alpha g(x)f(\frac{a+b}{2}) \exp\left[(x - \frac{a+b}{2})\frac{M(\frac{a+b}{2},f)}{f(\frac{a+b}{2})}\right] \\ &+\; \beta f(x)g(\frac{a+b}{2}) \exp\left[(x - \frac{a+b}{2})\frac{M(\frac{a+b}{2},g)}{g(\frac{a+b}{2})}\right],\end{aligned} \tag{3.21}$$

integrating both sides of (3.21) with respect to x from a to b, we get

$$\begin{aligned}\int_a^b f(x)g(x)dx \;&\geq\; \alpha f(\frac{a+b}{2}) \int_a^b g(x) \exp\left[(x - \frac{a+b}{2})\frac{M(\frac{a+b}{2},f)}{f(\frac{a+b}{2})}\right] dx \\ &+\; \beta g(\frac{a+b}{2}) \int_a^b f(x) \exp\left[(x - \frac{a+b}{2})\frac{M(\frac{a+b}{2},g)}{g(\frac{a+b}{2})}\right] dx\,.\end{aligned}$$

□

Theorem 3.8. *Let $f,\ g$ and $h : I \to (0,\infty)$ be sub E-functions on I and $a, b \in I$ with $a < b$. Then the following inequality holds*

$$\begin{aligned}
3\int_a^b f(x)g(x)h(x)dx \;&\geq\; f(\frac{a+b}{2})\int_a^b g(x)h(x)\exp\left[(x-\frac{a+b}{2})\frac{M(\frac{a+b}{2},f)}{f(\frac{a+b}{2})}\right]dx\\
&+\; g(\frac{a+b}{2})\int_a^b f(x)h(x)\exp\left[(x-\frac{a+b}{2})\frac{M(\frac{a+b}{2},g)}{g(\frac{a+b}{2})}\right]dx\\
&+\; h(\frac{a+b}{2})\int_a^b f(x)g(x)\exp\left[(x-\frac{a+b}{2})\frac{M(\frac{a+b}{2},h)}{h(\frac{a+b}{2})}\right]dx.
\end{aligned}$$

Proof. Since f, g and h are sub E-functions on I, from Definition 2.2, we have $\forall x, y \in I$

$$f(x) \;\geq\; f(y)\exp\left[(x-y)\frac{M_{y,f}}{f(y)}\right], \tag{3.22}$$

$$g(x) \;\geq\; g(y)\exp\left[(x-y)\frac{M_{y,g}}{g(y)}\right], \tag{3.23}$$

$$h(x) \;\geq\; h(y)\exp\left[(x-y)\frac{M_{y,h}}{h(y)}\right], \tag{3.24}$$

multiplying both sides of (3.22), (3.23) and (3.24) by $g(x)h(x)$, $f(x)h(x)$ and $f(x)g(x)$ respectively and adding the resulting inequalities

$$\begin{aligned}
3f(x)g(x)h(x) \;&\geq\; g(x)h(x)f(y)\exp\left[(x-y)\frac{M_{y,f}}{f(y)}\right]\\
&+\; f(x)h(x)g(y)\exp\left[(x-y)\frac{M_{y,g}}{g(y)}\right]\\
&+\; f(x)g(x)h(y)\exp\left[(x-y)\frac{M_{y,h}}{h(y)}\right].
\end{aligned} \tag{3.25}$$

Now, if we choose $y = \dfrac{a+b}{2}$ in (3.25), we obtain

$$\begin{aligned} 3f(x)g(x)h(x) &\geq g(x)h(x)f(\frac{a+b}{2})\exp\left[(x-\frac{a+b}{2})\frac{M(\frac{a+b}{2},f)}{f(\frac{a+b}{2})}\right] \\ &+ f(x)h(x)g(\frac{a+b}{2})\exp\left[(x-\frac{a+b}{2})\frac{M(\frac{a+b}{2},g)}{g(\frac{a+b}{2})}\right] \\ &+ f(x)g(x)h(\frac{a+b}{2})\exp\left[(x-\frac{a+b}{2})\frac{M(\frac{a+b}{2},h)}{h(\frac{a+b}{2})}\right]. \end{aligned} \tag{3.26}$$

Integrating both sides of (3.26) with respect to x from a to b, we get

$$\begin{aligned} 3\int_a^b f(x)g(x)h(x)dx &\geq f(\frac{a+b}{2})\int_a^b g(x)h(x)\exp\left[(x-\frac{a+b}{2})\frac{M(\frac{a+b}{2},f)}{f(\frac{a+b}{2})}\right]dx \\ &+ g(\frac{a+b}{2})\int_a^b f(x)h(x)\exp\left[(x-\frac{a+b}{2})\frac{M(\frac{a+b}{2},g)}{g(\frac{a+b}{2})}\right]dx \\ &+ h(\frac{a+b}{2})\int_a^b f(x)g(x)\exp\left[(x-\frac{a+b}{2})\frac{M(\frac{a+b}{2},h)}{h(\frac{a+b}{2})}\right]dx. \end{aligned}$$

Hence, the theorem follows. □

Theorem 3.9. *Let $f_1, f_2 ...., f_n$ and $h : I \to (0,\infty)$ be sub E-functions on I and $a, b \in I$ with $a < b$. Further, let $\alpha_1, \alpha_2, ..., \alpha_n > 0$ with $\sum_{i=1}^n \alpha_i = 1$. Then the following inequality holds*

$$\begin{aligned} \int_a^b \prod_{i=1}^n f_i(x)dx &\geq \alpha_1 f_1(\frac{a+b}{2})\int_a^b f_2(x)f_3(x)...f_n(x)\exp\left[(x-\frac{a+b}{2})\frac{M(\frac{a+b}{2},f_1)}{f_1(\frac{a+b}{2})}\right]dx \\ &+ \alpha_2 f_2(\frac{a+b}{2})\int_a^b f_1(x)f_3(x)...f_n(x)\exp\left[(x-\frac{a+b}{2})\frac{M(\frac{a+b}{2},f_2)}{f_2(\frac{a+b}{2})}\right]dx \\ &\vdots \\ &+ \alpha_n f_n(\frac{a+b}{2})\int_a^b f_1(x)f_2(x)...f_{n-1}(x)\exp\left[(x-\frac{a+b}{2})\frac{M(\frac{a+b}{2},f_n)}{f_n(\frac{a+b}{2})}\right]dx\,. \end{aligned}$$

Proof. Since $f_1, f_2, ..., f_n$ are sub E-functions on I, we have $\forall x, y \in I$

$$f_1(x) \geq f_1(y) \exp\left[(x-y)\frac{M_{y,f_1}}{f_1(y)}\right] \tag{3.27}$$

$$f_2(x) \geq f_2(y) \exp\left[(x-y)\frac{M_{y,f_2}}{f_2(y)}\right], \tag{3.28}$$

$$\vdots$$

$$f_n(x) \geq f_n(y) \exp\left[(x-y)\frac{M_{y,f_n}}{f_n(y)}\right]. \tag{3.29}$$

Multiplying both sides of (3.27), (3.28),... and (3.29) by $\alpha_1 f_2(x) f_3(x)...f_n(x)$, $\alpha_2 f_1(x) f_3(x)...f_n(x) \ldots$, and $\alpha_n f_1(x) f_2(x)...f_{n-1}(x)$ respectively and adding the resulting inequalities

$$\begin{aligned}\prod_{i=1}^{n} f_i(x) &\geq \alpha_1 f_2(x) f_3(x)...f_n(x) f_1(y) \exp\left[(x-y)\frac{M_{y,f_1}}{f_1(y)}\right] \\ &+ \alpha_2 f_1(x) f_3(x)...f_n(x) f_2(y) \exp\left[(x-y)\frac{M_{y,f_2}}{f_2(y)}\right] \\ &\vdots \\ &+ \alpha_n f_1(x) f_2(x)...f_{n-1}(x) f_n(y) \exp\left[(x-y)\frac{M_{y,f_n}}{f_n(y)}\right].\end{aligned} \tag{3.30}$$

Now, if we choose $y = \dfrac{a+b}{2}$ in (3.30), we obtain

$$\begin{aligned}\prod_{i=1}^{n} f_i(x) &\geq \alpha_1 f_2(x) f_3(x)...f_n(x) f_1(\frac{a+b}{2}) \exp\left[(x-\frac{a+b}{2})\frac{M_{\frac{a+b}{2},f_1}}{f_1(\frac{a+b}{2})}\right] \\ &+ \alpha_2 f_1(x) f_3(x)...f_n(x) f_2(\frac{a+b}{2}) \exp\left[(x-\frac{a+b}{2})\frac{M_{\frac{a+b}{2},f_2}}{f_2(\frac{a+b}{2})}\right] \\ &\vdots \\ &+ \alpha_n f_1(x) f_2(x)...f_{n-1}(x) f_n(\frac{a+b}{2}) \exp\left[(x-\frac{a+b}{2})\frac{M_{\frac{a+b}{2},f_n}}{f_n(\frac{a+b}{2})}\right].\end{aligned} \tag{3.31}$$

Integrating both sides of (3.31) with respect to x from a to b, we get the desired inequality. □

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