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QUASI - P-NORMAL COMPOSITION AND WEIGHTED COMPOSITION OPERATORS ON THE COMPLEX HILBERT SPACE AND n POWER CLASS (Q) OPERATORS AND COMPOSITION OPERATORS ON THE HILBERT SPACE

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ABSTRACT. In this paper we characterized quasi- P-normal operators, composition operators, weighted composition operators and almost similarity of operators in the complex Hilbert space.

1. INTRODUCTION

Let H be a Complex Hilbert Space. An operator T on H is called normal if $T^*T = TT^*$, quasi normal if $TT^*T = T^*TT$. i.e., T^*T commute with T, quasi P-normal if $T^*T(T+T^*) = (T+T^*)T^*T$, and it is denoted by [QPN], 2 power normal operator if. $T^2T^* = T^*T^2$ An operator T is class (Q) operator if ${T^*}^2T^2 = (T^*T)^2$, see [1–5].

Composition Operator: Let $L^2(\lambda) = L^2(X, \sum, \lambda)$ where $(X, \sum, \lambda)$ is a σ - finite measure space. The relation almost everywhere, denoted by a.e is an equivalence relation in $L^2(\lambda)$. Let T be a non singular measurable transformation from X onto itself. Then the composition operator C_T on the space $L^2(\lambda)$ induced by T is given by $C_Tf = f \circ T$ for all $f \in L^2(\lambda)$ and $C_T^*f = h.E(f) \circ T^{-1}$ for all $f \in L^2(\lambda)$.

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Weighted Composition Operator: A Weighted Composition Operator W is a linear transformation acting on a set of complex valued $\sum$ - measurable functions f of the form $Wf = \phi C_T f = \phi(f \circ T)$, where ϕ is a complex valued $\sum$- measurable function. When $\phi = 1$, we say W is a composition operator. The adjoint operator W^* is defined as $W^* f = h.E(\phi.f) \circ T^{-1}$ for $f \in L^2(\lambda)$. Also $\phi_n = \phi(\phi \circ T)(\phi \circ T^2)(\phi \circ T^3)...(\phi \circ T^{n-1})$. For $f \in L^2(\lambda)$, $W^n f = \phi_n.f \circ T^n$, $W^{*^n} f = h_n.E(\phi_n.f) \circ T^{-n}$.

2. Some characterizations on Quasi - P normal operators.

Theorem 2.1. *Every normal operator T is quasi P- normal operator.*

Theorem 2.2. *If T is a normal operator, then T^* is quasi P- normal operator.*

Theorem 2.3. *Let T be a quasi P - normal operator. If T is 2 power normal operator, then T^* is a quasi P normal operator.*

Proof. Given T is a quasi P - normal operator. Therefore, $[T + T^*, T^*T] = 0.$
i.e, $(T+T^*)T^*T = T^*T(T+T^*)$, which implies that $TT^*T + +T^*T^*T = T^*TT + T^*TT^*$.

$$TT^*T + T^{*^2}T = T^*T^2 + T^*TT^* \,. \tag{2.1}$$

Also, T is 2 - power normal operator. Therefore,

$$T^2T^* = T^*T^2 \,. \tag{2.2}$$

Taking adjoint on both sides,

$$TT^{*^2} = T^{*^2}T \,. \tag{2.3}$$

When we substitute (2.2) and (2.3) in (2.1), we get:

$$\begin{aligned}
&TT^*T + TT^{*^2} = T^2T^* + T^*TT^* \\
&\Rightarrow TT^*T + TT^*T^* = TTT^* + T^*TT^* \\
&\Rightarrow TT^*(T + T^*) = (T + T^*)TT^* \\
&\Rightarrow [T + T^*, TT^*] = 0 \,.
\end{aligned}$$

Hence, T^* is quasi P normal operator. □

2.1. **Quasi P normal Composition operators on the Complex Hilbert space.**

Theorem 2.4. *Let C_T be a composition operator on $L^2(\lambda)$. Then the following statements are equivalent.*

(i) *C_T^* is quasi - P - normal operator.*
(ii) $h^2.E(f)]\circ T^{-1}+h\circ T^2.E(f)\circ T=(h\circ T).E(h).E(f)\circ T^{-1}+h\circ T+E[f\circ T]$.

Proof. Suppose, for $f\in L^2(\mu)$, C_T^* is quasi - P - normal operator. Then $(C_T^*+C_T)(C_TC_T^*)f=(C_TC_T^*)(C_T^*+C_T)f$, and we have:

$$\begin{aligned}
(C_T^*+C_T)(C_TC_T^*)f &= C_T^*(C_TC_T^*)f+C_T(C_TC_T^*)f\\
&= C_T^*C_T(h.E(f)\circ T^{-1})+C_TC_T(h.E(f)\circ T^{-1})\\
&= C_T^*(h.E(f)\circ T^{-1})\circ T+C_T(h.E(f)\circ T^{-1})\circ T\\
&= C_T^*(h\circ T.E(f))+C_T(h\circ T.E(f))\\
&= h.E[h\circ T.E(f)]\circ T^{-1}+(h\circ T.E(f))\circ T\\
&= h.h.E(f)]\circ T^{-1}+h\circ T^2.E(f)\circ T\\
&= h^2.E(f)]\circ T^{-1}+h\circ T^2.E(f)\circ T\,.
\end{aligned}$$

$$\begin{aligned}
(C_TC_T^*)(C_T^*+C_T)f &= (C_TC_T^*)C_T^*f+(C_TC_T^*)C_Tf\\
&= C_TC_T^*(h.E(f)\circ T^{-1})+C_TC_T^*(f\circ T)\\
&= C_T[h.E[h.E(f)\circ T^{-1}]\circ T^{-1}]+C_TC_T^*(f\circ T)\\
&= C_T[h.E[h]\circ T^{-1}.E(f)\circ T^{-2}]+C_T[h.E[f\circ T]\circ T^{-1}]\\
&= [h.E[h]\circ T^{-1}.E(f)\circ T^{-2}]\circ T+[h.f]\circ T\\
&= \circ T.E(h).E(f)\circ T^{-1}+h\circ T.f\circ T\,.
\end{aligned}$$

If C_T is quasi - P - normal operator. Then we have,
$(C_T^*+C_T)(C_TC_T^*)f=(C_TC_T^*)(C_T^*+C_T)f$, and this implies that,
$h^2.E(f)]\circ T^{-1}+h\circ T^2.E(f)\circ T=h\circ T.E(h).E(f)\circ T^{-1}+h\circ T.f\circ T$. □

2.2. **Characterizations on Quasi P normal Weighted Composition operators on L^2 space.**

Theorem 2.5. *Let W be a weighted composition operator on $L^2\lambda)$. Then the following statements are equivalent.*

(i) *W is quasi - P - normal operator.*

(ii) $\phi.h \circ T.E[\phi^2].f \circ T + h.E[\phi] \circ T^{-1}.E[h] \circ T^{-1}.E[\phi^2] \circ T^{-2}.E(f) \circ T^{-1} =$ $h.E[\phi.\phi_2] \circ T^{-1}.\ f \circ T + h.E[\phi^2] \circ T^{-1}.h.E(\phi.f) \circ T^{-1}$.

Proof. For $f \in\in L^2(\mu)$, W is quasi - P - normal operator if $(W + W^*)(W^*W)f = (W^*W)(W + W^*)f$, and we have:

$$\begin{aligned}
(W + W^*)(W^*W)f &= W(W^*W)f + W^*(W^*W)f \\
&= WW^*(\phi.f \circ T) + W^*W^*(\phi.f \circ T) \\
&= W[h.E(\phi.\phi.f \circ T) \circ T^{-1}] + W^*[h.E(\phi.\phi.f \circ T) \circ T^{-1}] \\
&= W[h.E[\phi^2] \circ T^{-1}.f] + W^*[h.E[\phi^2] \circ T^{-1}.f] \\
&= \phi.(h.E[\phi^2] \circ T^{-1}.f) \circ T + h.E[\phi.h.E[\phi^2] \circ T^{-1}.f] \circ T^{-1} \\
&= \phi.h \circ T.E[\phi^2].f \circ T + h.E[\phi] \circ T^{-1}.E[h] \circ T^{-1}.E[\phi^2] \circ T^{-2}.E(f) \circ T^{-1} .
\end{aligned}$$

Consider,

$$\begin{aligned}
(W^*W)(W + W^*)f &= (W^*W)Wf + (W^*W)W^*f \\
&= (W^*W)(\phi.f \circ T) + (W^*W)h.E(\phi.f) \circ T^{-1} \\
&= W^*[\phi.(\phi.f \circ T) \circ T] + W^*[\phi.(h.E(\phi.f) \circ T^{-1}) \circ T] \\
&= W^*[\phi.\phi \circ T.f \circ T^2] + W^*\phi.h \circ T.E(\phi.f) \\
&= W^*[\phi_2.f \circ T^2] + W^*[\phi.h \circ T.E(\phi.f)] \\
&= = h.E[\phi.\phi_2.f \circ T^2] \circ T^{-1} + h.E[\phi.\phi.h \circ T.E(\phi.f)] \circ T^{-1} \\
&= = h.E[\phi.\phi_2] \circ T^{-1}.f \circ T + h.E[\phi^2] \circ T^{-1}.h.E(\phi.f) \circ T^{-1} .
\end{aligned}$$

Suppose, W is quasi - P - normal operator. Then,
$(W + W^*)(W^*W)f = (W^*W)(W + W^*)f$
$\Leftrightarrow \phi.h\circ T.E[\phi^2].f\circ T+h.E[\phi]\circ T^{-1}.E[h]\circ T^{-1}.E[\phi^2]\circ T^{-2}.E(f)\circ T^{-1} = h.E[\phi.\phi_2]\circ T^{-1}.f \circ T + h.E[\phi^2] \circ T^{-1}.h.E(\phi.f) \circ T^{-1}$. $\square$

Theorem 2.6. *Let W be a weighted composition operator on $L^2(\lambda)$. Then the following statements are equivalent.*

(i) *W^* is quasi - P - normal operator.*
(ii) $h.E[\phi^2]\circ T^{-1}.E(f).E(\phi.f)\circ T^{-1}+\phi_2.h\circ T^2.E(\phi.f)\circ T = \phi.h\circ T.E[\phi].E[h].$ $E(\phi.f) \circ T^{-1} + \phi.h \circ T.E[\phi^2].f \circ T$.

Proof. For $f \in\in L^2(\lambda)$, W^* is quasi - P - normal operator if $(W^*+W)(WW^*)f = (WW^*)(W^*+W)f$, and then we have:

$$\begin{aligned}
(W^*+W)(WW^*)f &= W^*(WW^*)f + W(WW^*)f \\
&= W^*Wh.E(\phi.f)\circ T^{-1} + WWh.E(\phi.f)\circ T^{-1} \\
&= W^*\phi.[h.E(\phi.f)\circ T^{-1}]\circ T + W\phi.[h.E(\phi.f)\circ T^{-1}]\circ T \\
&= W^*[\phi.h\circ T.E(\phi.f)] + W[\phi.h\circ T.E(\phi.f)] \\
&= h.E[\phi.\phi.h\circ T.E(\phi.f)]\circ T^{-1} + \phi.[\phi.h\circ T.E(\phi.f)]\circ T \\
&= .E[\phi^2]\circ T^{-1}.h.E(\phi.f)\circ T^{-1} + \phi.(\phi\circ T).h\circ T^2.E(\phi.f)\circ T \\
&= h.E[\phi^2]\circ T^{-1}.h.E(\phi.f)\circ T^{-1} + \phi_2.h\circ T^2.E(\phi.f)\circ T\,.
\end{aligned}$$

Consider,

$$\begin{aligned}
(WW^*)(W^*+W)f &= (WW^*)W^*f + (WW^*)Wf \\
&= (WW^*)(h.E(\phi.f)\circ T^{-1} + (WW^*)(\phi.f\circ T) \\
&= Wh.E[\phi.h.E(\phi.f)\circ T^{-1}]\circ T^{-1} + Wh.E(\phi.\phi.f\circ T)\circ T^{-1} \\
&= W[h.E[\phi]\circ T^{-1}.E[h]\circ T^{-1}.E(\phi.f)\circ T^{-2}] + W[h.E[\phi^2]\circ T^{-1}.f] \\
&= \phi.[h.E[\phi]\circ T^{-1}.E[h]\circ T^{-1}.E(\phi.f)\circ T^{-2}]\circ T + \phi.[h.E[\phi^2]\circ T^{-1}.f]\circ T \\
&= \phi.h\circ T.E[\phi].E[h].E(\phi.f)\circ T^{-1} + \phi.h\circ T.E[\phi^2].f\circ T\,.
\end{aligned}$$

Suppose, W^* is quasi - P - normal operator. Then,
$(W^*+W)(WW^*)f = (WW^*)(W^*+W)f$
$\Leftrightarrow h.E[\phi^2]\circ T^{-1}.E(f).E(\phi.f)\circ T^{-1} + \phi_2.h\circ T^2.E(\phi.f)\circ T = \phi.h\circ T.E[\phi].E[h].$
$E(\phi.f)\circ T^{-1} + \phi.h\circ T.E[\phi^2].f\circ T$ □

2.3. **Characterizations on Almost similar and similar operators.**

Definition 2.1. *Almost similar operators [6]: Two bounded linear operators T and S of a Banach Algebra on a Hilbert space H (i.e S and $T\in B(H)$) are said to be almost similar if there exists an invertible operator N such that the following two conditions are satisfied.* $S^*S = N^{-1}(T^*T)N$ *and* $S^*+S = N^{-1}(T^*+T)N$.

Remark 2.1. *Almost similarity of an operator is also an equivalence relation.*

Theorem 2.7. *Let S be a Quasi P - normal operator,* $T\in B(H)$ *and* $S\in B(H)$. *If S almost similar to T then T is Quasi P - normal operator.*

Proof. Since, ST almost, from the definition, we have: $S^*S = N^{-1}(T^*T)N$ and $S^* + S = N^{-1}(T^* + T)N$,

$$S^*S(S^* + S) = N^{-1}(T^*T)NN^{-1}(T^* + T)N = N^{-1}(T^*T)(T^* + T)N\,, \tag{2.4}$$

and

$$(S^* + S)S^*S = N^{-1}(T^* + T)NN^{-1}(T^*T)N = N^{-1}(T^* + T)(T^*T)N\,. \tag{2.5}$$

Since S is quasi - P normal operator, by definition we get,
$S^*S(S^* + S) = (S^* + S)S^*S$ and therefore from equations (2.4) and (2.5) we get,
$N^{-1}(T^*T)(T^* + T)N = N^{-1}(T^* + T)(T^*T)N$. Since N is invertible, pre multiply by N and post multiply by N^{-1}, we obtain $(T^*T)(T^* + T) = (T^* + T)(T^*T)$. Hence T is also Quasi P - normal operator. □

3. n POWER CLASS (Q) OPERATORS

3.1. **Characterizations on n power class (Q) operators on the Hilbert space.** Let H be a Hilbert space and L(H) be the algebra of all bounded linear operators acting on H. An operator T in L(H) is called normal if $T^*T = TT^*$, class (Q) if $T^{*^2}T^2 = (T^*T)^2$, n power class (Q) if $T^{*^2}T^{2n} = (T^*T^n)^2$. If T is not n power class (Q) operator but T^* is also n power class (Q) operator for some $n \in N$. We verify this result through the following example for n=2.

Example 1. *An operator T=* $\begin{pmatrix} i & 2 \\ 0 & -i \end{pmatrix}$ *acting on C^2 is not power class (Q) but 2 power class(Q) operator and the adjoint of T is 2 power class(Q) operator.*
Solution: Given, T = $\begin{pmatrix} i & 2 \\ 0 & -i \end{pmatrix}$*, from this we get* $T^* = \begin{pmatrix} -i & 0 \\ 2 & i \end{pmatrix}$*, By usual matrix calculations we get,* $T^{*^2}T^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $(T^*T)^2 = \begin{pmatrix} 5 & -12i \\ 12i & 29 \end{pmatrix}$,
$T^{*^2}T^2 \neq (T^*T)^2$*, Hence, T is not power class (Q) operator. Further, we have:*

$T^{*^2}T^4 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = (T^*T^2)^2$. *It follows that T is 2 power class(Q) operator. Also* $T^2T^(*^4) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = (TT^{*^2})^2$. *Hence* T^* *is also 2 power class (Q) operator.*

3.2. **Characterizations on n power class (Q) composition operators on the Hilbert space.**

Theorem 3.1. *Let* $C_T \in B(L^2(\mu))$. *Then* C_T *is in n power class (Q) if and only if* $h.E[h] \circ T^{-1}.f \circ T^{2n-2} = h.h \circ T^{n-1}.f \circ T^{2n-2}$.

Proof. C_T is in n power class (Q) $\Leftrightarrow C_T^{*^2} C_T^{2n} f = (C_T^* C_T^n)^2 f$.

$$
\begin{aligned}
C_T^{*^2} f \circ T^{2n} &= (C_T^* C_T^n) C_T^* C_T^n f \\
C_T^* h.E[f \circ T^{2n}] \circ T^{-1} &= (C_T^* C_T^n) C_T^* (f \circ T^n) \\
C_T^* h.f \circ T^{2n-1} &= (C_T^* C_T^n) h.E(f \circ T^n) \circ T^{-1} \\
C_T^* [h.f \circ T^{2n-1}] &= (C_T^* C_T^n) [h.f \circ T^{n-1}] \\
h.E[h.f \circ T^{2n-1}] \circ T^{-1} &= C_T^* C_T^{n-1} [h.f \circ T^{n-1}] \circ T \\
h.E[h] \circ T^{-1}.f \circ T^{2n-2} &= C_T^* C_T^{n-1} [(h \circ T).f \circ T^n] \\
&= C_T^* C_T^{n-2} [(h \circ T).(f \circ T^n)] \circ T \\
&= C_T^* C_T^{n-2} [(h \circ T^2).(f \circ T^(n+1))] \\
&= C_T^* C_T^(n-3)[(h \circ T^3).(f \circ T^(n+2))] \\
&= C_T^* C_T^(n-4)[(h \circ T^4).(f \circ T^(n+3))] \\
&\dots \\
&= C_T^* [(h \circ T^n).(f \circ T^{2n-1})] \\
&= h.E[(h \circ T^n).(f \circ T^{2n-1})] \circ T^{-1} \\
h.E[h] \circ T^{-1}.f \circ T^{2n-2} &= h.h \circ T^{n-1}.f \circ T^{2n-2} .
\end{aligned}
$$

□

Remark 3.1. *Let* $C_T \in B(L^2(\mu))$. *We can easily verify that* $C_T^{*^n} f = h.E[h] \circ T^{-1}.E[h] \circ T^{-2}$ ..... $E[h] \circ T^{-(n-1)} E(f) \circ T^n]$. *This remark is used for proving the following theorem.*

Theorem 3.2. *Let $C_T \in B(L^2(\mu))$. Then C_T^* is in n power class (Q) if and only if* $h \circ T^2.E[h] \circ T.E[h].E[h] \circ T^{-1}$ ..... $E[h] \circ T^{-(2n-3)}.E[f] \circ T^{-(2n-2)} = h \circ T.E[h].E[h] \circ T^{-1}.E[h] \circ T^{-2}$ ..... $E[h] \circ T^{-(n-2)}.E[h] \circ T^{-(n-2)}E[h] \circ T^{-n+1}$ ..... $E[h] \circ T^{-(2n-3)}.E[f] \circ T^{-(2n-2)}$.

Proof. C_T^* is in n power class $(Q) \Leftrightarrow C_T^2 C_T^{*^{2n}} f = (C_T C_T^{*^n})^2 f$. Then:

$$\begin{aligned} C_T^2 C_T^{*^{2n}} f &= C_T^2[h.E[h] \circ T^{-1}.E[h] \circ T^{-2} \ldots E[h] \circ T^{-(2n-1)}.E[f] \circ T^{-2n}] \\ &\ldots \\ &= h \circ T^2.E[h] \circ T.E[h].E[h] \circ T^{-1} \ldots E[h] \circ T^{-(2n-3)}.E[f] \circ T^{-(2n-2)} . \end{aligned}$$

Further,

$$\begin{aligned} (C_T C_T^{*^n})^2 f &= (C_T C_T^{*^n}) C_T C_T^{*^n} f \\ &\ldots \\ &= h \circ T.E[h].E[h] \circ T^{-1}.E[h] \circ T^{-2} \ldots \\ &\quad \ldots E[h] \circ T^{-(n-2)}.E[h] \circ T^{-(n-2)} E[h] \circ T^{-n+1} \\ &\quad \ldots E[h] \circ T^{-(2n-3)}.E[f] \circ T^{-(2n-2)} . \end{aligned}$$

□

3.3. **Characterizations on n power class (Q) weighted composition operators.**

Theorem 3.3. *Let W be a weighted composition operator on $B(L^2(\mu))$. Then W is n power class (Q) weighted composition operator if and only if* $h.E[\phi] \circ T^{-1}.E[h] \circ T^{-1}.E[\phi.\phi_{2n}] \circ T^{-2}.f \circ T^{2n-2} = h.E[\phi.\phi_n] \circ T^{-1}.h \circ T^{n-1}.E[\phi.\phi_n] \circ T^{n-2}.f \circ T^{2n-2}$.

Proof. Now consider,

$$\begin{aligned} W^{*^2} W^{2n} f &= W^{*^2}[\phi_{2n}.f \circ T^{2n}] \\ &= W^* h.E[\phi.\phi_{2n}.f \circ T^{2n}] \circ T^{-1} \\ &= W^* h.E[\phi.\phi_{2n}] \circ T^{-1}.f \circ T^{2n-1} \\ &= h.E[\phi.h.E[\phi.\phi_{2n}] \circ T^{-1}.f \circ T^{2n-1}] \circ T^{-1} \\ &= h.E[\phi] \circ T^{-1}.E[h] \circ T^{-1}.E[\phi.\phi_{2n}] \circ T^{-2}.f \circ T^{2n-2} . \end{aligned}$$

Next, consider,

$$\begin{aligned}
(W^*W^n)^2f &= (W^*W^n)W^*W^nf \\
&= (W^*W^n)W^*[\phi_n.f\circ T^n] \\
&= (W^*W^n)h.E[\phi.(\phi_n.f\circ T^n)]\circ T^{-1} \\
&= (W^*W^n)h.E[\phi.\phi_n]\circ T^{-1}.f\circ T^{n-1} \\
&= W^*W^n[h.E[\phi.\phi_n]\circ T^{-1}.f\circ T^{n-1}] \\
&= W^*\phi_n.[h.E[\phi.\phi_n]\circ T^{-1}.f\circ T^{n-1}]\circ T^n \\
&= W^*[\phi_n.h\circ T^n.E[\phi.\phi_n]\circ T^{n-1}.f\circ T^{2n-1}] \\
&= h.E[\phi.(\phi_n.h\circ T^n.E[\phi.\phi_n]\circ T^{n-1}.f\circ T^{2n-1})]\circ T^{-1} \\
&= h.E[\phi.\phi_n]\circ T^{-1}.h\circ T^{n-1}.E[\phi.\phi_n]\circ T^{n-2}.f\circ T^{2n-2}\,.
\end{aligned}$$

Given W is n power class (Q) weighted composition operator

$$\Leftrightarrow W^{*^2}W^{2n}f=(W^*W^n)^2f$$

$h.E[\phi]\circ T^{-1}.E[h]\circ T^{-1}.E[\phi.\phi_{2n}]\circ T^{-2}.f\circ T^{2n-2} = h.E[\phi.\phi_n]\circ T^{-1}.h\circ T^{n-1}.E[\phi.\phi_n]\circ T^{n-2}.f\circ T^{2n-2}$ □

Theorem 3.4. *Let W be a weighted composition operator on $B(L^2(\mu))$. Then W^* is n power class (Q) weighted composition operator if and only if* $\phi_2.h\circ T^2.E[\phi]\circ T.E[h]\circ T.E[\phi].E[h]$ .... $E[\phi]\circ T^{-(2n-3)}.E[h]\circ T^{-(2n-3)}E[\phi.f]\circ T^{-(2n-2)} = \phi.h\circ T.E[\phi].E[h].E[\phi]\circ T^{-1}.E[h]\circ T^{-1}E[\phi]\circ T^{-2}.E[h]\circ T^{-2}$ ..... $E[\phi]\circ T^{-(n-2)}.E[h]\circ T^{-(n-2)}E[\phi]\circ T^{-n+1}.E[\phi]\circ T^{-n+1}E[h]\circ T^{-(n-2)}.E[\phi]\circ T^{-n+1}.E[h]\circ T^{-n+1}.E[\phi]\circ T^{n}.E[h]\circ T^{n}$ ..... $E[\phi]\circ T^{-2n+1}.E[h]\circ T^{-2n+1}E[\phi.f]\circ T^{-(2n-2)}$.

Proof. Now if we consider,

$$\begin{aligned}
W^2W^{*^{2n}}f &= W^2[h.E[\phi]\circ T^{-1}.E[h]\circ T^{-1}.E[\phi]\circ T^{-2}.E[h]\circ T^{-2}\dots \\
&\quad \dots \\
&\quad E[\phi]\circ T^{-(2n-1)}.E[h]\circ T^{-(2n-1)}E[\phi.f]\circ T^{-2n} \\
&\quad \dots \\
&= \phi_2.h\circ T^2.E[\phi]\circ T.E[h]\circ T.E[\phi].E[h] \\
&\quad \dots \\
&\quad E[\phi]\circ T^{-(2n-3)}.E[h]\circ T^{-(2n-3)}E[\phi.f]\circ T^{-(2n-2)}\,,
\end{aligned}$$

and we consider,

$$\begin{aligned}
(WW^{*^n})^2 f &= (WW^{*^n})WW^{*^n} f \\
&= (WW^{*^n})W[h.E[\phi]\circ T^{-1}.E[h]\circ T^{-1}.E[\phi]\circ T^{-2}.E[h]\circ T^{-2} \\
&\quad \dots \\
&\quad [\phi]\circ T^{-(n-1)}.E[h]\circ T^{-(n-1)}E[\phi.f]\circ T^n] \\
&\quad \dots \\
&= \phi.h\circ T.E[\phi].E[h].E[\phi]\circ T^{-1}.E[h]\circ T^{-1}E[\phi]\circ T^{-2}.E[h]\circ T^{-2} \\
&\quad \dots \\
&\quad E[\phi]\circ T^{-(n-2)}.E[h]\circ T^{-(n-2)}E[\phi]\circ T^{-n+1}.E[\phi]\circ T^{-n+1} \\
&\quad \dot{E}[h]\circ T^{-(n-2)}.E[\phi]\circ T^{-n+1}.E[h]\circ T^{-n+1}.E[\phi]\circ T^n.E[h]\circ T^n \\
&\quad \dots \\
&\quad E[\phi]\circ T^{-2n+1}.E[h]\circ T^{-2n+1}E[\phi.f]\circ T^{-2n+2}.
\end{aligned}$$

Since, W is a weighted composition operator, by definition, $W^2W^{*^{2n}}f = (WW^{*^n})^2 f$ This implies that,
$\phi_2.h\circ T^2.E[\phi]\circ T.E[h]\circ T.E[\phi].E[h].....E[\phi]\circ T^{-(2n-3)}.E[h]\circ T^{-(2n-3)}E[\phi.f]\circ T^{-(2n-2)} = \phi.h\circ T.E[\phi].E[h].E[\phi]\circ T^{-1}.E[h]\circ T^{-1}E[\phi]\circ T^{-2}.E[h]\circ T^{-2}.....E[\phi]\circ T^{-(n-2)}.E[h]\circ T^{-(n-2)}E[\phi]\circ T^{-n+1}.E[\phi]\circ T^{-n+1}E[h]\circ T^{-(n-2)}.E[\phi]\circ T^{-n+1}.E[h]\circ T^{-n+1}.E[\phi]\circ T^n.E[h]\circ T^n.....E[\phi]\circ T^{-2n+1}.E[h]\circ T^{-2n+1}E[\phi.f]\circ T^{-(2n-2)}$. □

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