ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **9** (2020), no.3, 1129–1139
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.3.38 Spec. Issue on RDESTM-2020

THE STEADY-STATE ANALYSIS OF FINITE WAITING SPACE QUEUING SYSTEM OF MULTIPLE PARALLEL CHANNELS IN SERIES WITH BALKING AND RENEGING CONNECTED TO MULTIPLE PARALLEL NON-SERIAL SERVERS WITH BALKING AND RENEGING

MEENU GUPTA[1], MAN SINGH, AND DEEPAK GUPTA

ABSTRACT. We design a queuing model comprised of M serial service channels connected to N non-serial service channels both characterized with balking as well as reneging due to long queue. We introduce multiple parallel servers in serial as well as non-serial channels. The difference-differential equations for this model are formulated and the steady-state solutions for various cases are obtained. Here, the input process is Poisson and depends upon the queue size in serial and non-serial channels. The service time distribution is exponential and the service discipline follows SIRO-rule instead of FIFO-rule. Waiting space is finite.

1. INTRODUCTION

It is being realized to obtain steady-state solutions for the network of queuing process having reneging and balking, as the impatient customers play the important role in present society. In [1] the author introduced the network of queues with customers of different kinds. The notion of impatient customers was introduced by [2] in the steady-state analysis of serial queuing processes with impatient customers. The author in [3] constructed the queuing models with serial and non-serial structure and obtained the steady-state solutions and

[1]*corresponding author*
Key words and phrases. Probability, Mean service rate, Service phase, Multiple Parallel channel, Operator, Reneging, Balking, Steady-state equation, Steady-state solution.

numerical solutions for mean queue length. In [4] the steady-state solutions for queuing model having multiple parallel channels in series with impatient customers was analyzed. Authors in [5] and [6] introduced the concept of multiple parallel channels in series connected to non-serial channels with balking, reneging and feedback. In present model, we introduce multiple parallel serial channels connected to multiple parallel non-serial channels and obtained steady -state solutions.

2. FORMULATION OF MODEL

The Queuing model analyzed here consists of $Q_i\,(\,i\, = 1, 2, 3, \ldots, M)$ serial service phases where each service phase Q_i has c_i identical parallel service facilities with respective servers S_i, Poisson input rates λ_i, queue size n_i and mean service rate μ_i connected to $Q_{1j}\,($ j $= 1, 2, 3, \ldots, N)$ non-serial service phases where each service phase Q_{1j} has d_j identical parallel service facilities with respective servers S_{1j}, Poisson input rates λ_{1j}, queue size m_j and mean service rate μ_{1j} respectively. Due to Balking the Poisson input rates would be $\dfrac{\lambda_i}{n_i+1}$ and $\dfrac{\lambda_{1j}}{m_j+1}$. After the completion of service at Q_i, the customer either leaves the system with probability $p_i$or joins the next phase with probability $q_{in_{i+1}} = \dfrac{q_i}{n_{i+1}+1}$; such that $p_i + \dfrac{q_i}{n_{i+1}+1} = 1; (i = 1, 2, \ldots, M-1)$. After completion of service at Q_M, the customer either leaves the system with probability p_M or joins any of the Q_{1j} with probability $\dfrac{q_{Mj}}{m_j+1}$ such that $p_M + \displaystyle\sum_{j=1}^{N} \dfrac{q_{Mj}}{m_j+1} = 1$.

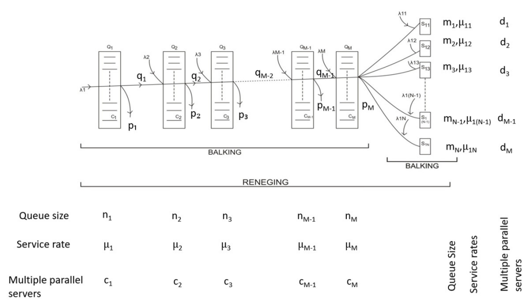

FIGURE 1. Diagrammatic Representation of the Queuing Model

3. Notations

$\tilde{n} = (n_1, n_2, \ldots\ldots\ldots\ldots, n_M)$	$\tilde{m} = (m_1, m_2, \ldots\ldots\ldots\ldots, m_N)$
$T_{\cdot i}(\tilde{n}) = (n_1, n_2, .., n_i{+}1, \ldots, n_M)$	$T_{\cdot i,i+1\cdot}(\tilde{n}) = (n_1, .., n_i + 1, n_{i+1} - 1, ., n_M)$
$\delta_{(n-c)} = \begin{bmatrix} 0 & ; & n < c \\ 1 & ; & n \geq c \end{bmatrix}$	$\delta_{(m_j - d_j)} = \begin{bmatrix} 0 & ; & m_j < d_j \\ 1 & ; & m_j \geq d_j \end{bmatrix}$
$\mu_{1j\,m_j} = \begin{bmatrix} m_j \mu_{1j} & ; & m_j < d_j \\ d_j \mu_{1j} & ; & m_j \geq d_j \end{bmatrix}$	$\lambda_{in_i} = \begin{bmatrix} \lambda_i & ; & n_i < c_i \\ \frac{\lambda_i}{n_i+1} & ; & n_i \geq c_i \end{bmatrix}$
$q_{in_i} = \begin{bmatrix} q_i & ; & n_i < c_i \\ \frac{q_i}{n_i+1} & ; & n_i \geq c_i \end{bmatrix}$	$r_{in_i} = \frac{\mu_i e^{\frac{-\mu_i T_{0i}}{n_i}}}{(1-e^{\frac{-\mu_i T_{0i}}{n_i}})} \quad ; i = 1, 2, \ldots\ldots M$
$T_{i\cdot(\tilde{n})} = (n_1, n_2, ., n_i\text{-}1, ., n_M)$	$T'_{\cdot M_i(\tilde{n})} = (n_1, n_2, ., n_{i-1}, ., n_M + 1)$
$\mu_{1j\,m_j} = \begin{bmatrix} m_j \mu_{1j} & ; & m_j < d_j \\ d_j \mu_{1j} & ; & m_j \geq d_j \end{bmatrix}$	$\lambda_{1jm_j} = \begin{bmatrix} \lambda_{1j} & ; & m_j < d_j \\ \frac{\lambda_{1j}}{m_j+1} & ; & m_j \geq d_j \end{bmatrix}$
$R_{jm_j} = \frac{\mu_{1j}\, e^{\frac{-\mu_{1j} T_{0j}}{m_j}}}{(1-e^{\frac{\mu_{1j} T_{0j}}{m_j}})}; \quad j = 1, 2, \ldots, N$	Here r_{in_i} and R_{jm_j} are the average rates at which the customers renege after a wait of certain time T_{0i} and T_{0j} whenever there are n_i and m_j customers in the queues Q_i and Q_{1j} respectively

4. Formulation of Equations

Define $P\,(n_1, n_2, n_3, \ldots, n_{M-1}, n_M;\; m_1, m_2, m_3, \ldots, m_{N-1}, m_N; t)$ as the probability that at time 't', there are n_i customers waiting in the Q_i service phase before the servers S_i, which may either leave the system after being serviced by the Q_i phase or join the next service phase; n_M customers waiting in the service channel Q_M, which may leave the system after being serviced by server S_M or join any of the queues Q_{1j}; m_j customers waiting before the servers S_{1j}, which may leave the system after the completion of the service. The customer may balk or renege anywhere in serial as well as non-serial channels. We write difference differential equations as:

$$\begin{aligned}
\frac{d}{dt}P(\tilde{n},\tilde{m},t) &= -\left[\begin{array}{l}\sum_{i=1}^{M}\lambda_{in_i}+\sum_{j=1}^{N}\lambda_{1j\,m_j}+\sum_{i=1}^{M}\delta\left(n_i\right)\left(\mu_{in_i}+\delta_{n_i-c_i}r_{in_i}\right)\\ +\sum_{j=1}^{N}\delta\left(m_j\right)\left(\mu_{1jm_j}+\delta_{m_j-d_j}R_{jm_j}\right)\end{array}\right]\\
&\quad P(\tilde{n},\tilde{m};t)\\
&+\sum_{i=1}^{M}\lambda_{i(n_i-1)}P(T_i\,.\left(\tilde{n}\right),\tilde{m};t)\\
&+\sum_{j=1}^{N}\lambda_{1j(m_j-1)}P(\tilde{n},T_j\,.\left(\tilde{m}\right);t)\\
&+\sum_{i=1}^{M}\delta_{n_i-c_i}(r_{i(n_i+1)})P(T.{}_i\left(\tilde{n}\right),\tilde{m};t)\\
&+\sum_{i=1}^{M-1}q_{i(n_{i+1}-1)}\mu_{i(n_i+1)}P(T.{}_{i\,,\,i+1}\,.\left(\tilde{n}\right),\tilde{m};t)\\
&+\sum_{i=1}^{M}p_i\mu_{i(n_i+1)}P(T.{}_i\;\left(\tilde{n}\right),\tilde{m};t)\\
&+\sum_{j=1}^{N}\mu_{M(n_M+1)}q_{Mj(m_j+1)}P(n_1,n_2,\ldots,n_M+1,T_j\,.\left(\tilde{m}\right);t)\\
&+\sum_{j=1}^{N}\left(\mu_{1j(m_j+1)}+\delta_{m_j-d_j}R_{j(m_j+1)}\right)P(\tilde{n},T.{}_j\left(\tilde{m}\right);t)
\end{aligned}\tag{4.1}$$

for $\left(\sum_{i=1}^{M}n_i+\sum_{j=1}^{N}m_j\right)<K$; and

$$\begin{aligned}
\frac{d}{dt}P(\tilde{n},\tilde{m};t) &= -\left[\sum_{i=1}^{M}\delta\left(n_i\right)\left(\mu_{in_i}+\delta_{n_i-c_i}r_{in_i}\right)\right.\\
&+\left.\sum_{j=1}^{N}\delta\left(m_j\right)\left(\mu_{1j\,m_j}+\delta_{m_j-d_j}R_{jm_j}\right)\right]P(\tilde{n},\tilde{m};t)\\
&+\sum_{i=1}^{M}\lambda_{i(n_i-1)}P(T_i\,.\left(\tilde{n}\right),\tilde{m};t)+\sum_{j=1}^{N}\lambda_{1j\,(m_j-1)}P(\tilde{n},T_j\,.\left(\tilde{m}\right);t)
\end{aligned}$$

$$+ \sum_{i=1}^{M-1} q_{i(n_{i+1}-1)}\mu_{i(n_i+1)}P(T._{i\ ,\ i+1}.(\tilde{n}),\tilde{m};t)$$

$$+ \sum_{j=1}^{N} \mu_{M(n_M+1)}q_{Mj(m_j+1)}P(n_1,n_2,\ldots,n_M+1,T_j.(\tilde{m});t) \tag{4.2}$$

for $\left(\sum_{i=1}^{M} n_i + \sum_{j=1}^{N} m_j\right) = K$.

4.1. **Steady-state equations.** Equating the time derivatives to zero in the equations (4.1) and (4.2) we have:

$$\begin{aligned}
&\left[\sum_{i=1}^{M}\lambda_{in_i} + \sum_{j=1}^{N}\lambda_{1j\ m_j} + \sum_{i=1}^{M}\delta(n_i)\left(\mu_{in_i} + \delta_{n_i-c_i}r_{in_i}\right)\right.\\
&\left.+\sum_{j=1}^{N}\delta(m_j)\left(\mu_{1j\ m_j} + \delta_{m_j-d_j}R_{jm_j}\right)\right]P(\tilde{n},\tilde{m})\\
=\ & \sum_{i=1}^{M}\lambda_{i(n_i-1)}P(T_i.(\tilde{n}),\tilde{m})\\
&+\sum_{j=1}^{N}\lambda_{1j\ (m_j-1)}P(\tilde{n},T_j.(\tilde{m})) + \sum_{i=1}^{M}\delta_{n_i-c_i}r_{i(n_i+1)}P(T._i(\tilde{n}),\tilde{m})\\
&+\sum_{i=1}^{M-1} q_{i(n\ _{i+1}-1)}\mu_{i(n\ _i\ +1)}P(T._{i\ ,\ i+1}.(\tilde{n}),\tilde{m}) + \sum_{i=1}^{M}p_i\mu_{i(n_i+1)}P(T._i(\tilde{n}),\tilde{m})\\
&+\sum_{j=1}^{N}\mu_{M(n_M+1)}q_{Mj(m_j+1)}P(n_1,n_2,\ldots,n_M+1,T_j.(\tilde{m}))\\
&+\sum_{j=1}^{N}\left(\mu_{1j\ (m_j+1)} + \delta_{m_j-d_j}R_{jm_j+1}\right)P(\tilde{n},T._j(\tilde{m}))
\end{aligned} \tag{4.3}$$

for $\left(\sum_{i=1}^{M} n_i + \sum_{j=1}^{N} m_j\right) < K$ and

$$\left[\ \sum_{i=1}^{M}\delta(n_i)\left(\mu_{in\ i} + \delta_{n_i-c_i}r_{in\ i}\right) + \sum_{j=1}^{N}\delta(m_j)\left(\mu_{1j\ m_j} + \delta_{m_j-d_j}R_{jm_j}\right)\ \right]P(\tilde{n},\tilde{m})$$

$$
\begin{aligned}
&= \sum_{i=1}^{M} \lambda_{i(n_i-1)} P(T_i\,.\,(\tilde{n})\,,\tilde{m}) + \sum_{j=1}^{N} \lambda_{1j\,(m_j-1)} P(\tilde{n}, T_j\,.\,(\tilde{m})) \\
&\quad + \sum_{i=1}^{M-1} q_{i(n_{i+1}-1)} \mu_{i(n_i+1)} P(T._{\,i\,,\,i+1}\,.\,(\tilde{n})\,,\tilde{m}) \\
&\quad + \sum_{j=1}^{N} \mu_{M(n_M+1)} q_{Mj(m_j+1)} P(n_1, n_2, ., n_M+1, T_j\,.\,(\tilde{m})) \qquad (4.4)
\end{aligned}
$$

for $\left(\sum\limits_{i=1}^{M} n_i + \sum\limits_{j=1}^{N} m_j\right) = K.$

4.2. **Case 1.** Steady- State Equations- (For $n_i{<}c_i$ and $m_j < d_j$): The equations (4.3) and (4.4) reduces to

$$
\begin{aligned}
&\left[\sum_{i=1}^{M} \lambda_i + \sum_{j=1}^{N} \lambda_{1j} + \sum_{i=1}^{M} \delta(n_i) n_i \mu_i + \sum_{j=1}^{N} \delta(m_j)\,(m_j \mu_{1j})\right] P(\tilde{n}, \tilde{m}) \\
&= \sum_{i=1}^{M} \lambda_i P(T_i\,.\,(\tilde{n})\,,\tilde{m}) + \sum_{j=1}^{N} \lambda_{1j} P(\tilde{n}, T_j\,.\,(\tilde{m})) \\
&\quad + \sum_{i=1}^{M-1} q_i \mu_i (n_i+1) P(T._{\,i\,,\,i+1}\,.\,(\tilde{n})\,,\tilde{m}) \\
&\quad + \sum_{i=1}^{M} p_i \mu_i (n_i+1) P(T._{\,i}\,(\tilde{n})\,,\tilde{m}) \\
&\quad + \sum_{j=1}^{N} \mu_M (n_M+1)\; q_{Mj} P(n_1, n_2, \ldots, n_M+1, T_j\,.\,(\tilde{m})) \\
&\quad + \sum_{j=1}^{N} ((m_j+1)\mu_{1j}) P(\tilde{n}, T._{\,j}\,(\tilde{m}))
\end{aligned}
$$

for $\left(\sum\limits_{i=1}^{M} n_i + \sum\limits_{j=1}^{N} m_j\right) < K$; and

$$
\left[\sum_{i=1}^{M} \delta(n_i) n_i \mu_i + \sum_{j=1}^{N} \delta(m_j)\,(m_j \mu_{1j})\right] P(\tilde{n}, \tilde{m})
$$

$$= \sum_{i=1}^{M} \lambda_i P(T_i\,.\,(\tilde{n})\,,\tilde{m}) + \sum_{j=1}^{N} \lambda_{1j} P(\tilde{n}, T_j\,.\,(\tilde{m}))$$
$$+ \sum_{i=1}^{M-1} q_i \mu_i (n_i+1) P(T.\,_{i\,,\,i+1}\,.\,(\tilde{n})\,,\tilde{m})$$
$$+ \sum_{j=1}^{N} \mu_M (n_M+1) q_{Mj} P(n_1, n_2, \ldots, n_M+1, T_j\,.\,(\tilde{m})) \tag{4.5}$$

for $\left(\sum_{i=1}^{M} n_i + \sum_{j=1}^{N} m_j\right) = K$.

4.2.1. *Steady-State Solutions for* $n_i<c_i$ *and* $m_j < d_j$.

$$P(\tilde{n},\tilde{m}) = P\left(\tilde{0},\tilde{0}\right)\left[\frac{1}{\underline{|n_1}}\left(\frac{\lambda_1}{\mu_1}\right)^{n_1}\right]\left[\frac{1}{\underline{|n_2}}\left(\frac{\lambda_2+q_1\alpha_1^{'}}{\mu_2}\right)^{n_2}\right]\left[\frac{1}{\underline{|n_3}}\left(\frac{\lambda_3+q_2\alpha_2^{'}}{\mu_3}\right)^{n_3}\right]$$
$$\ldots\left[\frac{1}{\underline{|n_M}}\left(\frac{\lambda_M+q_{M-1}\alpha_{M-1}^{'}}{\mu_M}\right)^{n_M}\right]\left[\frac{1}{\underline{|m_1}}\left\{\frac{(\lambda_{11}+\mu_M q_{M1}\rho_M)}{(\mu_{11})}\right\}^{m_1}\right]$$
$$\left[\frac{1}{\underline{|m_2}}\left\{\frac{(\lambda_{12}+\mu_M q_{M2}\rho_M)}{(\mu_{12})}\right\}^{m_2}\right]\ldots\left[\frac{1}{\underline{|m_N}}\left\{\frac{(\lambda_{1N}+\mu_M q_{MN}\rho_M)}{(\mu_{1N})}\right\}^{m_N}\right] \tag{4.6}$$

where $\rho_M = \dfrac{\lambda_M + q_{M-1}\alpha_{M-1}^{'}}{\mu_M}; \alpha_1^{'} = \lambda_1; \alpha_k^{'} = \lambda_k + q_{k-1}\alpha_{k-1}^{'}; k = 2, 3, \ldots, M-1$.

4.3. **Case 2.** Steady-State Equations (For $n_i \geq c_i$ and $m_j \geq d_j$)
The equations (4.3) and (4.4) reduce to:

$$\left[\sum_{i=1}^{M}\frac{\lambda_i}{n_i+1} + \sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j+1} + \sum_{i=1}^{M}(c_i\mu_i + r_{in_i}) + \sum_{j=1}^{N}\left(d_j\mu_{1j} + R_{jm_j}\right)\right]P(\tilde{n},\tilde{m})$$
$$= \sum_{i=1}^{M}\frac{\lambda_i}{n_i}P(T_i\,.\,(\tilde{n})\,,\tilde{m}) + \sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j}P(\tilde{n},T_j\,.\,(\tilde{m})) + \sum_{i=1}^{M} r_{i(n_i+1)}P(T.\,_i\,(\tilde{n})\,,\tilde{m})$$
$$+ \sum_{i=1}^{M-1}\frac{q_i}{n_{i+1}}\mu_i c_i P(T.\,_{i\,,\,i+1}\,.\,(\tilde{n})\,,\tilde{m}) + \sum_{i=1}^{M} p_i\mu_i c_i P(T.\,_i\,(\tilde{n})\,,\tilde{m})$$
$$+ \sum_{j=1}^{N}\mu_M c_M \frac{q_{Mj}}{m_j}P(n_1, n_2, \ldots, n_M+1, T_j\,.\,(\tilde{m}))$$

$$+\sum_{j=1}^{N}\left(d_j\mu_{1j}+R_{j(m_j+1)}\right)P(\tilde{n},T._j\,(\tilde{m}))\,; \tag{4.7}$$

for $\left(\sum_{i=1}^{M} n_i+\sum_{j=1}^{N} m_j\right)<K$; and

$$\left[\sum_{i=1}^{M}\left(c_i\mu_i+r_{in_i}\right)+\sum_{j=1}^{N}\left(d_j\mu_{ij}+R_{jm_j}\right)\right]P(\tilde{n},\tilde{m})$$
$$=\sum_{i=1}^{M}\frac{\lambda_i}{n_i}P(T_i\,.\,(\tilde{n})\,,\tilde{m})+\sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j}P(\tilde{n},T_j\,.\,(\tilde{m}))+\sum_{i=1}^{M-1}\frac{q_i}{n_{i+1}}c_i\mu_iP(T._{\,i\,,\,i+1}\,.\,(\tilde{n})\,,\tilde{m})$$
$$+\sum_{j=1}^{N}c_M\mu_M\frac{q_{Mj}}{m_j}P(n_1,n_2,\ldots,n_M+1,T_j\,.\,(\tilde{m})) \tag{4.8}$$

for $\left(\sum_{i=1}^{M} n_i+\sum_{j=1}^{N} m_j\right)=K\,.$

4.3.1. *Steady-State Solution (for $n_i\geq c_i$ and $m_j\geq d_j$).*

$$P\left(\tilde{n},\tilde{m}\right)=P\left(\tilde{0},\tilde{0}\right)\left[\frac{1}{\underline{|n_1}}\left(\frac{(\lambda_1)^{n_1}}{\prod_{i=1}^{n_1}\left(c_1\mu_1+r_{1i}\right)}\right)\right]$$
$$\left[\frac{1}{\underline{|n_2}}\left(\frac{\left\{\lambda_2\left(c_1\mu_1+r_{1(n_1+1)}\right)+c_1q_1\mu_1\alpha_1\right\}^{n_2}}{\prod_{i=1}^{n_2}\left(c_2\mu_2+r_{2i}\right)\left(c_1\mu_1+r_{1(n_1+1)}\right)^{n_2}}\right)\right]$$
$$\left[\frac{1}{\underline{|n_3}}\left(\frac{\left\{\lambda_3\prod_{i=1}^{2}\left(c_i\mu_i+r_{i(n_i+1)}\right)+c_2q_2\mu_2\alpha_2\right\}^{n_3}}{\prod_{i=1}^{n_3}\left(c_3\mu_3+r_{3i}\right)\prod_{i=1}^{2}\left(c_i\mu_i+r_{i(n_i+1)}\right)^{n_3}}\right)\right].$$
$$\left[\frac{1}{\underline{|n_M}}\left(\frac{\left\{\lambda_M\prod_{i=1}^{M-1}\left(c_i\mu_i+r_{i(n_i+1)}\right)+c_{M-1}q_{M-1}\mu_{M-1}\alpha_{M-1}\right\}^{n_M}}{\prod_{i=1}^{n_M}\left(c_M\mu_M+r_{Mi}\right)\prod_{i=1}^{M-1}\left(c_i\mu_i+r_{i(n_i+1)}\right)^{n_M}}\right)\right].$$

$$\left[\frac{\left\{\lambda_{11}+c_M q_{M1}\mu_M\rho_M^{'}\right\}^{m_1}}{\underline{|m_1}\prod_{j=1}^{m_1}\left(d_1\mu_{11}+R_{1j}\right)}\right]\left[\frac{\left\{\lambda_{12}+c_M q_{M2}\mu_M\rho_M^{'}\right\}^{m_2}}{\underline{|m_2}\prod_{j=1}^{m_2}\left(d_2\mu_{12}+R_{2j}\right)}\right].$$

$$\left[\frac{\left\{\lambda_{1N}+c_M q_{MN}\mu_M\rho_M^{'}\right\}^{m_N}}{\underline{|m_N}\prod_{j=1}^{m_N}\left(d_N\mu_{1N}+R_{Nj}\right)}\right]. \tag{4.9}$$

$$\alpha_1=\frac{\lambda_1}{n_1+1}$$

$$\rho_M^{'}=\frac{\lambda_M\prod_{i=1}^{M-1}\left(c_i\mu_i+r_{i(n_i+1)}\right)+c_{M-1}\mu_{M-1}q_{M-1}\alpha_{M-1}}{\left[\left(c_M\mu_M+r_{M(n_M+1)}\right)\prod_{i=1}^{M-1}\left(c_i\mu_i+r_{i(n_i+1)}\right)\right]\left[n_M+1\right]};$$

$$\alpha_k=\frac{\lambda_k\prod_{i=1}^{k-1}\left(c_iu_i+r_{i(n_i+1)}\right)+q_{k-1}\alpha_{k-1}u_{k-1}c_{k-1}}{n_k+1};$$

for $k=2,3,\ldots,M-1$.

Here , it is mentioned that the customers leave the system at constant rate as long as there is a line provided that the customers are served in the order in which they arrive. Putting $R_{jm_j}=R_j\ j=1,2,\ldots N$ in equations (4.5),(4.7) and (4.8) and $r_{in_i}=r_i$ in equations (4.7) and (4.8), the steady-state solutions (4.6) and (4.9) reduce to

$$P\left(\tilde{n},\tilde{m}\right)$$
$$=P\left(\tilde{0},\tilde{0}\right)\left[\frac{1}{\underline{|n_1}}\left(\frac{\lambda_1}{\mu_1}\right)^{n_1}\right]\left[\frac{1}{\underline{|n_2}}\left(\frac{\lambda_2+q_1\alpha_1^{'}}{\mu_2}\right)^{n_2}\right]\left[\frac{1}{\underline{|n_3}}\left(\frac{\lambda_3+q_2\alpha_2^{'}}{\mu_3}\right)^{n_3}\right]\cdots$$
$$\left[\frac{1}{\underline{|n_M}}\left(\frac{\lambda_M+q_{M-1}\alpha_{M-1}^{'}}{\mu_M}\right)^{n_M}\right]\left[\frac{1}{\underline{|m_1}}\left(\frac{\lambda_{11}+\mu_M q_{M1}\rho_M}{\mu_{11}}\right)^{m_1}\right]$$
$$\left[\frac{1}{\underline{|m_2}}\left\{\frac{\left(\lambda_{12}+\mu_M q_{M2}\rho_M\right)}{\left(\mu_{12}\right)}\right\}^{m_2}\right]\cdots\left[\frac{1}{\underline{|m_N}}\left(\frac{\lambda_{1M}+\mu_M q_{MN}\rho_M}{\mu_{1N}}\right)^{m_N}\right];$$

for Case 1 and

$$P\left(\tilde{n},\tilde{m}\right)=P\left(\tilde{0},\tilde{0}\right)\left[\frac{\lambda_1}{\left(c_1\mu_1+r_1\right)}\right]^{n_1}\left[\frac{\lambda_2\left(c_1\mu_1+r_1\right)+c_1q_1\mu_1\alpha_1}{\left(c_1\mu_1+r_1\right)\left(c_2\mu_2+r_2\right)}\right]^{n_2}$$

$$\left[\frac{\lambda_3 \prod\limits_{i=1}^{2}(c_i\mu_i + r_i) + c_2 q_2 \mu_2 \alpha_2}{\prod\limits_{i=1}^{3}(c_i\mu_i + r_i)}\right]^{n_3} \cdots$$

$$\cdots \left[\frac{\lambda_M \prod\limits_{i=1}^{M-1}(c_i\mu_i + r_i) + c_{M-1} q_{M-1} \mu_{M-1} \alpha_{M-1}}{\prod\limits_{i=1}^{M}(c_i\mu_i + r_i)}\right]^{n_M}$$

$$\left[\frac{1}{\lfloor m_1}\left(\frac{\lambda_{11} + c_M q_{M1} \mu_M \rho_M^{'}}{(d_1\mu_{11} + R_1)}\right)^{m_1}\right]$$

$$\left[\frac{1}{\lfloor m_2}\left(\frac{\lambda_{12} + c_M q_{M2} \mu_M \rho_M^{'}}{(d_2\mu_{12} + R_2)}\right)^{m_2}\right]$$

$$\cdots \left[\frac{1}{\lfloor m_N}\left(\frac{\lambda_{1N} + c_M q_{MN} \mu_M \rho_M^{'}}{(d_N\mu_{1N} + R_N)}\right)^{m_N}\right];$$

for Case 2.

Here, $P(\tilde{n}, \tilde{m}; t) = 0$; if any of the arguments is negative. We obtain $P\left(\tilde{0}, \tilde{0}\right)$ from the normalizing condition $\sum\limits_{\tilde{n}=\tilde{0}}^{K} \sum\limits_{\tilde{m}=\tilde{0}}^{K} P(\tilde{n}, \tilde{m}) = 1$and $\sum\limits_{i=1}^{M} n_i + \sum\limits_{l=1}^{N} m_j = K$ and with the restrictions that the traffic intensity of each service channel of the system is less than unity.

REFERENCES

[1] F. P. KELLY: *Network of queues with customers of different type*, Journal of Applied Probability, **12**(3) (1975), 542 – 554.

[2] M. SINGH: *Steady-state behavior of serial queuing processes with impatient customers*, Math Operations forsh, U.Statist. Ser. Statist., **15**(2) (1984), 289 – 298.

[3] M. SINGH, U. SINGH: *Network of serial and Non-serial queuing processes with impatient customers*, JISSOR, **15**(1-4) (1994), 81–96.

[4] M. SINGH, A. AHUJA: *The steady-state solution of multiple parallel channel in series with impatient customers*, Intl. J. Mgmt. Syst., **11**(2) (1995), 81–96.

[5] M. GUPTA, M. SINGH, D. GUPTA: *The steady-state solution of multiple parallel channels in series and non-serial servers both with balking and reneging due to long queue and some urgent message* , Int.j.of Applied Engg. Research, **6**(23) (2012), 2685–2694.

[6] M. GUPTA, M. SINGH, D. GUPTA: *Allocating Feedback Strategies in General Multichannel queuing problem with balking and reneging*, Int.j.of Engg. and Scientific Research, **4**(9) (2016), 42–57.

Department of Mathematics
Dr. B.R. Ambedkar Govt. College, Kaithal
Kaithal, Haryana, India
E-mail address: meenugupta0@gmail.com

Department of Mathematics and Statistics
Ch.Charan Singh Haryana Agricultural University, Hisar
Kaithal, Haryana, India
E-mail address: drmansingh742@gmail.com

Department of Mathematics
Maharishi Markandeshwar University, Mullana
Jagadhari, Haryana, India
E-mail address: guptadeepak2003@yahoo.co.in