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A DEFINITION OF DIRAC DELTA FUNCTIONS

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ABSTRACT. Physicists used to avoid the theory of generalized functions developed by Sobolev and the theory of distributions developed by Schwartz. A simplified mathematical definition for the Dirac delta functions is proposed in this article. This rigourous definition will be acceptable to physicists, because it depends only on elementary concepts, and not on functional analysis. For example, the usual one dimensional Dirac delta function with mass at the origin is identified as the collection of all real valued functions f on the real line having the property $f(b) - f(a) = 1$ whenever $a < 0 < b$.

1. INTRODUCTION

One has to understand unbounded linear operators on an inner product space to study quantum mechanics. One has to know about the Dirac delta function, while he studies quantum mechanics. The concept of the Kronecker delta function for the discrete case was generalized to a concept of a delta function for the continuous case by Dirac [2]. This is now known as the Dirac delta function. Dirac [2] called this function as an improper function, because it is not a classical function, and he listed a set of properties of this special improper function δ. The definition given by him for δ was also an improper one to mathematicians. S. L. Sobolev [6,7] proposed a definition. L. Schwartz [5] refined

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this definition. These two definitions for the Dirac delta function is universally accepted in the world of mathematicians. The definitions of Dirac and Schwartz are given in terms of integrations. Schwartz specifies classes of functions for which integrations are considered when they are multiplied by the Dirac delta function. Specifications of classes fix Dirac delta functions, mathematically. To understand this proper definition of Schwartz for the Dirac delta function, one should understand the concept of continuous linear functionals on locally convex topological vector spaces in terms of inductive limit, in addition to classical calculus. It seems that learning all these functional analysis concepts, directly or indirectly, is a heavy work to the students of physics who study quantum mechanics. The following are stated in section 1.5.2 in the text book [3] of D. J. Grifffths : "In the mathematical literature, it is known as a generalized function, or distribution. It is, if you like, the limit of a sequence of functions, such as $\cdots$". It looks like a general comment. But, this statement is an indirect declaration on sufficiency of an improper definition of the Dirac delta function to go through the course. A definition which can be included in such text books is proposed in this article. It may be considered that there are articles which try to provide new definitions for the Dirac delta function (for example, see [1]). Dirac defines an improper extended real valued function δ on the real line R^1 as an improper function having the following major property: $\int\limits_{-\infty}^{+\infty} f(x)\delta(x)dx = f(0)$, for any continuous real valued function f on the real line R^1. Dirac (section 75, in [2]) also considered three dimensional improper delta function δ_3 on R^3, which was defined by the relation $\delta_3((x,y,z)) = \delta(x)\ \delta(y)\ \delta(z)$, $\forall (x,y,z) \in R^3$, when δ is the one dimensional improper delta function defined on R^1. The definition proposed in this article agrees with this relation. Excellent works done by S. L. Sobolev and L. Schwartz are not sufficient to universally satisfy the following relation mentioned in the book [3] in section 1.5.3: $\nabla.(\frac{\hat{\mathbf{r}}}{r^2}) = 4\pi\delta^3(\mathbf{r})$. Here δ^3 represents the three dimensional improper Dirac delta function, $\mathbf{r} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, $\hat{\mathbf{r}}$ = unit vector for $\mathbf{r}$, and $r^2 = x^2 + y^2 + z^2$ in [3]. This property is a motivation to the definition presented in this article. Another motivation for this definition is based on a simple theorem on integration by parts for multivariables.

2. DIRAC CLOSED SURFACES

The definition for Dirac closed surfaces is presented in this section. This concept is introduced only for a convenient way of introducing the improper Dirac delta function.

Let $\gamma : [a,b] \to R^2$ be a continuous function such that $\gamma(a) = \gamma(b)$ and γ is one - to - one on (a,b). It is called a closed curve. Let us call it positively oriented, if γ determines only two regions, and the bounded region lies in the left side of γ, when t of $\gamma(t)$ transverses from a to b. A function f: $\{\gamma(t) : a \le t \le b\} \to R^1$ is said to be Riemann integrable, if $f \circ \gamma$ is a Riemann integrable function on $[a,b]$. Let $\gamma : [a,b] \to R^2$ be a positively oriented closed curve in R^2 such that

(a) the set γ is the boundary of the bounded open region D determined by γ;
(b) $D \cup \gamma$ is a countable union of closed rectangles whose sides are parallel to the axes (say x-axis and y-axis), and the interior of these rectangles are pairwise disjoint; and
(c) the line integrals for Riemann integrable real valued functions on γ can be defined over γ.

Let us call such closed curves as *Dirac closed curves*.

Lemma 2.1. *Let $f : \Omega - \{p\} \to R^1$ be a continuous real valued function, where Ω is a nonempty simply connected open subset of R^2 and $p \in \Omega$. Suppose that the line integral $\int\limits_{\partial R} f = c$, a constant, for every positively oriented boundary ∂R of a closed rectangular subset R in Ω, for which the sides are parallel to the axes and for which p is in the interior of R. Then for every closed rectangular region R_1 in Ω, with positively oriented boundary ∂R_1, for which the sides are parallel to the axes, and for which p is not in R_1, it is true that $\int\limits_{\partial R_1} f = 0$.*

Proof. Consider two rectangles R and R_1 in Ω, for which the sides are parallel to the axes, and such that $R_1 \cap R$ is a side of R_1 as well as a side of R. Suppose p is in the interior of R such that p is not in R_1. If $R_2 = R \cup R_1$ and ∂R_2 is the positively oriented boundary of R_2, then $\int\limits_{\partial R} f = c = \int\limits_{\partial R_2} f = \int\limits_{\partial R} f + \int\limits_{\partial R_1} f = c + \int\limits_{\partial R_1} f$. Thus $\int\limits_{\partial R_1} f = 0$. □

Lemma 2.2. *Let f, Ω, p and c be as in the previous lemma 2.1. Let γ be a Dirac closed curve in Ω such that the bounded open region enclosed by γ contains p. Then the value of the line integral $\int_{\gamma} f$ is also c.*

Proof. Apply the previous lemma 2.1 and the property (b) given in the definition for Dirac closed curves, along with the continuity of f in $\Omega - \{p\}$. □

The curves in the plane R^2 are considered as 1-dimensional surfaces. One may refer to the book [4] for the definition of $(n-1)$-dimensional positively oriented sufaces in the n-dimensional space R^n, for $n \geq 2$. One can now define $(n-1)$-dimensional Dirac closed surfaces in R^n with an understanding that closed surfaces have zero oriented boundaries. See the book [4] for the meaning of zero oriented boundaries defined through integration.

3. Dirac delta functions

Definition 3.1. *Let $n \geq 2$. Let $p = (p_1, p_2, \cdots, p_n) \in R^n$. Let $c \in R^1$. Let $c\delta_{n(p)}$ denote the collection of all functions f on R^n such that f is Riemann integrable over $(n-1)$-dimensional Dirac closed surfaces and such that the surface integral $\int_S f = c$ for every $(n-1)$-dimensional Dirac closed surface S for which p is in the bounded open region determined by S; or equivalently (in view of the Lemma 2.2 for the case R^n), for every $(n-1)$-dimensional positively oriented closed rectangular surface S of the form*

$$\bigcup_{j=1}^{n} \{(z_1, z_2, \cdots, z_{j-1}, x_j, z_{j+1}, \cdots, z_n) : a_k \leq z_k \leq b_k \ \forall\ k \neq j,\ and\ x_j = a_j\ or\ b_j\},$$

when $a_i < p_i < b_i$, $\forall\ i = 1, 2, \cdots, n$. Let us write $1\delta_{n(p)}$ as $\delta_{n(p)}$. Let the collection $\delta_{n(p)}$ be called the n-dimensional improper Dirac delta function at p.

Let us consider $c\delta_{n(p)}$ as a "scalar multiple" of $\delta_{n(p)}$; a multiple of the scalar c. For any open set U containing p in R^n and for any continuous real valued function f on U, the Dirac integral for $fc\delta_{n(p)}$, or, for $c\delta_{n(p)}f$, over U is denoted by $\int_U fc\delta_{n(p)}$ and defined as the number $cf(p)$. For the constant function $f = 1$, $\int_U c\delta_{n(p)} = c$. If f is a Riemann (Lebesgue) integrable function f on an open set U containing p in R^n, let us define $\int_U (f + c\delta_{n(p)})$ as the number $c + \int_U f(x)dx$, where $\int_U f(x)dx$ is the usual Riemann (Lebesgue) integral.

Definition 3.2. *For $n = 1$, let $p \in R^1$ and $c \in R^1$. Let $c\delta_{1(p)}$ be the collection of all real valued functions f on R^1 such that whenever $a < p < b$, it is true that $f(b) - f(a) = c$. The collection $1\delta_{1(p)}$ is denoted by $\delta_{1(p)}$. $\delta_{1(p)}$ is called the one dimensional improper Dirac delta function. For any open set U containing p in R^1, and for any continuous real valued function f on U, the Dirac integral for $fc\delta_{1(p)}$, or, for $c\delta_{1(p)}f$ over U is denoted by $\int\limits_U fc\delta_{1(p)}$ and defined as the number $cf(p)$. For the constant function $f = 1$, $\int\limits_U c\delta_{1(p)} = c$. Similarly, if in addition f is a Riemann (Lebesgue) integrable function, let us define $\int\limits_U (f + c\delta_{1(p)})$ as the number $c + \int\limits_U f(x)dx$.*

Define $H_{1(p)} : R^1 \to R^1$ by

$$H_{1(p)}(x) = \begin{cases} 0 & \text{if } x < p \\ 1 & \text{if } x \geq p \end{cases}.$$

Then the Heaviside function $H_{1(p)} \in \delta_{1(p)}$ and $cH_{1(p)} \in c\delta_{1(p)}$, when $(cH_{1(p)})(x) = c(H_{1(p)}(x)),\ \forall x \in R^1$. Moreover, the function $g : R^1 \to R^1$ defined by

$$g(x) = \begin{cases} e & \text{if } x < p \\ e+1 & \text{if } x \geq p \end{cases}$$

is also in $\delta_{1(p)}$, for any $e \in R^1$. Define $H_{n(p)} : R^n \to R^1$ by

$$H_{n(p)}(x_1, x_2, \cdots, x_n) = \begin{cases} 0 & \text{if } x_i < p_i \ for\ some\ i \\ 1 & \text{if } x_i \geq p_i \ for\ all\ i \end{cases},$$

when $p = (p_1, p_2, \cdots, p_n)$. Then $H_{n(p)} = H_{1(p)}H_{1(p)} \cdots H_{1(p)} \in \delta_{n(p)}$, when the product has n factors.

For a specific region $U = \prod\limits_{i=1}^{n}(a_i, b_i)$ satisfying $a_i < p_i < b_i$, and for a continuous function $f : U \to R^1$, the Dirac integration $\int\limits_{(a_i,b_i)} f\delta_{1(p_i)} = \int\limits_{a_i}^{b_i} f(x_1, x_2, \cdots, x_n)\delta_{1(p_i)}$ has the value $f(x_1, x_2, \cdots, x_{i-1}, p_i, x_{i+1}, \cdots, x_n)$ for $x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n \in R^1$. Thus the iterated integral

$$\int_{a_n}^{b_n}[\int_{a_{n-1}}^{b_{n-1}}[\cdots[\int_{a_2}^{b_2}[\int_{a_1}^{b_1} f(x_1, x_2, \cdots, x_n)\delta_{1(p_1)}dx_1]\delta_{1(p_2)}dx_2]\cdots]\delta_{1(p_{n-1})}dx_{n-1}]\delta_{1(p_n)}dx_n$$

can be defined, and it is seen that it coincides with $f(p_1, p_2, \cdots, p_n) = \int\limits_U f\ \delta_{n(p)}$.

For the case $n = 1$, the definition depends on the treatment of considering $\delta_{1(p)}$ as a "derivative" of $H_{1(p)}$ and based on "integration by parts" (see [2]). For the general case $\delta_{n(p)}$ is considered as the "partial derivative $\dfrac{\partial^n H_{n(p)}}{\partial x_1\ \partial x_2\ \cdots\ \partial x_n}$". What is the required "integration by parts" for the general case? There are articles which provide multidimensional integration by parts (for example, see [8, 9]). The following obvious theorem 3.1 is sufficient for motivation in defining $\delta_{n(p)}$.

Theorem 3.1. *Let* $f : \prod\limits_{i=1}^{n}[a_i, b_i] \to R^1$ *and* $g : \prod\limits_{i=1}^{n}[a_i, b_i] \to R^1$ *be continuous functions for which* $\dfrac{\partial^n f}{\partial x_1\ \partial x_2\ \cdots\ \partial x_n}$ *and* $\dfrac{\partial^n g}{\partial x_1\ \partial x_2\ \cdots\ \partial x_n}$ *exist and these derivatives are Riemann integrable on* $\prod\limits_{i=1}^{n}[a_i, b_i]$. *Here* $a_i < b_i$, $\forall\ i$. *Then*

$$\int\limits_{a_n}^{b_n}\int\limits_{a_{n-1}}^{b_{n-1}}\cdots\int\limits_{a_2}^{b_2}\int\limits_{a_1}^{b_1} g\frac{\partial^n f}{\partial x_1\ \partial x_2\ \cdots\ \partial x_n}dx_1dx_2\cdots dx_{n-1}dx_n$$
$$= \sum(-1)^{k(z_1,z_2,\cdots,z_n)}f(z_1, z_2, \cdots, z_n)g(z_1, z_2, \cdots, z_n)$$
$$-\int\limits_{a_n}^{b_n}\int\limits_{a_{n-1}}^{b_{n-1}}\cdots\int\limits_{a_2}^{b_2}\int\limits_{a_1}^{b_1} f\frac{\partial^n g}{\partial x_1\ \partial x_2\ \cdots\ \partial x_n}dx_1dx_2\cdots dx_{n-1}dx_n,$$

when the sum is taken over all $(z_1, z_2, \cdots, z_n)$, *when each* z_i *is either* a_i *or* b_i, *and when* $k(z_1, z_2, \cdots, z_n)$ *is the number of* z_i *which are equal to* a_i.

Corollary 3.1. *For f,* $\sum$, *and* $k(z_1, z_2, \cdots, z_n)$ *given as in the Theorem 3.1, it is true that*

$$\int\limits_{a_n}^{b_n}\int\limits_{a_{n-1}}^{b_{n-1}}\cdots\int\limits_{a_2}^{b_2}\int\limits_{a_1}^{b_1} \frac{\partial^n f}{\partial x_1\ \partial x_2\ \cdots\ \partial x_n}dx_1dx_2\cdots dx_{n-1}dx_n$$
$$= \sum(-1)^{k(z_1,z_2,\cdots,z_n)}f(z_1, z_2, \cdots, z_n).$$

Corollary 3.2. *Let f be given as in the theorem 3.3. Let* $x = (x_1, x_2, \cdots, x_n)$ *and* $c = (c_1, c_2, \cdots, c_n)$ *be such that* $x_i \neq c_i, \forall\ i$. *Let* z_i *be* x_i *or* c_i, *for every i, and let* $k(z_1, z_2, \cdots, z_n)$ *denote the number of* z_i *which are equal to* c_i. *Then*

$$\lim_{x\to c}\frac{\sum(-1)^{k(z_1,z_2,\cdots,z_n)}f(z_1, z_2, \cdots, z_n)}{\prod\limits_{i=1}^{n}(x_i - c_i)}$$

exists, when the sum is taken over all the possible $(z_1, z_2, \cdots, z_n)$*, and this limit is equal to* $\dfrac{\partial^n f(c_1, c_2, \cdots, c_n)}{\partial x_1\, \partial x_2\, \cdots\, \partial x_n}$*, when* $\dfrac{\partial^n f}{\partial x_1\, \partial x_2\, \cdots\, \partial x_n}$ *is continuous at c.*

4. CONCLUSIONS

The mathematicians may put additional conditions on the members of $\delta_{n(p)}$, according to their mathematical problems. For example, it may be assumed that all members of $\delta_{n(p)}$ are differentiable (see [4]) almost everywhere, after excluding p. So there is a problem in fixing and characterizing members of $\delta_{n(p)}$. However,the following simplified definition is sufficient to explain fundamental problems in physics, because the integrals defined in the previous section helps one to define integral transforms for the improper Dirac delta functions. Let us call improper Dirac delta functions simply as Dirac delta functions, for the following definition for physicists.

The n-dimensional Dirac delta function with mass at the origin can be defined as the collection of all real valued mappings on R^n whose Riemann integral (so, functions are integrable) value over any $(n-1)$-dimensional positively oriented rectangular closed surface having the origin inside the bounded region determined by the surface is one. The one dimensional Dirac delta function with mass at the origin can be defined as the collection of all real valued functions f on the real line R^1 having the property $f(b) - f(a) = 1$ whenever $a < 0 < b$.

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