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WEYL FRACTIONAL INTEGRAL AND MULTI-INDEX DZRBASHJAN-GELFOND-LEONTIEV (D-G-L) DIFFERENTIATION AND INTEGRATION WITH MULTI-INDEX MITTAG-LEFFLER FUNCTION

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ABSTRACT. The objective of this paper is first to explore linkage that prevails between the Weyl fractional integral and multi-index Dzrbashjan-Gelfond-Leontiev (D-G-L) differentiation and integration with multi-index Mittag-Leffler function. Later, certain striking special cases are attained. The findings of the paper may find application in several fractional differential and integral problems where the D-G-L operators and the multi-index Mittag-Leffler functions are involved.

1. INTRODUCTION AND PRELIMINARIES

Fractional calculus is the area of mathematical analysis that attends to study, inquiry, applications of derivatives and integrals with respect to arbitrary order. Recently, the subject has gain much attention and is being broadly utilized in almost every domain of ordinary fractional calculus, q-transform analysis, solutions of the q-differential and q-integral equations to say the least.

Kiryakova [4,5,6] introduced the multi-index Mittag-Leffler (M-L) function. Also, the multi-index D-G-L differentiation and integration, which is generated by the multi-index M-L function are presented and examined by Kiryakova [7].

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Let n>1 be an integer, $\delta_1, \cdots, \delta_n > 0$ and $\sigma_1, \cdots, \sigma_n$ be arbitrary real numbers then the multi-index Mittag-Leffler function is

$$F_{(\frac{1}{\delta_i}),(\sigma_i)}(x) = \sum_{m=0}^{\infty} \phi_m x^m = \sum_{m=0}^{\infty} \frac{x^m}{\prod_{j=1}^{n} \Gamma(\sigma_j + \frac{m}{\delta_j})}. \tag{1.1}$$

For n=1, the equation (1.1) is the classical Mittag-Leffler function $F_{(\frac{1}{\delta}),(\sigma)}(x)$ and for $\frac{1}{\delta} = \alpha > 0, \sigma = \beta > 0$, (1.1) reduces to Mittag-Leffler function $F_{\alpha,\beta}(x)$ owing to Wiman 1905 and Agarwal 1953.

Let $\Delta_R = \{|y| < R\}$ be a disk and $g(y)$ is an analytic function in it and $\delta_i > 0, \sigma_i \in \Re(i = 1, \cdots, n)$ be parameters which are arbitrary, then the congruity:

$$g(y) = \sum_{m=0}^{\infty} a_m y^m \longmapsto \widetilde{D}g(y) = D_{(\delta_i),(\sigma_i)}g(y), \widetilde{L}g(y) = L_{(\delta_i),(\sigma_i)}g(y),$$

defined by

$$\widetilde{D}g(y) = \sum_{m=1}^{\infty} a_m \frac{\Gamma(\sigma_1 + \frac{m}{\delta_1}) \cdots \Gamma(\sigma_n + \frac{m}{\delta_n})}{\Gamma(\sigma_1 + \frac{m-1}{\delta_1}) \cdots \Gamma(\sigma_n + \frac{m-1}{\delta_n})} y^{m-1}, \tag{1.2}$$

$$\widetilde{L}g(y) = \sum_{m=0}^{\infty} a_m \frac{\Gamma(\sigma_1 + \frac{m}{\delta_1}) \cdots \Gamma(\sigma_n + \frac{m}{\delta_n})}{\Gamma(\sigma_1 + \frac{m+1}{\delta_1}) \cdots \Gamma(\sigma_n + \frac{m+1}{\delta_n})} y^{m+1} \tag{1.3}$$

are titled as the multi-index D-G-L differentiations and integrations respectively which are spawned from multi-index M-L functions.

When n=1, the operators given by equation (1.2) and (1.3) reduces to the D-G-L differentiation and integration, studied by Kiryakova [7], Dimovski and Kiryakova [1,2]:

$$D_{\delta,\sigma}g(y) = \sum_{m=1}^{\infty} a_m \frac{\Gamma(\sigma + \frac{m}{\delta})}{\Gamma(\sigma + \frac{m-1}{\delta})} y^{m-1},$$

$$L_{\delta,\sigma}g(y) = \sum_{m=0}^{\infty} a_m \frac{\Gamma(\sigma + \frac{m}{\delta})}{\Gamma(\sigma + \frac{m+1}{\delta})} y^{m+1}.$$

The Weyl fractional integral operator [3] for $Re(\alpha) > 0$ is given by

$$W_{+}^{\alpha} g(y) = \frac{1}{\Gamma(\alpha)} \int_{y}^{\infty} (z - y)^{\alpha-1} g(z) dz. \tag{1.4}$$

Lemma 1.1. *[5,6] The following relations holds true for multi-index Mittag-Leffler function for $\lambda \neq 0$:*

$$D_{(\delta_i),(\sigma_i)} F_{(\frac{1}{\delta_i}),(\sigma_i)}(\lambda y) = \lambda F_{(\frac{1}{\delta_i}),(\sigma_i)}(\lambda y), \tag{1.5}$$

$$L_{(\delta_i),(\sigma_i)} F_{(\frac{1}{\delta_i}),(\sigma_i)}(\lambda y) = \frac{1}{\lambda} F_{(\frac{1}{\delta_i}),(\sigma_i)}(\lambda y) - \frac{1}{\lambda \prod_{j=1}^{n} \Gamma(\sigma_j)}. \tag{1.6}$$

The lemma can be verified by using (1.1), (1.2) and (1.3).

2. Main Result

In this segment, we bring forth the following relations that exist between Weyl fractional integral and Multi-index Dzrbashjan-Gelfond-Leontiev (D-G-L) operators for Differentiation and Integration with multi-index Mittag-Leffler function.

Theorem 2.1. *Let $W_{+}^{\alpha} g(y)$ be the Weyl fractional integral operator (1.4) and let $\alpha > 0, \delta_i > 0, \sigma_i \in \Re(i = 1, \cdots, n), \lambda \neq 0$, then there holds the formula*

$$\begin{aligned} &\left(W_{+}^{\alpha} \left[z^{-\alpha-\sigma_1} \widetilde{D} F_{(\frac{1}{\delta_i}),(\sigma_i)}(\lambda z^{-\frac{1}{\delta_1}}) \right] \right)(y) \\ = \;& y^{-\sigma_1+\frac{1}{\delta_1}} \left[F_{(\frac{1}{\delta_i}),(\sigma_1+\alpha-\frac{1}{\delta_1}, \sigma_2-\frac{1}{\delta_2}, \cdots, \sigma_n-\frac{1}{\delta_n})}(\lambda y^{-\frac{1}{\delta_1}}) - \frac{1}{\Gamma(\sigma_1+\alpha-\frac{1}{\delta_1}) \prod_{j=2}^{n} \Gamma(\sigma_j - \frac{1}{\delta j})} \right]. \end{aligned} \tag{2.1}$$

Proof. By virtue of (1.4) and (1.5), we have

$$\begin{aligned} \Phi \equiv &\left(W_{+}^{\sigma} \left[z^{-\alpha-\sigma_1} \widetilde{D} F_{(\frac{1}{\delta_i}),(\sigma_i)}(\lambda z^{-\frac{1}{\delta_1}}) \right] \right)(y) \\ = &\frac{1}{\Gamma(\alpha)} \int_{y}^{\infty} (z-y)^{\alpha-1} z^{-\alpha-\sigma_1} \lambda \sum_{m=0}^{\infty} \frac{\lambda^m z^{-\frac{m}{\delta_1}}}{\prod_{j=1}^{n} \Gamma(\sigma_j + \frac{m}{\delta_j})} dz\,. \end{aligned}$$

Swapping the integration and summation order and then assessing with help of beta function formula,

$$\begin{aligned}\Phi &= y^{-\sigma_1}\sum_{m=1}^{\infty}\frac{\lambda^m y^{-\frac{(m-1)}{\delta_1}}}{\lambda(\sigma_1+\alpha+\frac{m-1}{\delta_1})\prod_{j=2}^{n}\Gamma(\sigma_j+\frac{m-1}{\delta_j})}\\
&= y^{-\sigma_1+\frac{1}{\delta_1}}\Bigg[\sum_{m=0}^{\infty}\frac{\lambda^m y^{-\frac{m}{\delta_1}}}{\lambda((\sigma_1+\alpha-\frac{1}{\delta_1})+\frac{m}{\delta_1})\prod_{j=2}^{n}\Gamma((\sigma_j-\frac{1}{\delta j})+\frac{m}{\delta_j})}-\\
&\qquad -\frac{1}{\Gamma(\sigma_1+\alpha-\frac{1}{\delta_1})\prod_{j=2}^{n}\Gamma(\sigma_j-\frac{1}{\delta j})}\Bigg]\\
&= y^{-\sigma_1+\frac{1}{\delta_1}}\left[F_{(\frac{1}{\delta_i}),(\sigma_1+\alpha-\frac{1}{\delta_1},\sigma_2-\frac{1}{\delta_2},\cdots,\sigma_n-\frac{1}{\delta_n})}(\lambda y^{-\frac{1}{\delta_1}})-\frac{1}{\Gamma(\sigma_1+\alpha-\frac{1}{\delta_1})\prod_{j=2}^{n}\Gamma(\sigma_j-\frac{1}{\delta j})}\right]\end{aligned}$$

□

Corollary 2.1. *For* $\alpha>0,\delta_i>0,\sigma_i\in\Re(i=1,\cdots,n),\lambda\neq 0$ *and let* $\dfrac{1}{\delta_1}=\alpha$*, then (2.1) reduces to*

$$\begin{aligned}&\left(W_+^{\alpha}\left[z^{-\alpha-\sigma_1}\widetilde{D}F_{(\alpha,\frac{1}{\delta_2},\cdots,\frac{1}{\delta_n}),(\sigma_i)}(\lambda z^{-\alpha})\right]\right)(y)\\
&= y^{-\sigma_1+\alpha}\left[F_{(\alpha,\frac{1}{\delta_1},\cdots,\frac{1}{\delta_n}),(\sigma_1,\sigma_2-\frac{1}{\delta_2},\cdots,\sigma_n-\frac{1}{\delta_n})}(\lambda y^{-\alpha})-\frac{1}{\Gamma(\sigma_1)\prod_{j=2}^{n}\Gamma(\sigma_j-\frac{1}{\delta j})}\right].\end{aligned}$$

Corollary 2.2. *For* $\alpha>0,\delta>0,\sigma\in\Re,\lambda\neq 0$ *and let* $n=1,\dfrac{1}{\delta_1}=\alpha,\sigma_1=\sigma$ *there holds the formula*

$$\left(W_+^{\alpha}\left[z^{-\alpha-\sigma}D_{\delta,\sigma}F_{\alpha,\sigma}(\lambda z^{-\alpha})\right]\right)(y)=y^{-\sigma+\alpha}\left[F_{\alpha,\sigma}(\lambda y^{-\alpha})-\frac{1}{\Gamma(\sigma)}\right].$$

Theorem 2.2. *Let* $W_+^{\alpha}g(y)$ *be the Weyl fractional integral operator (1.4) and let* $\alpha>0,\delta_i>0,\sigma_i\in\Re(i=1,\cdots,n),\lambda\neq 0$*, then there holds the formula*

$$\begin{aligned}&\left(W_+^{\alpha}\left[z^{-\alpha-\sigma_1}\widetilde{L}F_{(\frac{1}{\delta_i}),(\sigma_i)}(\lambda z^{-\frac{1}{\delta_1}})\right]\right)(y)\\
&=\frac{1}{\lambda}y^{-\sigma_1}\left[F_{(\frac{1}{\delta_i}),(\sigma_1+\alpha,\sigma_2,\cdots,\sigma_n)}(\lambda y^{-\frac{1}{\delta_1}})-\frac{1}{\Gamma(\sigma_1+\alpha)\prod_{j=2}^{n}\Gamma(\sigma_j)}\right].\end{aligned}$$

Proof. By virtue of (1.4) and (1.6), we have

$$\begin{aligned}\Phi &\equiv \left(W_+^{\sigma}\left[z^{-\alpha-\sigma_1}\widetilde{L}F_{(\frac{1}{\delta_i}),(\sigma_i)}(\lambda z^{-\frac{1}{\delta_1}})\right]\right)(y)\\ &= \frac{1}{\Gamma(\alpha)}\int_y^{\infty}(z-y)^{\alpha-1}z^{-\alpha-\sigma_1}\cdot\frac{1}{\lambda}\left[\sum_{m=0}^{\infty}\frac{\lambda^m z^{-\frac{m}{\delta_1}}}{\prod_{j=1}^{n}\Gamma(\sigma_j+\frac{m}{\delta_j})}-\frac{1}{\prod_{j=1}^{n}\Gamma(\sigma_j)}\right]dz\,.\end{aligned}$$

Now the integral is split into two integrals and we interchange the sequence of summation and integration in first integral and then evaluate using beta function formula, we obtain:

$$\begin{aligned}\Phi &= \frac{1}{\lambda}y^{-\sigma_1}\left[\sum_{m=0}^{\infty}\frac{\lambda^m y^{-\frac{m}{\delta_1}}}{\lambda((\sigma_1+\alpha)+\frac{m}{\delta_1})\prod_{j=2}^{n}\Gamma(\sigma_j+\frac{m}{\delta_j})}-\frac{1}{\Gamma(\sigma_1+\alpha)\prod_{j=2}^{n}\Gamma(\sigma_j)}\right]\\ &= \frac{1}{\lambda}y^{-\sigma_1}\left[F_{(\frac{1}{\delta_i}),(\sigma_1+\alpha,\sigma_2,\cdots,\sigma_n)}(\lambda y^{-\frac{1}{\delta_1}})-\frac{1}{\Gamma(\sigma_1+\alpha)\prod_{j=2}^{n}\Gamma(\sigma_j)}\right].\end{aligned}$$

□

Corollary 2.3. *For* $\alpha>0,\delta>0,\sigma\in\Re,\lambda\neq 0$ *and let* $n=1,\frac{1}{\delta_1}=\beta,\sigma_1=\sigma$ *there holds the formula:*

$$\left(W_+^{\alpha}\left[z^{-\alpha-\sigma}L_{\delta,\sigma}F_{\beta,\sigma}(\lambda z^{-\beta})\right]\right)(y)=\frac{1}{\lambda}y^{-\sigma}\left[F_{\beta,\sigma+\alpha}(\lambda y^{-\beta})-\frac{1}{\Gamma(\sigma+\alpha)}\right].$$

REFERENCES

[1] I. DIMOVSKI, V. KIRYAKOVA: *Convolution and commutant of Gelfond-Leontiev operator of integration*, Proc. Constr. Funct. Theory. Varna 1981, Publ. House BAS, Sofia, (1983), 288-294.

[2] I. DIMOVSKI, V. KIRYAKOVA: *Convolution and differential property of the Borel-Dzrbashjan transform*, Proc. Complex Anal. Appl., Varna 1981, Publ. House BAS, Sofia, (1984), 148-156.

[3] K. S. MILLER: *The Weyl fractional calculus., Fractional calculus and its application*, Lecture notes in Math., Springer-Verlag, New york, **457** (1975), 80-89.

[4] V. KIRYAKOVA: *Multi-index Mittag-Leffler functions, related Gelfond-Leontiev operators and Laplace type integral transforms*, Fract. Calc. Appl. Anal., **2**(4) (1999), 445-462.

[5] V. KIRYAKOVA: *Multiple (multi-index) Mittag-Leffler functions and relations to generalized fractional calculus*, J. Comput. Appl. Math., **118** (2000), 241-259.

[6] V. KIRYAKOVA: *The Multi-index Mittag-Leffler functions as an important class of special functions of fractional calculus*, Comput. Math. Appl., **59**(5) (2010), 1885-1895.

[7] V. KIRYAKOVA: *Generalized fractional calculus and applications* (Pitman Research Notes in Mathematics Vol. 301). Proc. Edinburg. Math. Soc., **38**(1) (1995), 189-190.

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