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IMPULSIVE FRACTIONAL DIFFERENCE EQUATION ON hZ

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ABSTRACT. In this paper, we analyze the classical difference equation with impulse

$$\begin{cases} v_{(m+1)h} = v_{mh} + g(mh, v_{mh}), m \in \mathbb{N}_0, m \neq m_j, h > 0 \\ v_{(m_j+1)h} = \bar{v}(m_j+1)h + c_j, m = m_j, j \in \mathbb{N}_1 \\ v(0) = \eta; \end{cases} \tag{0.1}$$

where $\bar{v}_{(m_j+1)h} = v_{m_jh} + f(m_j + v_{m_jh}), 0 < m_0, m_1 < ... < m_2 < ... < m_j < m_{j+1} < ..., 1 < m_{j+1} - m_j$, c_j is a constant and $t = a + (m_j+1)h$ are the impulse points. Since fractional calculus holds the memory effects for $\alpha \neq 1$ and this features can be more easily understood from the sum equation where the kernel function provides coefficients of memory effects. Much more important, many techniques from ordinary differential equations can be applied. And we also generalize the equation (0.1) for the fractional order and obtain more results on discrete fractional case.

1. INTRODUCTION

Fractional calculus, usually determined as fractional order calculus, it is grown from L'Hospital and Leibniz when they have raised a question in the year 1695. It is developed for any order in fractional calculus and has been interesting to scientists and mathematicians for having many applications. The major aim for this results is, they found so many results on derivatives and

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integrals which has many real world phenomena, one can refer [1, 3, 4, 7, 8, 11–14, 21, 22, 25, 30].

In 1989, Miller and Ross [20] initiated the process to develop the theory for fractional finite differences. Further developments took place in 2007 and 2012 when the authors [2,6] put forth several results and a discrete transform method for fractional order difference equation. In the year 2009, Laplace transform is an important concept to find the solution of fractional order equations in the field of stability and chaos [9, 10, 15–19, 31]. For recent developments of fractional difference operators one can refer [24]- [29].

2. Basic Results

Here, we given some of the basic definitions of difference operator and some basic results which will be use further.

Definition 2.1. *For $v(t),\ t \in [0, \infty)$, and $h > 0$, we define*

$$\Delta_h v(t) = \frac{v(t+h) - v(t)}{h},$$

and also defined as

$$\nabla_h v(t) = \frac{v(t) - v(t-h)}{h}.$$

The shifting operators $\sigma_h(t) = t+h$ and $\rho_h(t) = t-h$ are defined for forward and backward difference operator on $h\mathbb{Z}$.

Definition 2.2. *For the real valued functions $v(t)$ and $w(t)$, we defined as*

$$w(t) - w(t-mh) = h\sum_{r=1}^{m} v(t-rh), m \in Z^+$$

and also

$$\Delta_h^{-1} v(t) = h\sum_{r=0}^{\infty} v(t+rh).$$

Definition 2.3. *[5] For $h > 0$ and the real value α, we define*

$$t_h^{(\alpha)} = h^\alpha \frac{\Gamma(\frac{t}{h}+1)}{\Gamma(\frac{t}{h}+1-\alpha)}.$$

Definition 2.4. *Let v be defined on co-domain R and $0 < \alpha$. The finite $h-$sum is defined by*

$$\Delta_{h:a}^{-\alpha} v(t) = \frac{h^{\alpha}}{\Gamma(\alpha)} \sum_{s=0}^{(t-a)/h-\alpha} ((t-a)/h - s - 1)^{(\alpha-1)} v(a+sh).$$

Property 2.1. *For the function $v(t)$, we have*

(i) $\Delta_{h(s)}(t-s)_h^{(\alpha)} = (-\alpha)(t-s-h)_h^{(\alpha-1)}$.

(ii) $\Delta_{h:a+h-\alpha h}^{-\alpha} {}^{C}\Delta_{h:a}^{\alpha} v(t) = v(t) - v(a), 0 < \alpha \leq 1$.

(iii) *The relation between the Caputo and RL sum:*

$$\begin{aligned} {}^{C}\Delta_{h:a}^{\alpha} v(t) &= \Delta_{h:a}^{\alpha}(v(t) - v(a)) \\ &= \Delta_{h:a}^{\alpha} v(t) + v(a)\frac{h^{-\alpha}}{\Gamma(1-\alpha)}((t-a)/h - s)^{-\alpha}\Big|_{s=0}^{(t-a)/h+\alpha} \end{aligned} \tag{2.1}$$

Proof.

(i) From the definition of Δ_h, taking difference with respect to s,

$$\begin{aligned} \Delta_{h(s)}(t-s)_h^{(\alpha)} &= [(t-s-h)_h^{(\alpha)} - (t-s)_h^{(\alpha)}]/h \\ &= h^{\alpha-1}\left\{\frac{\Gamma(\frac{t}{h}-\frac{s}{h})}{\Gamma(\frac{t}{h}-\frac{s}{h}-\alpha)} - \frac{\Gamma(\frac{t}{h}-\frac{s}{h}+1)}{\Gamma(\frac{t}{h}-\frac{s}{h}+1-\alpha)}\right\} \\ &= (-\alpha)(t-s-h)_h^{(\alpha-1)}. \end{aligned}$$

(ii) For $0 < \alpha < 1, m = 1$,

$$\Delta_{h:a+h-\alpha h}^{-\alpha} {}^{C}\Delta_{h:a}^{\alpha} v(t) = \Delta_{h:a+h-\alpha h}^{-\alpha} \Delta_{h:a}^{\alpha-1} \Delta_h v(t) = \Delta_{h:a}^{-1} \Delta_h v(t) = v(t) - v(a).$$

(iii) From Caputo equation, we have

$$\begin{aligned} {}^{C}\Delta_{h:a}^{\alpha} v(t) &= \Delta_{h:a}^{-(m-\alpha)} \Delta_h^m v(t) \\ &= \Delta_{h:a}^{\alpha}(v(t) - v(a)) \\ &= \Delta_{h:a}^{\alpha} v(t) + \frac{v(a)h^{-\alpha}}{\Gamma(-\alpha)}((t-a)/h - s)^{(-\alpha)}\Big|_{s=0}^{(t-a)/h+\alpha}, \end{aligned}$$

which completes the proof. □

Corollary 2.1. *For the function v, we have*

$${}^{C}\Delta_{1:a}^{\alpha} v(t) = \Delta_{1:a}^{\alpha}(v(t) - v(a)) = \Delta_{1:a}^{\alpha} v(t) + \frac{v(a)}{\Gamma(1-\alpha)}(t-a)^{-\alpha} + v(a).$$

Proof. Taking $h = 1$ in (2.1), we get the proof. □

Definition 2.5. *(Fractional Integration) The fractional integration of $v(t)$ is defined by*

$$\int_{t:a}^{(\alpha)} v(s)ds = \Delta_{h\to 0:a}^{-\alpha} v(t) = \lim_{h\to 0} \frac{h^{\alpha}}{\Gamma(\alpha)} \sum_{s=0}^{(t-a)/h-\alpha} ((t-a)/h - s - 1)^{(\alpha-1)} v(a+sh).$$

Definition 2.6. *(Fractional $h:a$ Integration) The fractional $h:a$ integration of a function $v(t): a+\mathbb{N}h \to \mathbb{R}$ for $0<\alpha$ is defined as below: for the function $v(t)$ such that $\Delta_{h:a}^{-\alpha} v(t) = w(t)$, then $w(t)$ is called as $h:a$ integration of $v(t)$ and it is denoted as $\int\limits_{h:a}^{(\alpha)} v(t)$.*

Remark 2.1. *For the function $v(t)$, we have $\lim\limits_{h\to 0} \int\limits_{h:a}^{(\alpha)} v(t) = \int\limits_{t:a}^{(\alpha)} v(s)ds$.*

Example 1. *Let $v(t) = 2^t, h = 1, a = 0, \alpha = 2$. Then*

(i) $\int\limits_{1:0}^{(2)} 2^t = 2^t - 2^0 t - 2^0, t \in 0+2+N,$

(ii) $\int\limits_{1:0}^{(3)} 2^t = 2^t - 2^0 \dfrac{t_1^{(2)}}{2!} - 2^0 \dfrac{t_1^{(1)}}{1!} - 2^0, t \in 0+3+N.$

3. Impulsive $h-$difference equation

Note that $v_{(m+1)h}$ is a sequence for $m = m_j$ and $m \neq m_j$ piecewisely. For example, it starts from $m=0$, we get $v_{mh} = v_{(m-1)h} + g((m-1)h, v_{(m-1)h})$ and $v_{m_1 h} = v_{(m_1-1)h} + g((m_1-1)h, v_{(m_1-1)h})$. We have $v_{(m_1+1)h} = \bar{v}_{(m_1+1)h} + c_1$ for $mh = (m_1+1)h$, obtained by the above relationship at the impulse point.

We obtain the $h-$summation form, for the $h-$difference equation (0.1),

$$\begin{cases} v_{(m+1)h} = v(0) + \sum\limits_{k=0}^{m} g(kh, v_{kh}), m \in \mathbb{N}_0, m \neq m_j, h > 0 \\ v_{(m_j+1)h} = \bar{v}_{(m_j+1)h} + c_j, m = m_j, j \in \mathbb{N}_1 \,. \end{cases} \tag{3.1}$$

We modified the equation as follows,

$$\begin{cases} v_{mh} = \eta + \sum\limits_{k=0}^{m-1} g(kh, v_{kh}), \\ \bar{v}_{(m_1+1)h} = \eta + \sum\limits_{k=0}^{m_1} g(kh, v_{kh}) \end{cases} \tag{3.2}$$

$$v_{(m_1+1)h} = \bar{v}_{(m_1+1)h} + c_1, \tag{3.3}$$

for $m \in \{0, 1, ..., m_1\}$. In the similar manner, for $m \in \{m_1+1, m_1+2, ..., m_2\}$, we seek that

$$\begin{aligned} v_{mh} &= v_{(m_1+1)h} + \sum_{k=m_1+1}^{m-1} g(kh, v_{kh}) \\ &= \bar{v}_{(m_1+1)h} + c_1 + \sum_{k=m_1+1}^{m-1} g(kh, v_{kh}) \\ &= \eta + c_1 + \sum_{k=0}^{m_1} g(kh, v_{kh}) + \sum_{k=m_1+1}^{m-1} g(k, v_{kh}) \\ &= \eta + c_1 + \sum_{k=0}^{m-1} g(kh, v_{kh}), m \in \{m_1+1, m_1+2, ..., m_2\} \end{aligned}$$

and

$$\bar{v}_{(m_2+1)h} = \eta + c_1 + \sum_{k=0}^{m_2} g(kh, v_{kh}). \tag{3.4}$$

In generally, we obtain an equivalent sum equation as follows

$$v_{mh} = \begin{cases} \eta + \sum\limits_{k=0}^{m-1} g(kh, v_{kh}), m \in \{0, 1, 2, ..., m_1\} \\ \eta + c_1 + \sum\limits_{k=0}^{m-1} g(kh, v_{kh}), m \in \{m_1+1, m_1+2, ..., m_2\} \\ \eta + \sum\limits_{i=1}^{j} c_i + \sum\limits_{k=0}^{m-1} g(kh, v_{kh}), m \in \{m_2+1, m_2+2, ..., m_{j+1}\} \\ \eta + \sum\limits_{i=1}^{N} c_i + \sum\limits_{k=0}^{m-1} g(kh, v_{kh}), m \in \{m_N+1, m_N+2, ...\}, N \to \infty \end{cases} . \tag{3.5}$$

In this research article, we use (3.5), since it holds the memory effects for $\alpha \neq 1$.

4. Concepts of impulsive fractional difference equations

Let us take the difference equation

$$h^{\alpha C}\Delta^{\alpha}_{h:a} v(t) = g(t + \alpha h, v(t + \alpha h)), x(a) = x_0, \tag{4.1}$$

it can be rewritten as follows

$$h^{\alpha}\Delta^{\alpha}_{h:a}(v(t) - v(a)) = g(t + \alpha h, v(t + \alpha h))$$

$$i.e \;\; v(t)-v(a)\frac{1}{h^{\alpha}}\Delta_{h:a}^{-\alpha}g(t+\alpha h, v(t+\alpha h))$$

$$v(t)=v(a)+\frac{1}{\Gamma(\alpha)}\sum_{s=1-\alpha}^{(t-a)/h-\alpha}((t-a)/h-s-1)^{(\alpha-1)}g(a+\alpha h+sh, v(a+\alpha h+sh))$$

(4.2)

$$\begin{aligned} v(t) &= v(a)\\ &+\frac{1}{\Gamma(\alpha)}\sum_{s=1-\alpha}^{(t-a)/h-\alpha}\frac{\Gamma((t-a)/h-s)}{\Gamma((t-a)/h-s+1-\alpha)}g(a+\alpha h+sh, v(a+\alpha h+sh)).\end{aligned}$$

Let us consider that $a=0$, for extending the concepts for the difference equations with impulsive

$$\begin{cases} h^{\alpha C}\Delta_{h:a}^{\alpha}v(t)=g(t+\alpha h, v(t+\alpha h)),\\ \qquad t\in a+h-\alpha h+Nh, t\neq a+m_jh+h-\alpha h\\ v_{(m_j+1)h}=\bar{v}_{(m_j+1)h}+c_j, t=a+(m_j+1)h-\alpha h, j\in\mathbb{N}_1\\ v_0=\eta \end{cases} \tag{4.3}$$

or

$$\begin{cases} v(t+h)=v_0\\ \quad +\dfrac{1}{\Gamma(\alpha)}\displaystyle\sum_{s=1-\alpha}^{(t-a)/h+1-\alpha}\frac{\Gamma((t-a)/h+1-s)}{\Gamma((t-a)/h-s+2-\alpha)}g(\alpha h+sh, v(a+\alpha h+sh))\\ v_{(m_j+1)h}=\bar{v}_{(m_j+1)h}+c_j, t=a+m_jh, j\in\mathbb{N}_1\\ v_0=\eta. \end{cases}$$

Since $v_{(m_j+1)h}$ solves the equation ${}^{C}\Delta_{h:a}^{\alpha}v(t)=g(t+\alpha h, v(t+\alpha h))$ with all of the past known information $v(a), v(a+h), ..., v(a+m_jh)$. In the above cases (3.5), (4.1)-(4.3), we take each sum on the interval from $a+(m_j+1)h$ to $a+m_{j+1}h$.

The following lemma proves the solution of fractional difference Cauchy problem for the fractional sum equation.

Lemma 4.1. *For the real valued function* $v(t)$ *is a solution of*

$$\begin{aligned} v(t) &= v(t^*) \\ &- \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{(t^*-a)/h-\alpha} ((t^*-a)/h - s - 1)^{(\alpha-1)} g(a+sh+\alpha h, v(a+sh+\alpha h)) \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{(t-a)/h-\alpha} ((t-a)/h - s - 1)^{(\alpha-1)} g(a+sh+\alpha h, v(a+sh+\alpha h)), \end{aligned}$$

$t \in a + h + \mathbb{N}h$, *iff* $v(t)$ *is also a solution of* $h^{\alpha C}\Delta^{\alpha}_{h:a} v(t) = g(t+\alpha h, v(t+\alpha h))$, $t \in a + h - \alpha h, 0 < \alpha \leq 1,$ *for the constraints*

$$v(a) = v(t^*) - \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{(t^*-a)/h-\alpha} ((t^*-a)/h - s - 1)^{(\alpha-1)} g(a+sh+\alpha h, v(a+sh+\alpha h)).$$

Proof. By (4.2) and taking

$$v_0 = v(a) = v(t^*) - \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{(t^*-a)/h-\alpha} ((t^*-a)/h - s - 1)^{(\alpha-1)} g(a+sh+\alpha h, v(a+sh+\alpha h)).$$

in (4.2), we get the proof. □

Lemma 4.2. *For the function* $v(t)$ *is a solution of* (4.4) *iff* $v(t)$ *is a solution of*

$$v(t) = \begin{cases} v_0 + h^{-\alpha}\Delta^{-\alpha}_{h:a+h-\alpha h} g(t+\alpha h, v(t+\alpha h)), \\ v_0 + c_1 + h^{-\alpha}\Delta^{-\alpha}_{h:a+h-\alpha h} g(t+\alpha h, v(t+\alpha h)), \\ v_0 + \sum\limits_{i=1}^{j} c_i + h^{-\alpha}\Delta^{-\alpha}_{h:a+h-\alpha h} g(t+\alpha h, v(t+\alpha h)), \\ v_0 + \sum\limits_{i=1}^{N} c_i + h^{-\alpha}\Delta^{-\alpha}_{h:a+h-\alpha h} g(t+\alpha h, v(t+\alpha h)), \end{cases} \tag{4.4}$$

Proof. For $t \in \{a+h, ..., a+m_1 h\}$, we have

$$v(t) = v_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{(t-a)/h-\alpha} ((t-a)/h - s - 1)^{(\alpha-1)} g(a+sh+\alpha h, v(a+sh+\alpha h))$$

and

$$\bar{v}_{(m_1+1)h} = v_0 + h^{-\alpha}\Delta^{-\alpha}_{h:a+h-\alpha h} g(t+\alpha h, v(t+\alpha h))\Big|_{t=a+(m_1+1)h}. \tag{4.5}$$

By using Lemma 4.1 and (4.5), for $t \in \{a+(m_1+1)h, a+(m_1+2)h..., a+m_2h\}$,

$$\begin{aligned} v(t) &= v_{(m_1+1)h} - h^{-\alpha}\Delta^{-\alpha}_{h:a+h-\alpha h} g(t+\alpha h, v(t+\alpha h))\Big|_{t=a+(m_1+1)h} \\ &\quad + h^{-\alpha}\Delta^{-\alpha}_{h:a+h-\alpha h} g(t+\alpha h, v(t+\alpha h)) \\ &= v_0 + c_1 + h^{-\alpha}\Delta^{-\alpha}_{h:a+h-\alpha h} g(t+\alpha h, v(t+\alpha h)). \end{aligned}$$

For $t \in \{a+(m_k+1)h, ..., a+m_{k+1}h\}$, we get

$$v(t) = v_0 + \sum_{i=1}^{k} c_i + h^{-\alpha}\Delta^{-\alpha}_{h:a+h-\alpha h} g(t+\alpha h, v(t+\alpha h))$$

For $t \in \{a+(m_{k+1}+1)h, ..., a+m_{k+2}h\}$, which yields

$$\begin{aligned} v(t) &= v_{(m_{k+1}+1)h} - h^{-\alpha}\Delta^{-\alpha}_{h:a+h-\alpha h} g(t+\alpha h, v(t+\alpha h))\Big|_{t=a+(m_{k+1}+1)h} \\ &\quad + h^{-\alpha}\Delta^{-\alpha}_{h:a+h-\alpha h} g(t+\alpha h, v(t+\alpha h)) \\ &= c_{k+1} + \bar{v}_{(m_{k+1}+1)h} - h^{-\alpha}\Delta^{-\alpha}_{h:a+h-\alpha h} g(t+\alpha h, v(t+\alpha h))\Big|_{t=a+(m_{k+1}+1)h} \\ &\quad + h^{-\alpha}\Delta^{-\alpha}_{h:a+h-\alpha h} g(t+\alpha h, v(t+\alpha h)) \\ &= v_0 + \sum_{i=1}^{k+1} c_i + h^{-\alpha}\Delta^{-\alpha}_{h:a+h-\alpha h} g(t+\alpha h, v(t+\alpha h)). \end{aligned}$$

By mathematical induction, we get the proof. □

5. Conclusion

The impulsive difference equation given in (0.1) exists for $\alpha \neq 1$, in this research work we studied and analysed (0.1) by using h-difference operators. Also, we found the solutions for the impulsive difference equations to the fractional sum.

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