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HYBRID IDEALS IN SEMIRINGS

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ABSTRACT. In this paper, we introduce the notions of hybrid ideals, hybrid $k-$ideals and hybrid $k-$ closure in semirings, and investigate some of their properties.

1. PRELIMINARIES

In 1965, Zadeh [12] introduced the concept of fuzzy subsets and studied their properties on the parallel line to set theory. In 1967, Rosenfeld [10] defined the fuzzy subgroup and gave some of its properties. Rosenfeld's definition of a fuzzy group is a turning point for pure mathematicians. Since then, several authors have been pursued the study of fuzzy algebraic structure in many directions such as groups, rings, modules, vector spaces and so on. As a new mathematical tool for dealing with uncertainties, Molodtsov [9] introduced the soft set theory. In the past few years, the fundamentals of soft set theory have been studied by various researchers. As a parallel circuit of fuzzy sets and soft sets, Jun, Song, and Muhiuddin [7] introduced the notion of hybrid structure in a set of parameters over an initial universe set, and applied it to BCK/BCI-algebras and linear spaces.

Later in 2017, S. Anis, M. Khan and Y. B. Jun introduced the notions of hybrid sub-semigroups and hybrid left(resp., right) ideals in semigroups and obtained several properties [2]. The notion of hybrid structures and its properties were

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applied to semigroups and obtained many useful results (see [3], [4], [5] and [8]). Following [2], in this paper, we apply the notion of hybrid structure to semiring and introduce the notion of hybrid sub-semirings and different ideal structures in semirings, and obtain their several properties.

Definition 1.1. *[11] Let $\mathscr{R} \neq \phi$ and $+$ and $.$ two binary operations defined on $\mathscr{R}$. $\mathscr{R}$ is called a semiring if*

(i) $(\mathscr{R}, +)$ *is commutative semigroup,*
(ii) $(\mathscr{R}, .)$ *is semgroup,*
(iii) $a_1'.(a_2' + a_3') = a_1'.a_2' + a_1'.a_3'$; $(a_1' + a_2').a_3' = a_1'.a_3' + a_2'.a_3'$ *for all* $a_1', a_2', a_3' \in \mathscr{R}$.

Throughout this paper $\mathscr{R}$ denotes a semiring and the power set of S is $\mathscr{P}(S)$.

Definition 1.2. *Let $J \in \mathscr{P}(\mathscr{R})$. J is called a left (resp., right) ideal of $\mathscr{R}$ if it satisfies*

(i) $j_1 + j_2 \in J$
(ii) $xj_1 \in J$ *(resp.,* $j_1 x \in J$*) for all* $j_1, j_2 \in J$ *and* $x \in \mathscr{R}$.

If J is both a left and a right ideal of $\mathscr{R}$), then J is called an ideal of $\mathscr{R}$.

Definition 1.3. *[2] Let $I = [0,\ 1]$. A hybrid structure in $\mathscr{R}$ over $\mathbb{U}$ is a mapping*
$\tilde{g}_\mu := (\tilde{g}, \mu) : \mathscr{R} \to \mathscr{P}(\mathbb{U}) \times I,\ r_1 \mapsto (\tilde{g}(r_1), \mu(r_1))$, *where* $\tilde{g} : \mathscr{R} \to \mathscr{P}(U)$ *and* $\mu : \mathscr{R} \to I$ *are mappings.*

The collection of all hybrid structures in $\mathscr{R}$ over $\mathbb{U}$ is denoted by $H(\mathscr{R})$. Clearly $(H(\mathscr{R}), \ll)$ is a poset where relation $\ll$ defined on $H(\mathscr{R})$ as follows : For all $\tilde{g}_\mu, \tilde{h}_\beta \in H(\mathscr{R})$, $\tilde{g}_\mu \ll \tilde{h}_\beta$ if and only if $\tilde{g} \mathrel{\tilde{\subseteq}} \tilde{h}, \mu \succeq \beta$, where $\tilde{g} \mathrel{\tilde{\subseteq}} \tilde{h}$ means that $\tilde{g}(t) \subseteq \tilde{h}(a_1')$ and $\mu \succeq \beta$ means that $\mu(a_1') \geq \beta(a_1')$ for all $a_1' \in \mathscr{R}$.

Definition 1.4. *[2] Let $\mathscr{R}$ be a non-empty set with a binary operation $*$ and $\tilde{g}_\mu \in H(\mathscr{R})$. $\tilde{g}_\mu$ is called hybrid subsemigroup with respect to $*$ if it satisfies the conditions $\tilde{g}(s' * s'') \supseteq \tilde{g}(s') \cap \tilde{g}(s'')$ and $\mu(s' * s'') \leq \mu(s') \vee \mu(s'')$ for all $s', s'' \in \mathscr{R}$.*

Definition 1.5. *Let $\tilde{g}_\mu \in H(\mathscr{R})$. $\tilde{g}_\mu$ is called hybrid sub-semiring if $\tilde{g}_\mu$ is a hybrid sub-semigroup of $\mathscr{R}$ over $\mathbb{U}$ with respect to both the binary operations addition and multiplication.*

Definition 1.6. *Let $\tilde{g}_\lambda \in H(\mathscr{R})$. $\tilde{g}_\lambda$ is hybrid left (resp., right) ideal if it satisfies the following assertions: for all $a_1, a_2 \in \mathscr{R}$,*

(i) $\tilde{g}_\lambda$ *is a hybrid sub-semigroup of* $\mathscr{R}$ *with respect to* $+$,
(ii) $\tilde{g}(a_1a_2) \supseteq \tilde{g}(a_2)$ *(resp.,* $\tilde{g}(a_1a_2) \supseteq \tilde{f}(a_1)$*)*,
(iii) $\lambda(a_1a_2) \leq \lambda(a_2)(resp., \lambda(a_1a_2) \leq \lambda(a_1))$.

$\tilde{f}_\lambda$ *is hybrid ideal if it is both a hybrid left and a hybrid right ideal of* $\mathscr{R}$.

Clearly every hybrid left(resp., right) ideal is hybrid sub-semiring, however hybrid sub-semirings need not be hybrid left(resp., right) ideals as can be seen by the following example.

Example 1. *Let* $\mathscr{R} = \mathbb{N}$ *and* $\tilde{g}_\lambda \in H(\mathscr{R})$ *defined by*

$$g(x) = \begin{cases} 0 & if\ x \in (0,\ 5) \\ [0, 0.5] & if\ x \in [5,\ 7) \\ [0, 1] & if\ 7 \leq x. \end{cases}$$

Then for any constant mapping $\lambda : \mathscr{R} \to I$, $\tilde{g}_\lambda$ *is a hybrid sub-semiring of* $\mathscr{R}$, *but it is neither a hybrid left nor a hybrid right ideal of* $\mathscr{R}$. □

Definition 1.7. *[2] For* $A \in \mathscr{P}(\mathscr{R})$ *and* $\tilde{g}_\lambda \in H(\mathscr{R})$, $\chi_A(\tilde{g}_\lambda)$ *is the characteristic hybrid structure of* A, *defined as follows:* $\chi_A(\tilde{g}_\lambda) = (\chi_A(\tilde{g}), \chi_A(\lambda))$, *where*

$$\chi_A(\tilde{g}) : \mathscr{R} \to \mathscr{P}(\mathbb{U}), a \mapsto \begin{cases} \mathbb{U} & if\ a \in A \\ \phi & otherwise \end{cases}$$

and

$$\chi_A(\lambda) : \mathscr{R} \to I, a \mapsto \begin{cases} 0 & if\ a \in A \\ 1 & otherwise. \end{cases}.$$

2. HYBRID STRUCTURES IN SEMIRINGS

Theorem 2.1. *For any* $E \in \mathscr{P}(\mathscr{R})$, E *is a sub-semigroup with respect to* $+$ *if and only if* $\chi_E(\tilde{h}_\mu)$ *is a hybrid sub-semigroup in* $\mathscr{R}$ *over* $\mathbb{U}$ *with respect to* $+$.

Proof. Let E be a sub-semigroup of $\mathscr{R}$ with respect to $+$. For any $z_1, z_1 \in \mathscr{R}$.

If either $y_1 \notin E$ or $z_1 \notin E$ (say $y_1 \notin E$), then $\chi_E(\tilde{h})(y_1 + z_1) \supseteq \phi = \chi_E(\tilde{h})(y_1) \supseteq \chi_E(\tilde{h})(y_1) \cap \chi_E(\tilde{h})(z_1)$ and $\chi_E(\mu)(y_1 + z_1) \leq 1 = \chi_E(\mu)(y_1) \leq \bigvee\{\chi_E(\mu)(y_1), \chi_E(\mu)(z_1)\}$.

If $y_1 \in E$ and $z_1 \in E$, then $y_1+b_2 \in E$ and so $\chi_E(\tilde{h})(y_1+z_1) \supseteq \phi = \chi_E(\tilde{h})(y_1)\cap \chi_E(\tilde{h})(z_1)$ and $\chi_E(\mu)(y_1+z_1) = 0 = \bigvee\{\chi_E(\mu)(y_1), \chi_E(\mu)(z_1)\}$. Therefore $\chi_E(\tilde{h}_\mu)$ is a hybrid sub-semigroup in $\mathscr{R}$ over $\mathbb{U}$ with respect to $+$. $\square$

Theorem 2.2. *For any $L \in \mathscr{P}(\mathscr{R})$ and $\tilde{f}_\lambda \in H(\mathscr{R})$, we have L is a left(resp., right) ideal of $\mathscr{R}$) if and only if $\chi_L(\tilde{f}_\lambda)$ is hybrid left(resp., right) ideal in $\mathscr{R}$.*

Proof. By Theorem 2.1, we have $a_1 + a_2 \in \chi_L(\tilde{f}_\lambda)$ for any $a_1, a_2 \in \chi_L(\tilde{f}_\lambda)$. Let $b_1, b_2 \in \mathscr{R}$. If $b_2 \notin L$, then $\chi_L(\tilde{f})(b_1b_2) \supseteq \phi = \chi_L(\tilde{f})(b_2)$ and $\chi_L(\lambda)(b_1b_2) \le 1 = \chi_L(\lambda)(b_2)$. If $b_2 \in L$, then $\chi_L(\tilde{f})(b_1b_2) = \mathbb{U} = \chi_L(\tilde{f})(b_2)$ and $\chi_L(\lambda)(b_1b_2) = 0 = \chi_L(\lambda)(b_2)$. Therefore $\chi_L(\tilde{f}_\lambda)$ is hybrid left ideal.

For converse, by Theorem 2.1, we have $l_1+l_2 \in L$ for any $l_1, l_2 \in L$. For $x_1 \in \mathscr{R}$ and $y_1 \in L$, we have $\chi_L(\tilde{f})(y_1) = \mathbb{U}$ and $\chi_L(\lambda)(y_1) = 0$ imply $\chi_L(\tilde{f})(x_1y_1) \supseteq \chi_L(\tilde{f})(y_1) = \mathbb{U}$ and $\chi_L(\lambda)(x_1y_1) \le 0 = \chi_L(\lambda)(y_1) = 0$. Thus $x_1y_1 \in L$ and hence L is a left ideal of $\mathscr{R}$. $\square$

Theorem 2.3. *For $\tilde{g}_\lambda \in H(\mathscr{R})$, $\tilde{g}_\lambda$ is hybrid subsemiring if and only if $\phi \ne L_g^\epsilon := \{x_1 \in \mathscr{R} \,:\, \tilde{g}(x_1) \supseteq \epsilon\}$ and $\phi \ne L_\lambda^{t_1} := \{x_1 \in \mathscr{R} \,:\, \lambda(x_1) \le t_1\}$ are subsemirings for any $\epsilon \in \mathscr{P}(\mathbb{U})$ and $t_1 \in I$.*

Proof. Suppose $\tilde{g}_\lambda \in H(\mathscr{R})$ is hybrid subsemiring of $\mathscr{R}$ over $\mathbb{U}$. Assume that $L_g^\epsilon \ne \phi$ and $L_\lambda^{t_1} \ne \phi$ for any $\epsilon \in \mathscr{P}(\mathbb{U})$ and $t_1 \in I$. Let $s', s'' \in L_g^\epsilon \cap L_\lambda^{t_1}$. Then $\tilde{g}(s') \supseteq \epsilon, \tilde{g}(s'') \supseteq \epsilon, \lambda(s') \le t_1$ and $\lambda(y) \le t_1$. Since $\tilde{g}(s'+s'') \supseteq \tilde{g}(s') \cap \tilde{g}(s'') \supseteq \epsilon$ and $\lambda(s'+s'') \le \lambda(s') \vee \lambda(s'') \le t_1$ imply $s'+s'' \in L_g^\epsilon \cap L_\lambda^{t_1}$. Also $\tilde{g}(s's'') \supseteq \tilde{g}(s') \cap \tilde{g}(s'') \supseteq \epsilon$ and $\lambda(s's'') \le \lambda(s') \vee \lambda(s'') \le t_1$ imply $s's'' \in L_g^\epsilon \cap L_\lambda^{t_1}$. Therefore L_g^ϵ and $L_\lambda^{t_1}$ are subsemirings of $\mathscr{R}$ for any $\epsilon \in \mathscr{P}(\mathbb{U})$ and $t_1 \in I$.

Conversely, assume that $L_g^\epsilon \ne \phi$ and $L_\lambda^{t_1} \ne \phi$ are subsemirings of $\mathscr{R}$ for all $(\epsilon, t_1) \in \mathscr{P}(\mathbb{U}) \times I$. For $a_1, a_2 \in \mathscr{R}$, let $\tilde{g}(a_1) = \epsilon_{a_1}$ and $\tilde{g}(a_2) = \epsilon_{a_2}$.

If we take $\epsilon := \epsilon_{a_1} \cap \epsilon_{a_2}$, then $a_1, a_2 \in L_g^\epsilon$ which implies $\tilde{g}(a_1+a_2) \supseteq \epsilon = \epsilon_{a_1} \cap \epsilon_{a_2} = \tilde{g}(a_1) \cap \tilde{g}(a_2)$ and $\tilde{g}(a_1a_2) \supseteq \epsilon = \epsilon_{a_1} \cap \epsilon_{a_2} = \tilde{g}(a_1) \cap \tilde{g}(a_2)$. Also for any $s', s'' \in \mathscr{R}$, let $\lambda(s') = t_{s'}$ and $\lambda(s'') = t_{s''}$. Taking $t_1 := t_{s'} \vee t_{s''}$ implies that $s', s'' \in L_\lambda^{t_1}$. So $s's'', s'+s'' \in L_\lambda^{t_1}$ implies that $\lambda(s's'') \le t_1 = t_{s'} \vee t_{s''} = \lambda(s') \vee \lambda(s'')$ and $\lambda(s'+s'') \le t_1 = t_{s'} \vee t_{s''} = \lambda(s') \vee \lambda(s'')$. Therefore $\tilde{g}_\lambda$ is hybrid subsemiring. $\square$

Theorem 2.4. *Let $\tilde{g}_\lambda \in H(\mathscr{R})$. $\tilde{g}_\lambda$ is hybrid left (resp., right) ideal if and only if $\phi \ne L_g^\epsilon := \{x_1 \in \mathscr{R} \,:\, \tilde{g}(x) \supseteq \epsilon\}$ and $\phi \ne L_\lambda^{t_1} := \{x_1 \in \mathscr{R} \,:\, \lambda(x) \le t_1\}$ are left (resp., right) ideal for $\epsilon \in \mathscr{P}(\mathbb{U})$ and $t_1 \in I$.*

Definition 2.1. *For any $\tilde{g}_\lambda$, $\tilde{h}_\gamma \in H(\mathscr{R})$, the hybrid sum of $\tilde{g}_\lambda$ and $\tilde{h}_\gamma$ in $\mathscr{R}$ is $\tilde{g}_\lambda \oplus \tilde{h}_\gamma = (\tilde{g} + \tilde{h}, \lambda + \gamma) \in H(\mathscr{R})$, where*

$$(\tilde{g} + \tilde{h})(a') = \begin{cases} \bigcup\limits_{a'=p_i+q_j} \{\tilde{g}(p_i) \cap \tilde{h}(q_j)\} & if\ \exists\ p_i, q_j \in \mathscr{R} \text{ such that } a' = p_i + q_j \\ \phi & otherwise \end{cases}$$

and

$$(\lambda + \gamma)(a') = \begin{cases} \bigwedge\limits_{a'=p_i+q_j} \bigvee\{\lambda(p_i), \lambda(q_j)\} & if\ \exists\ p_i, q_j \in \mathscr{R} \text{ such that } a' = p_i + q_j \\ 1 & otherwise \end{cases}$$

for all $a' \in \mathscr{R}$.

Theorem 2.5. *Let $\tilde{h}^1_\lambda = (\tilde{h}_1, \lambda_1), \tilde{h}^2_\lambda = (\tilde{h}_2, \lambda_2), \tilde{h}^3_\mu = (\tilde{h}_3, \mu_1)$ and $\tilde{h}^4_\mu = (\tilde{h}_4, \mu_2)$ be elements in $H(\mathscr{R})$. If $\tilde{h}^1_\lambda \ll \tilde{h}^3_\mu$ and $\tilde{h}^2_\lambda \ll \tilde{h}^4_\mu$, then $\tilde{h}^1_\lambda \oplus \tilde{h}^2_\lambda \ll \tilde{h}^3_\mu \oplus \tilde{h}^4_\mu$.*

Proof. Let $a' \in \mathscr{R}$. If a' is not expressed as $a' = s'' + s''$ for $s', s'' \in \mathscr{R}$, then clearly $\tilde{h}^1_\lambda \oplus \tilde{h}^2_\lambda \ll \tilde{h}^3_\mu \oplus \tilde{h}^4_\mu$. Assume that $a' = s' + s''$ for some $s', s'' \in \mathscr{R}$. Then

$$(\tilde{h}_1 + \tilde{h}_2)(a') = \bigcup_{a'=s'+s''} \{\tilde{h}_1(s') \cap \tilde{h}_2(s'')\} \subseteq \bigcup_{a'=s'+s''} \{\tilde{h}_3(s') \cap \tilde{h}_4(s'')\} = (\tilde{g}_1 + \tilde{g}_2)(a').$$

Also $(\lambda_1 + \lambda_2)(a') = \bigwedge\limits_{a'=s'+s''} \bigvee\{\lambda_1(s'), \lambda_2(s'')\} \geq \bigwedge\limits_{a'=s'+s''} \bigvee\{\mu_1(s'), \mu_2(s'')\} = (\mu_1 + \mu_2)(a')$. Therefore $\tilde{h}^1_\lambda \oplus \tilde{h}^2_\lambda \ll \tilde{h}^3_\mu \oplus \tilde{h}^4_\mu$. □

Theorem 2.6. *For $\tilde{h}_\lambda \in H(\mathscr{R})$, $\tilde{h}_\lambda$ is hybrid sub-semigroup with respect to $+$ if and only if $\tilde{h}_\lambda \oplus \tilde{h}_\lambda \ll \tilde{h}_\lambda$.*

Proof. Let $\tilde{h}_\gamma \in H(\mathscr{R})$ be a hybrid sub-semigroup of $\mathscr{R}$. Then $\tilde{h}(s) \supseteq \tilde{h}(s) \cap \tilde{h}(a_i)$ and $\lambda(s) \cap \lambda(a_i) \leq \lambda(s) \vee \lambda(a_i)\}$ for all $r \in \mathscr{R}$ with $s = a_i + b_j$.

Now $\tilde{h}(s) \supseteq \bigcup\limits_{s=a_i+b_j} \{\tilde{h}(a_i) \cap \tilde{h}(b_j)\} = (\tilde{h} + \tilde{h})(s)$ and $\lambda(s) = \bigwedge\limits_{s=a_i+b_j} \{\lambda(a_i) \vee \lambda(b_j)\} = (\lambda + \lambda)(s)$ for all $s \in \mathscr{R}$. So $\tilde{h} + \tilde{h} \subseteq \tilde{h}$ and $\lambda \leq \lambda + \lambda$. Therefore $\tilde{h}_\lambda \oplus \tilde{h}_\lambda \ll \tilde{h}_\lambda$.

Conversely, if $\tilde{h}_\lambda \oplus \tilde{h}_\lambda \ll \tilde{h}_\lambda$, then $\tilde{h}(b_j + a_i) \supseteq \tilde{h}(b_j) \cap \tilde{h}(a_i)$ and $\lambda(b_j + a_i) \leq \lambda(b_j) \vee \lambda(a_i)$ for any $b_j, a_i \in \mathscr{R}$. □

Theorem 2.7. *For any $\tilde{h}_\lambda \in H(\mathscr{R})$, $\tilde{h}_\lambda$ is hybrid sub-semiring if and only if $\tilde{h}_\lambda \oplus \tilde{h}_\lambda \ll \tilde{h}_\lambda$ and $\tilde{h}_\lambda \odot \tilde{h}_\lambda \ll \tilde{h}_\lambda$.*

Proof. It follows from Theorem 2.6 and Theorem 3.12 of [2]. □

Theorem 2.8. *For any* $S, T \in \mathscr{P}(\mathscr{R})$, $\chi_S(\tilde{h}_\lambda) \oplus \chi_T(\tilde{h}_\lambda) = \chi_{(S+T)}(\tilde{h}_\lambda)$.

Proof. For any $r_1 \in \mathscr{R}$, if $r_1 \in S+T$, then $r_1 = s_1 + t_1$ for some $s_1 \in S$ and $t_1 \in T$. Now $(\chi_S(\tilde{h}) + \chi_T(\tilde{h}))(r_1) = \bigcup\limits_{r_1 = y_1 + z_1} \{\chi_S(\tilde{h})(y_1) \cap \chi_S(\tilde{h})(z_1)\} \supseteq \chi_S(\tilde{h})(s_1) \cap \chi_S(\tilde{h})(t_1) = \mathbb{U}$ and $(\chi_S(\lambda) + \chi_T(\lambda))(r_1) = \bigwedge\limits_{r_1 = y_1 + z_1} \bigvee\{\chi_S(\lambda)(y_1), \chi_T(\lambda(z_1))\} \leq \bigvee\{\chi_S(\lambda)(s_1), \chi_T(\lambda)(t_1)\} = 0$.

So $(\chi_S(\tilde{h}) + \chi_T(\tilde{h}))(r_1) = \chi_{(S+T)}(\tilde{h}_\lambda)$ and $(\chi_S(\lambda) + \chi_T(\lambda))(r_1) = 0 = \chi_{(S+T)}(\lambda)(r_1)$.

Suppose $r_1 \notin S+T$. Then $r_1 \neq s_1 + t_1$ for all $s_1 \in S$ and $t_1 \in T$. If $r_1 = y_1 + z_1$ for some $y_1, z_1 \in \mathscr{R}$, then $y_1 \notin S$ or $z_1 \notin T$.

$$(\chi_S(\tilde{h}) + \chi_T(\tilde{h}))(r_1) = \bigcup_{r_1 = y_1 + z_1} \{\chi_S(\tilde{h})(y_1) \cap \chi_S(\tilde{h})(z_1)\} = \phi = \chi_{(S+T)}(\tilde{h}_\lambda),$$

$$(\chi_S(\lambda) + \chi_T(\lambda))(r_1) = \bigwedge_{r_1 = y_1 + z_1} \bigvee\{\chi_S(\lambda)(y_1), \chi_T(\lambda(z_1))\} = 1 = \chi_{(S+T)}(\lambda)(r_1).$$

If $r_1 \neq y_1 + z_1$ for all $y_1, z_1 \in \mathscr{R}$, then $(\chi_S(\tilde{h}) + \chi_T(\tilde{h}))(r_1) = \phi = \chi_{(S+T)}(\tilde{h}_\lambda)$, and $(\chi_S(\lambda) + \chi_T(\lambda))(r_1) = 1 = \chi_{(S+T)}(\lambda)(r_1)$.

In both cases, we have $\chi_S(\tilde{h}) + \chi_T(\tilde{h})(r_1) = \chi_{(S+T)}(\tilde{h}_\lambda)$ and $(\chi_S(\lambda) + \chi_T(\lambda))(r_1) = 0 = \chi_{(S+T)}(\lambda)(r_1)$ for all $r_1 \in \mathscr{R}$.

So $\chi_S(\tilde{h}_\lambda) \oplus \chi_T(\tilde{h}_\lambda) = (\chi_S(\tilde{h}) + \chi_T(\tilde{h}), \chi_S(\lambda) + \chi_T(\lambda)) = ((\chi_{(S+T)}(\tilde{h}), \chi_{(S+T)}(\lambda)) = \chi_{(S+T)}(\tilde{h}_\lambda)$. ☐

Following [6], a left (resp., right) ideal J of $\mathscr{R}$ is called left $k-$ideal (resp., right $k-$ideal) if $i \in J$ and $v_1' \in \mathscr{R}$ and if $i + v_1' \in J$ or $v_1' + i \in J$, then $v_1' \in J$.

If J is both a left and a right $k-$ ideal of $\mathscr{R}$, then J is called a $k-$ ideal of $\mathscr{R}$.

Definition 2.2. *Let* $\tilde{g}_\mu \in H(\mathscr{R})$. $\tilde{g}_\mu$ *is said to be hybrid* $k-$*left (resp.,* $k-$*right) ideal of* $\mathscr{R}$ *if* $\tilde{g}_\mu$ *is hybrid left(resp., right) ideal in* $\mathscr{R}$ *and satisfies the below conditions:*

(i) $\tilde{g}(p_i) \supseteq (\tilde{g}(p_i + s_l) \cup \tilde{g}(s_l + p_i)) \cap \tilde{g}(s_l) = \bigcap\{\bigcup\{\tilde{g}(p_i + s_l), \tilde{g}(s_l + p_i)\}, \tilde{g}(s_l)\}$

(ii) $\mu(p_i) \leq \bigvee\{\mu(p_i + s_l) \wedge \mu(s_l + p_i), \mu(s_l)\}$ *for any* $p_i, s_l \in \mathscr{R}$.

If $\mathscr{R}$ is additively commutative, then the condition reduces to $\tilde{g}_\mu(p_i) \supseteq \tilde{g}_\mu(p_i + q_j) \cap \tilde{g}_\mu(q_j)$ and $\mu(p_i) \leq \bigvee\{\mu(p_i + q_j), \mu(q_j)\}$.

Example 2. *Consider the semiring* $B = B(5,3)$ *(F.E.Alarcon and D.Polkoska [1]), where* $+$ *and* . *are defined in Table 1 and Table 2 respectively. Let* $\tilde{f}_\lambda \in H(\mathscr{R})$ *where* $\mathbb{U} = \{c_1, c_2, c_3, c_4\}$*, which is given in Table 3.*

$+$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	3
2	2	3	4	3	4
3	3	4	3	4	3
4	4	3	4	3	4

.	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	4	4
3	0	3	4	3	4
4	0	4	4	4	4

$\mathscr{R}$	$\tilde{f}$
0	$\mathbb{U}$
1	$\{c\}$
2	$\{c_1, c_3\}$
3	$\{c_2, c_3\}$
4	$\{c_1, c_2, c_3\}$

For any constant mapping $\lambda : \mathscr{R} \to I$, $\tilde{f}_\lambda$ is a hybrid ideal in $\mathscr{R}$, but not hybrid $k-$ideal. $\square$

Definition 2.3. *[2] Let $\tilde{g}_\lambda, \tilde{h}_\gamma \in H(\mathscr{R})$. Then $\tilde{f}_\lambda \Cap \tilde{h}_\gamma \in H(\mathscr{R})$ is the hybrid intersection of $\tilde{g}_\lambda$ and $\tilde{h}_\gamma$ which is defined as follows:*

$$\tilde{g}_\lambda \Cap \tilde{h}_\gamma : \mathscr{R} \to \mathscr{P}(\mathbb{U}) \times I, r_1 \mapsto ((\tilde{g} \cap \tilde{h})(r_1), (\lambda \vee \Upsilon)(r_1)),$$

where $\tilde{g} \cap \tilde{h} : \mathscr{R} \longrightarrow \mathscr{P}(\mathbb{U}), r_1 \mapsto \tilde{g}(r_1) \cap \tilde{h}(r_1)$ and $\lambda \vee \gamma : \mathscr{R} \to I, r_1 \mapsto \lambda(r_1) \vee \gamma(r_1)$.

Theorem 2.9. *For any $\tilde{f}_\lambda$, $\tilde{h}_\gamma \in H(\mathscr{R})$. If $\tilde{f}_\lambda$ and $\tilde{h}_\gamma$ are hybrid $k-$ideals, then $\tilde{f}_\lambda \Cap \tilde{h}_\gamma$ is also hybrid $k-$ideal.*

Proof. For $p_i, s_l \in \mathscr{R}$, $(\tilde{f} \cap \tilde{h})(p_i + s_l) \cap (\tilde{f} \cap \tilde{h})(s_l) = \tilde{f}(p_i + s_l) \cap \tilde{h}(p_i + s_l) \cap \tilde{f}(s_l) \cap \tilde{h}(s_l) \subseteq \tilde{f}(p_i) \cap \tilde{h}(p_i) = (\tilde{f} \cap \tilde{h})(p_i)$.

Also, $\bigvee\{(\lambda \vee \gamma)(p_i + s_l), (\lambda \vee \gamma)(s_l)\} = \bigvee\{\vee\{\lambda(p_i + s_l), \gamma(p_i + s_l)\}, \vee\{\lambda(s_l), \gamma(s_l)\}\} = \bigvee\{\vee\{\lambda(p_i + s_l), \lambda(s_l)\}, \vee\{\gamma(p_i + s_l), \gamma(s_l)\}\}$. So $\tilde{f}_\lambda \Cap \tilde{h}_\gamma$ is hybrid $k-$ideal. $\square$

Definition 2.4. *Let $\tilde{f}_\gamma \in H(\mathscr{R})$. Then the hybrid $k-closure$ $\overline{\tilde{f}_{\overline{\gamma}}}$ of $\tilde{f}_\gamma$ is defined as follows: For $p_i \in \mathscr{R}$,*

$\overline{\tilde{f}}(p_i) = \bigcup_{q_j \in \mathscr{R}} \{\tilde{f}(p_i + q_j) \cap \tilde{f}(q_j)\}$, *and* $\overline{\gamma(p_i)} = \bigwedge_{q_j \in \mathscr{R}} \{\gamma(p_i + q_j) \vee \gamma(q_j)\} \forall q_j \in H(\mathscr{R})$.

Clearly, $\tilde{f} \subseteq \overline{\tilde{f}}$ and $\gamma \subseteq \overline{\gamma}$.

Lemma 2.1. *Let $\tilde{f}_\gamma \in H(\mathscr{R})$. Then $\overline{\tilde{f}}_{\overline{\gamma}}$ is a hybrid ideal in $\mathscr{R}$ over $\mathbb{U}$.*

Proof. For any $p_i, q_j \in \mathscr{R}$, we have

$$\begin{aligned}
\overline{\tilde{f}}(p_i) \cap \overline{\tilde{f}}(q_j) &= \bigcup_{s_l \in \mathscr{R}} \{\tilde{f}(p_i + s_l) \cap \tilde{f}(s_l)\} \cap \bigcup_{r_k \in \mathscr{R}} \{\tilde{f}(p_i + r_k) \cap \tilde{f}(r_k)\} \\
&= \bigcup_{s_l, r_k \in \mathscr{R}} \{\tilde{f}(p_i + s_l) \cap \tilde{f}(s_l)\} \cap \{\tilde{f}(p_i + r_k) \cap \tilde{f}(r_k)\} \\
&= \bigcup_{s_l, r_k \in \mathscr{R}} \{\tilde{f}(p_i + s_l) \cap \tilde{f}(p_i + r_k) \cap \tilde{f}(s_l) \cap \tilde{f}(r_k)\}
\end{aligned}$$

$$\subseteq \bigcup_{s_l, r_k \in \mathscr{R}} \{\tilde{f}(p_i + q_j + (s_l + r_k)) \cap \tilde{f}(s_l + r_k) = \overline{\tilde{f}}(p_i + q_j),$$

Also $\overline{\tilde{f}}(p_i q_j) = \bigcup_{t \in \mathscr{R}} \{\tilde{f}(p_i q_j + x_i) \cap \tilde{f}(x_i) \supseteq \bigcup_{x_i \in \mathscr{R}} \{\tilde{f}(p_i q_j + x_i q_j)) \cap \tilde{f}(x_i q_j) \supseteq \bigcup_{x_i \in \mathscr{R}} \{\tilde{f}(p_i + x_i)) \cap \tilde{f}(x_i) = \overline{\tilde{f}}(p_i)$.

Similarly, $\overline{\tilde{f}}(p_i q_j) \supseteq \overline{\tilde{f}}(q_j)$.

Also $\bigvee\{\overline{\gamma}(p_i), \overline{\gamma}(q_j)\} = \bigvee\{ \bigwedge_{s_l \in \mathscr{R}} \{\gamma(p_i + s_l) \vee \gamma(s_l)\}\}, \bigwedge_{r_k \in \mathscr{R}} \{\gamma(q_j + r_k) \vee \gamma(r_k)\}\}$

$$= \bigvee\{ \bigwedge_{s_l, r_k \in \mathscr{R}} \{\gamma(p_i + s_l) \vee \gamma(s_l) \vee \gamma(q_j + r_k) \vee \gamma(r_k)\}\}$$

$$\leq \bigvee\{ \bigwedge_{s_l, r_k \in \mathscr{R}} \{\gamma(p_i + q_j + s_l + r_k) \vee \gamma(s_l + r_k) = \overline{\gamma}(p_i + q_j).$$

Hence $\overline{\tilde{f}}$ is hybrid ideal. □

Lemma 2.2. *If $\tilde{g}_\gamma$ is a hybrid $k-$ideal of $\mathscr{R}$ over $\mathbb{U}$, then $\overline{\tilde{g}}_{\overline{\gamma}} = \tilde{g}_\gamma$.*

Proof. Let $\tilde{g}_\gamma$ be hybrid $k-$ideal. Then $\tilde{g}(p_i) \supseteq \tilde{g}(p_i + q_j) \cap \tilde{g}(q_j)$ and $\gamma(p_i) \leq \gamma(p_i + q_j) \vee \gamma(q_j)$ for all $p_i, q_j \in \mathscr{R}$ which imply $\overline{\tilde{g}(p_i)} = \bigcup_{q_j \in \mathscr{R}} \{\tilde{g}(p_i + q_j) \cap \tilde{g}(q_j)\} = \tilde{g}_\gamma(p_i)$ and $\overline{\gamma(p_i)} = \bigwedge_{q_j \in \mathscr{R}} \{\gamma(p_i + q_j) \vee \gamma(q_j)\} \geq \gamma(p_i)$. So $\overline{\tilde{g}} \subseteq \tilde{g}$ and $\overline{\gamma(p_i)} \subseteq \gamma(p_i)$. □

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