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AN APPLICATION OF PASCAL DISTRIBUTION SERIES ON CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS

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ABSTRACT. In this paper, we examine the connections between the new subclasses of $\gamma-$ parabolic starlike and $\gamma-$ uniformly convex functions of order δ by using a convolution operator involving the Pascal distribution series. Further we point out significance of our main results .

1. INTRODUCTION

Denote the class of analytic functions by $\mathcal{A}$ comprising functions of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \quad z \in \mathbb{U}. \tag{1.1}$$

As usual, we denote by $\mathbf{S}$ the subclass of $\mathbf{A}$ whose members are normalized by $f(0) = 0 = f'(0) - 1$ and also univalent in $\mathbb{U}$. Denote by $\mathbf{T}$ the subclass of $\mathbf{A}$ comprising of functions whose nonzero coefficients from second on, is given by $f(z) = z - \sum_{j=2}^{\infty} |a_j| z^j, z \in \mathbb{U}$ is introduced and studied by Silverman [12]. For functions $f \in \mathbf{A}$ given by (1.1) and $g \in \mathbf{A}$ given by $g(z) = z + \sum_{j=2}^{\infty} b_j z^j$, we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z + \sum_{j=2}^{\infty} a_j b_j z^j, \quad z \in \mathbb{U}.$$

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Inspired by the earlier work of Rnning [11], Bharathi et al., [1] defined the following subclass of $\mathbf{S}$ namely $\gamma-$ parabolic starlike and $\gamma-$ uniformly convex functions $(-1 < \delta \leq 1, \gamma \geq 0)$:

(1) $\mathbf{S}_P(\delta,\gamma) = \left\{ f \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)} - \delta\right) > \gamma\left|\frac{zf'(z)}{f(z)} - 1\right|, z \in \mathbb{U}\right\}$

(2) $\mathbf{UCV}(\delta,\gamma) = \left\{ f \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)} - \delta\right) > \gamma\left|\frac{zf'(z)}{f(z)} - 1\right|, z \in \mathbb{U}\right\}$.

We note that $f \in \mathbf{UCV}(\delta,\gamma) \Leftrightarrow zf' \in \mathbf{S}_P(\delta,\gamma)$.

Remark 1.1. *It is of importance to state that* $\mathbf{UCV}(\delta,0) = \mathbf{K}(\delta)$ *and* $\mathbf{S}_P(\delta,0) = \mathbf{S}^*(\delta)$*(See [12]).*

We consider the following subclasses of $\mathbf{A}$ studied in [5].

For $0 \leq \mu < 1$, $0 \leq \delta < 1, \gamma \geq 0$, and $f \in \mathbf{A}$ we let $\mathbf{S}_P(\mu,\delta,\gamma)$ be the subclass of $\mathbf{A}$ satisfying the analytic criterion

$$\Re\left(\frac{zf'(z)}{(1-\mu)f(z)+\mu zf'(z)} - \delta\right) > \gamma\left|\frac{zf'(z)}{(1-\mu)f(z)+\mu zf'(z)} - 1\right|,\ z \in \mathbb{U},$$

and also, let $\mathbf{UCV}(\mu,\delta,\gamma)$ be the subclass of $\mathbf{A}$ satisfying the analytic criterion

$$\Re\left(\frac{f'(z)+zf''(z)}{f'(z)+\mu zf''(z)} - \delta\right) > \gamma\left|\frac{f'(z)+zf''(z)}{f'(z)+\mu zf''(z)} - 1\right|,\ z \in \mathbb{U}.$$

By suitably specializing the parameters μ,δ,γ as mentioned in Murugusundaramoorthy and Magesh [5], we get various special classes of $\mathbf{S}$ and $\mathbf{T}$. Further we recall the following results for f to be in the subclasses $\mathbf{S}_P(\mu,\delta,\gamma)$, $\mathbf{TS}_P(\mu,\delta,\gamma)$, $\mathbf{UCV}(\mu,\delta,\gamma)$ and $\mathbf{UCT}(\mu,\delta,\gamma)$ as given in [5].

Lemma 1.1. *Let f be given by (1.1) is in $\mathbf{S}_P(\mu,\delta,\gamma)$ if*

$$\sum_{j=2}^{\infty}[j(1+\gamma) - (\delta+\gamma)(1+j\mu-\mu)]\ |a_j| \leq 1-\delta.$$

Lemma 1.2. *A function $f(z) = z - \sum_{j=2}^{\infty}|a_j|z^j$, is in $\mathbf{TS}_P(\mu,\delta,\gamma)$ if and only if*

$$\sum_{j=2}^{\infty}[j(1+\gamma) - (\delta+\gamma)(1+j\mu-\mu)]\ |a_j| \leq 1-\delta.$$

Lemma 1.3. *Let f be given by (1.1) is in $\mathbf{UCV}(\mu,\delta,\gamma)$ if*

$$\sum_{j=2}^{\infty} j[j(1+\gamma) - (\delta+\gamma)(1+j\mu-\mu)]\ |a_j| \leq 1-\delta.$$

Lemma 1.4. *A function* $f(z) = z - \sum_{j=2}^{\infty} |a_j| z^j$, *is in* $\mathbf{UCT}(\mu, \delta, \gamma)$ *if and only if*

$$\sum_{j=2}^{\infty} j[j(1+\gamma) - (\delta+\gamma)(1+j\mu-\mu)]\, |a_j| \leq 1-\delta.$$

A variable χ is said to be *Pascal distribution* if it takes the values $0, 1, 2, 3, \ldots$ with probabilities
$(1-q)^{\vartheta}$, $\frac{q\vartheta(1-q)^{\vartheta}}{1!}$, $\frac{q^2\vartheta(\vartheta+1)(1-q)^{\vartheta}}{2!}$, $\frac{q^3\vartheta(\vartheta+1)(\vartheta+2)(1-q)^{\vartheta}}{3!}, \ldots$
respectively, where q and ϑ are called the parameters, and thus

$$P(\chi = k) = \sum_{j=2}^{\infty} \binom{k+\vartheta-1}{\vartheta-1} q^k (1-q)^{\vartheta}, k = 0, 1, 2, 3, \ldots.$$

Lately, El-Deeb [3] gave a power series representation , whose coefficients are probabilities of Pascal distribution

$$\Psi_q^{\vartheta}(z) = z + \sum_{j=2}^{\infty} \binom{j+\vartheta-2}{\vartheta-1} \cdot q^{j-1}(1-q)^{\vartheta} z^j, \qquad z \in \mathbb{U},$$

where $\vartheta \geq 1; 0 \leq q \leq 1$ and we note that, by ratio test the radius of convergence of above series is infinity. Recently Murugusundaramoorthy et al., [7] using the concept of convolution or hadamard product, defined the linear tranformation $\mathbf{I}_q^{\vartheta}(z) : \mathbf{A} \to \mathbf{A}$ defined by the

$$\mathbf{I}_q^{\vartheta} f(z) = \Psi_q^{\vartheta}(z) * f(z) = z + \sum_{j=2}^{\infty} \binom{j+\vartheta-2}{\vartheta-1} \cdot q^{j-1}(1-q)^{\vartheta} a_j z^j, \qquad z \in \mathbb{U},$$

and also defined

$$\Phi_q^{\vartheta}(z) = 2z - \Psi_q^{\vartheta}(z) = z - \sum_{j=2}^{\infty} \binom{j+\vartheta-2}{\vartheta-1} \cdot q^{j-1}(1-q)^{\vartheta} z^j, \qquad z \in \mathbb{U}.$$

Motivated by results on connections between various subclasses of analytic univalent functions by using Poisson distribution series (see [6, 8, 10] and references cited there in), in this paper we discussed connections between the new subclasses $\mathbf{S}_P(\mu, \delta, \gamma)$, and $\mathbf{UCV}(\mu, \delta, \gamma)$ by using a convolution operator defined through the Pascal distribution series. Finally, we give conditions for the integral operator $\mathbf{G}(\vartheta, z) = \int_0^z \frac{\mathbf{I}_q^{\vartheta} f(t)}{t} dt$ belonging to the classes $\mathbf{TS}_P(\mu, \delta, \gamma)$ and $\mathbf{UCT}(\mu, \delta, \gamma)$.

2. Inclusion Results

For convenience throughout in the sequel, we use the following notations:

$$\sum_{j=0}^{\infty}\binom{j+\vartheta-1}{\vartheta-1}q^j=\frac{1}{(1-q)^{\vartheta}},\qquad \sum_{j=0}^{\infty}\binom{j+\vartheta-2}{m-2}q^j=\frac{1}{(1-q)^{\vartheta-1}},$$

$$\sum_{j=0}^{\infty}\binom{j+\vartheta}{\vartheta}q^j=\frac{1}{(1-q)^{\vartheta+1}}\ \text{ and }\ \sum_{j=0}^{\infty}\binom{j+\vartheta+1}{\vartheta+1}q^j=\frac{1}{(1-q)^{\vartheta+2}}. \tag{2.1}$$

By simple computation we state the following:

$$\sum_{j=2}^{\infty}\binom{j+\vartheta-2}{\vartheta-1}q^{j-1}=\sum_{j=0}^{\infty}\binom{j+\vartheta-1}{\vartheta-1}q^j-1, \tag{2.2}$$

$$\sum_{j=2}^{\infty}(j-1)\binom{j+\vartheta-2}{\vartheta-1}q^{j-1}=q\,m\sum_{j=0}^{\infty}\binom{j+\vartheta}{\vartheta}q^j \tag{2.3}$$

and

$$\sum_{j=2}^{\infty}(j-1)(j-2)\binom{j+\vartheta-2}{\vartheta-1}q^{j-1}=q^2\,m(m+1)\sum_{n=0}^{\infty}\binom{j+m+1}{m+1}q^j. \tag{2.4}$$

Throughout this paper unless otherwise stated, we let

$$0\le\mu<1, 0\le\delta<1, \gamma\ge 0,\quad\text{and}\quad 0\le q\le 1.$$

Theorem 2.1. *If $\vartheta\ge 1$, then $\Phi_q^m(z)\in\mathbf{TS}_P(\mu,\delta,\gamma)$ if and only if*

$$[1+\gamma-\mu(\delta+\gamma)]\frac{q\,\vartheta}{(1-q)^{\vartheta+1}}\le 1-\delta \tag{2.5}$$

Proof. Since $\Phi_q^{\vartheta}(z)=2z-\Psi_q^{\vartheta}(z)=z-\sum\limits_{j=2}^{\infty}\binom{j+\vartheta-2}{\vartheta-1}\cdot q^{j-1}(1-q)^{\vartheta}z^j$, according to Lemma 1.2, we must show that

$$\Lambda_1(\mu,\delta,\gamma)=\sum_{j=2}^{\infty}[j(1+\gamma)-(\delta+\gamma)(1+j\mu-\mu)]\binom{j+\vartheta-2}{\vartheta-1}q^{j-1}(1-q)^{\vartheta}\le(1-\delta).$$

By writing $j = j - 1$ we get

$$\begin{aligned}\Lambda_1(\mu,\delta,\gamma) &= [1+\gamma-\mu(\delta+\gamma)]\sum_{j=2}^{\infty}(j-1)\binom{j+\vartheta-2}{\vartheta-1}q^{j-1}(1-q)^{\vartheta}\\ &\quad + (1-\delta)\sum_{j=2}^{\infty}\binom{j+\vartheta-2}{\vartheta-1}q^{j-1}(1-q)^{\vartheta}.\end{aligned}$$

From (2.2) and (2.3) we get

$$\begin{aligned}\Lambda_1(\mu,\delta,\gamma) &= [1+\gamma-\mu(\delta+\gamma)]q\,\vartheta(1-q)^{\vartheta}\sum_{n=0}^{\infty}\binom{j+\vartheta}{\vartheta}q^{j}\\ &\quad + (1-\delta)(1-q)^{\vartheta}\left[\sum_{n=0}^{\infty}\binom{j+\vartheta-1}{\vartheta-1}q^{j}-1\right]\\ &= [1+\gamma-\mu(\delta+\gamma)]\frac{q\,\vartheta}{1-q}+(1-\delta)\left[1-(1-q)^{\vartheta}\right].\end{aligned}$$

But $\Lambda_1(\mu,\delta,\gamma)$ is bounded above by $1-\delta$ if and only if (2.5) holds. □

Theorem 2.2. *If $\vartheta \geq 1$, then $\Phi_q^{\vartheta}(z) \in \mathbf{UCT}(\mu,\delta,\gamma)$ if and only if*

$$[1+\gamma-\mu(\delta+\gamma)]\frac{q^2\,\vartheta(\vartheta+1)}{(1-q)^{\vartheta+2}}+[3+2\gamma-\delta-2\mu(\delta+\gamma)]\frac{q\,\vartheta}{(1-q)^{\vartheta+1}} \leq 1-\delta. \tag{2.6}$$

Proof. In view of Lemma 1.4,we need to show that

$$\Lambda_2(\mu,\delta,\gamma) = \sum_{j=2}^{\infty}j[j(1+\gamma)-(\delta+\gamma)(1+j\mu-\mu)]\binom{j+\vartheta-2}{\vartheta-1}q^{j-1}(1-q)^{\vartheta} \leq (1-\delta).$$

Now

$$\begin{aligned}\Lambda_2(\mu,\delta,\gamma) &= [1+\gamma-\mu(\delta+\gamma)]\sum_{j=2}^{\infty}j^2\binom{j+\vartheta-2}{\vartheta-1}q^{j-1}(1-q)^{\vartheta}\\ &\quad -(\delta+\gamma)(1-\mu)\sum_{j=2}^{\infty}j\binom{j+\vartheta-2}{\vartheta-1}q^{j-1}(1-q)^{\vartheta}\end{aligned}$$

Writing $j^2 = (j-1)(j-2) + 3(j-1) + 1$ and $j = (j-1) + 1$, we get

$$\Lambda_2(\mu, \delta, \gamma) = [1 + \gamma - \mu(\delta + \gamma)] \sum_{j=2}^{\infty} (j-1)(j-2) \binom{j + \vartheta - 2}{\vartheta - 1} q^{j-1}(1-q)^{\vartheta}$$

$$+ [3 + 2\gamma - \delta - 2\mu(\delta + \gamma)] \sum_{j=2}^{\infty} (j-1) \binom{j + \vartheta - 2}{\vartheta - 1} q^{j-1}(1-q)^{\vartheta}$$

$$+ (1-\delta) \sum_{j=2}^{\infty} \binom{j + \vartheta - 2}{\vartheta - 1} q^{j-1}(1-q)^{\vartheta}.$$

From (2.2), (2.3) and (2.4),we get

$$\Lambda_2(\mu, \delta, \gamma) = [1 + \gamma - \mu(\delta + \gamma)](1-q)^{\vartheta} q^2\ \vartheta(\vartheta + 1) \sum_{j=0}^{\infty} \binom{j + \vartheta + 1}{\vartheta + 1} q^j$$

$$+ [3 + 2\gamma - \delta - 2\mu(\delta + \gamma)] q\ \vartheta(1-q)^{\vartheta} \sum_{j=0}^{\infty} \binom{j + \vartheta}{\vartheta} q^j$$

$$+ (1-\delta)(1-q)^{\vartheta} \left[\sum_{j=0}^{\infty} \binom{j + \vartheta - 1}{\vartheta - 1} q^j - 1 \right].$$

Further by using (2.1) we get

$$\Lambda_2(\mu, \delta, \gamma) = [1 + \gamma - \mu(\delta + \gamma)] \frac{q^2\ \vartheta(\vartheta + 1)}{(1-q)^2}$$

$$+ [3 + 2\gamma - \delta - 2\mu(\delta + \gamma)] \frac{q\ \vartheta}{(1-q)} + (1-\delta)[1 - (1-q)^{\vartheta}].$$

Then $\Lambda_2(\mu, \delta, \gamma)$ is bounded above by $(1-\delta)$ if and only if (2.6) holds. $\square$

3. Inclusion Properties

Let $f \in \mathbf{A}$ and f is in $\mathbf{R}^{\kappa}(\mathrm{A}, \mathrm{B})$, $(\kappa \in \mathbb{C}\backslash\{0\},\ -1 \leq \mathrm{B} < \mathrm{A} \leq 1)$, if

$$\left| \frac{f'(z) - 1}{(\mathrm{A} - \mathrm{B})\kappa - \mathrm{B}[f'(z) - 1]} \right| < 1, \quad z \in \mathbb{U},$$

holds. The class $\mathbf{R}^{\kappa}(\mathrm{A}, \mathrm{B})$ was defined earlier by Dixit and Pal [4]. By fixing $\kappa = 1,\ \mathrm{A} = \gamma$ and $\mathrm{B} = -\gamma\ (0 < \gamma \leq 1)$, (among others)Padmanabhan [9] and Caplinger and Causey [2] deduce the class of functions $f \in \mathbf{A}$ satisfying the inequality $\left| \frac{f'(z)-1}{f'(z)+1} \right| < \gamma \quad (z \in \mathbb{U}; 0 < \gamma \leq 1)$. Making use of the following lemma, we will study the action of the Pascal distribution on the classes $\mathbf{UCT}(\mu, \delta, \gamma)$.

Lemma 3.1. *[4] If $f \in \mathbf{R}^{\kappa}(\mathrm{A},\mathrm{B})$ is of form (1.1), then*

$$|a_j| \leq \frac{(\mathrm{A}-\mathrm{B})\,|\kappa|}{j}, \quad j \in \mathbb{N}\setminus\{1\}.$$

The result is sharp.

Theorem 3.1. *Let If $\vartheta \geq 1$. If $f \in \mathbf{R}^{\kappa}(\mathrm{A},\mathrm{B})$, and if the inequality*

$$\begin{aligned}(\mathrm{A}-\mathrm{B})|\kappa| \Big\{ & [1+\gamma-\mu(\delta+\gamma)][1-(1-q)^{\vartheta}] \\ & - \frac{(\delta+\gamma)(1-\mu)}{q(\vartheta-1)}\left[(1-q)-[1+q(\vartheta-1)](1-q)^{\vartheta}\right]\Big\} \leq 1-\delta. \end{aligned} \tag{3.1}$$

is satisfied, then $\mathbf{I}_q^{\vartheta} f(z) \in \mathbf{TS}_P(\mu,\delta,\gamma)$.

Proof. Let f be given by(1.1)is in $\mathbf{R}^{\kappa}(\mathrm{A},\mathrm{B})$. By asset of Lemma 1.2, it enough to show that

$$\Lambda_3(\mu,\delta,\gamma) = \sum_{j=2}^{\infty}[j(1+\gamma)-(\delta+\gamma)(1+j\mu-\mu)]\binom{j+\vartheta-2}{\vartheta-1}q^{j-1}(1-q)^{\vartheta}|a_j| \leq 1-\delta.$$

Since $f \in \mathbf{R}^{\kappa}(\mathrm{A},\mathrm{B})$ then by Lemma 3.1 we have

$$|a_j| \leq \frac{(A-B)\,|\kappa|}{j},$$

$$\begin{aligned}\Lambda_3(\mu,\delta,\gamma) &\leq (\mathrm{A}-\mathrm{B})|\kappa| \left[\sum_{j=2}^{\infty}\frac{1}{j}[j(1+\gamma)-(\delta+\gamma)(1+j\mu-\mu)] \right. \\ &\qquad \left. \times \binom{j+\vartheta-2}{\vartheta-1}q^{j-1}(1-q)^{\vartheta}\right] \\ &= (\mathrm{A}-\mathrm{B})|\kappa| \left[[1+\gamma-\mu(\delta+\gamma)]\sum_{j=2}^{\infty}\binom{j+\vartheta-2}{\vartheta-1}q^{j-1}(1-q)^{\vartheta} \right. \\ &\qquad \left. -(\delta+\gamma)(1-\mu)\sum_{j=2}^{\infty}\frac{1}{j}\binom{j+\vartheta-2}{\vartheta-1}q^{j-1}(1-q)^{\vartheta}\right].\end{aligned}$$

Hence by using (2.1) and (2.3),we get

$$\begin{aligned}\Lambda_3(\mu,\delta,\gamma) \leq (\mathrm{A}-\mathrm{B})|\kappa| \Big\{ & [1+\gamma-\mu(\delta+\gamma)][1-(1-q)^{\vartheta}] \\ & - \frac{(\delta+\gamma)(1-\mu)}{q(\vartheta-1)}\left[(1-q)-[1+q(\vartheta-1)](1-q)^{\vartheta}\right]\Big\}.\end{aligned}$$

But $\Lambda_3(\mu,\delta,\gamma)$ is bounded above by $1-\delta$ if and only if (3.1) holds. □

Theorem 3.2. *Let* $m \geq 1$ *then* $\mathbf{G}(\vartheta, z) = \int_0^z \frac{\mathbf{I}_q^\vartheta f(t)}{t} dt$ *is in* $\mathbf{TS}_P(\mu, \delta, \gamma)$ *if and only if inequality*

$$[1+\gamma-\mu(\delta+\gamma)][1-(1-q)^\vartheta] - \frac{(\delta+\gamma)(1-\mu)}{q(\vartheta-1)} \left[(1-q) - [1+q(\vartheta-1)](1-q)^\vartheta\right] \leq 1-\delta.$$

Proof. Since $\mathbf{G}(\vartheta, z) = z - \sum_{j=2}^{\infty} \binom{j+\vartheta-2}{\vartheta-1} q^{j-1}(1-q)^\vartheta \frac{z^j}{j}$, then by Lemma 1.2 we need only to show that

$$\Lambda_4(\mu, \delta, \gamma) = \sum_{j=2}^{\infty} [j(1+\gamma) - (\delta+\gamma)(1+j\mu-\mu)] \binom{j+\vartheta-2}{\vartheta-1} q^{j-1}(1-q)^\vartheta \leq 1-\delta.$$

Now,

$$\begin{aligned}
\Lambda_4(\mu, \delta, \gamma) &= \sum_{j=2}^{\infty} \frac{1}{j}[j(1+\gamma) - (\delta+\gamma)(1+j\mu-\mu)] \binom{j+\vartheta-2}{\vartheta-1} q^{j-1}(1-q)^\vartheta \\
&= \sum_{j=2}^{\infty} \Bigg[[1+\gamma-\mu(\delta+\gamma)] \sum_{j=2}^{\infty} \binom{j+\vartheta-2}{\vartheta-1} q^{j-1}(1-q)^\vartheta \\
&\quad -(\delta+\gamma)(1-\mu) \sum_{j=2}^{\infty} \frac{1}{j} \binom{j+\vartheta-2}{\vartheta-1} q^{j-1}(1-q)^\vartheta \Bigg].
\end{aligned}$$

Following the similar technics employed in Theorem 3.1, we have

$$\begin{aligned}
\Lambda_4(\mu, \delta, \gamma) &= [1+\gamma-\mu(\delta+\gamma)][1-(1-q)^\vartheta] \\
&\quad - \frac{(\delta+\gamma)(1-\mu)}{q(\vartheta-1)} \left[(1-q) - [1+q(\vartheta-1)](1-q)^\vartheta\right]
\end{aligned}$$

which is bounded above by $1-\delta$ if and only if (3.2) holds. □

Theorem 3.3. *Let If* $\vartheta \geq 1$. *If* $f \in \mathbf{R}^\kappa(\mathrm{A}, \mathrm{B})$, *and if*

$$(\mathrm{A}-\mathrm{B})|\kappa| \left\{ [1+\gamma-\mu(\delta+\gamma)] \frac{q\,\vartheta}{1-q} + (1-\delta)\left[1-(1-q)^\vartheta\right] \right\} \leq 1-\delta \tag{3.2}$$

holds, then $\mathbf{I}_q^\vartheta f(z) \in \mathbf{UCT}(\mu, \delta, \gamma)$.

Proof. Let f be given by (1.1) and $f \in \mathbf{R}^\kappa(\mathrm{A}, \mathrm{B})$. By virtue of Lemma 1.4, it suffices to show that

$$\Lambda_5(\mu, \delta, \gamma) = \sum_{j=2}^{\infty} j[j(1+\gamma) - (\delta+\gamma)(1+j\mu-\mu)] \binom{j+\vartheta-2}{\vartheta-1} q^{j-1}(1-q)^\vartheta |a_j| \leq 1-\delta.$$

Since $f \in \mathbf{R}^{\kappa}(\mathrm{A},\mathrm{B})$ then by Lemma 3.1 we have $|a_j| \leq \frac{(A-B)|\kappa|}{j}$.

$$\begin{aligned}
&\Lambda_5(\mu,\delta,\gamma) \\
&\leq (\mathrm{A}-\mathrm{B})|\kappa| \left[\sum_{j=2}^{\infty}[j(1+\gamma)-(\delta+\gamma)(1+j\mu-\mu)] \binom{j+\vartheta-2}{\vartheta-1} \cdot q^{j-1}(1-q)^{\vartheta}\right] \\
&\leq (\mathrm{A}-\mathrm{B})|\kappa| \left[[1+\gamma-\mu(\delta+\gamma)] \sum_{j=2}^{\infty}(j-1)\binom{j+\vartheta-2}{\vartheta-1} q^{j-1}(1-q)^{\vartheta}\right. \\
&\quad \left. +(1-\delta)\sum_{j=2}^{\infty} \binom{j+\vartheta-2}{\vartheta-1} q^{j-1}(1-q)^{\vartheta}\right].
\end{aligned}$$

Further, proceeding as in Theorem 2.1

$$\Lambda_5(\mu,\delta,\gamma) \leq (\mathrm{A}-\mathrm{B})|\kappa| \left\{[1+\gamma-\mu(\delta+\gamma)]\frac{q\,\vartheta}{1-q} + (1-\delta)\left[1-(1-q)^{\vartheta}\right]\right\}.$$

But $\Lambda_5(\mu,\delta,\gamma)$ is bounded above by $1-\delta$ if and only if (3.2) holds. $\square$

Theorem 3.4. *Let $\vartheta \geq 1$ then $\mathbf{G}(\vartheta, z) = \int_0^z \frac{\mathbf{I}_q^{\vartheta} f(t)}{t} dt$ is in $\mathbf{UCT}(\mu,\delta,\gamma)$ if and only if inequality*

$$[1+\gamma-\mu(\delta+\gamma)]\frac{q\,\vartheta}{(1-q)^{\vartheta+1}} \leq 1-\delta. \tag{3.3}$$

Proof. Since $\mathbf{G}(\vartheta, z) = z + \sum_{j=2}^{\infty} \binom{j+\vartheta-2}{\vartheta-1} q^{j-1}(1-q)^{\vartheta}\frac{z^j}{n}$ then by Lemma 1.4, we need only to show that

$$\Lambda_6(\mu,\delta,\gamma) = \sum_{j=2}^{\infty} j[j(1+\gamma)-(\delta+\gamma)(1+j\mu-\mu)] \binom{j+\vartheta-2}{\vartheta-1} q^{j-1}(1-q)^{\vartheta} \leq 1-\delta.$$

Now,

$$\begin{aligned}
\Lambda_6(\mu,\delta,\gamma) &= \sum_{j=2}^{\infty} j[j(1+\gamma)-(\delta+\gamma)(1+j\mu-\mu)] \binom{j+\vartheta-2}{\vartheta-1} q^{j-1}(1-q)^{\vartheta}\frac{1}{j} \\
&= \sum_{j=2}^{\infty}[j(1+\gamma)-(\delta+\gamma)(1+j\mu-\mu)] \binom{j+\vartheta-2}{\vartheta-1} q^{j-1}(1-q)^{\vartheta}.
\end{aligned}$$

Further, proceeding as in Theorem 2.1

$$\Lambda_6(\mu,\delta,\gamma) = [1+\gamma-\mu(\delta+\gamma)]\frac{q\,\vartheta}{1-q} + (1-\delta)\left[1-(1-q)^{\vartheta}\right],$$

which is bounded above by $1-\delta$ if and only if (3.3) holds. $\square$

Concluding Remarks. By taking $\gamma = 0$ and specializing the parameter μ in above theorems we obtain analogous results for the subclasses listed in [5].

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