Advances in Mathematics: Scientific Journal **9** (2020), no.5, 2467–2476 ISSN: 1857-8365 (printed); 1857-8438 (electronic) https://doi.org/10.37418/amsj.9.5.10

STRONG FUZZY BI-IDEALS OF BCK-ALGEBRAS

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ABSTRACT. We consider the strong fuzzification of the concept of sub-algebras and strong Bi-ideals in BCK - algebras are defined and explored some of their properties. We also fuzzify the notion of equivalence relations on the family of all strong fuzzy Bi-ideals of a BCK-algebras are discussed and some of the properties are investigated.

1. INTRODUCTION

After the initiation of the concept of fuzzy set by Zadeh in [5], Imaiand Iseki introduced two classes of abstract algebras, BCK-algebras and BCI-algebras. BCK-algebras where studied by many researches.

In this paper, using the Atanassov, [1], and Young Bae Jun, [2] we establish the notion of equivalence relations on the family of all strong fuzzification of the concept of sub-algebras and strong fuzzy Bi-ideals of a BCK-algebra and investigate some of the properties.

2. Preliminaries

The following definition is given in [3].

Definition 2.1. A BCK-algebra is a non-empty set X with a binary operation * and a constant 0 satisfying the following condition:

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²⁰¹⁰ Mathematics Subject Classification. 54A40.

Key words and phrases. Fuzzy Bi-ideals, Strong Fuzzy Bi-ideals, homomorphism.

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- (a) ((x * y) * (x * z) * (z * y)) = 0;
- (b) (x * (x * y)) * y) = 0;
- (c) x * x = 0;
- (d) 0 * x = 0;
- (e) x * y = 0 and y * x = 0 imply that x = y for all $x, y, z \in X$.

The following definition is given in [5].

Definition 2.2. A partial ordering " \leq " on X can be defined by $x \leq y$ iff x * y = 0.

The following definition is given in [3].

Definition 2.3. A non-empty subset S of a BCK-algebra X is called a sub-algebra of X if $x * y \in S$ whenever $x, y \in S$.

The next definition is given in [1].

Definition 2.4. A non-empty subset I of a BCK-algebra X is called an ideal of X if

- (a) $0 \in I$;
- (b) $x * y \in I$ and $y \in I$ imply that $x \in I$ for all $x, y \in X$.

The following two definitions are given in [5].

Definition 2.5. A fuzzy set α in a non-empty set X we mean a function $\alpha : X \rightarrow [0, 1]$.

Definition 2.6. The complement of α , denoted by $\bar{\alpha}$ is the fuzzy set in X given by $\bar{\alpha}(x) = 1 - \alpha(x)$ for all $x \in X$.

A fuzzy set α in a BCK-algebra X is called a strong fuzzy sub-algebra of X if $\alpha(y * z) \ge \min\{\alpha(y), \alpha(z)\}$ for all $y, z \in Y$.

In [2] is given the next definition.

Definition 2.7. A fuzzy set μ in a BCK-algebra X is called a fuzzy ideal of X if

- (a) $\alpha(0) \ge \alpha(x)$ for all $x \in X$.
- (b) $\alpha(x) \ge \min\{\alpha(x * y), \alpha(y)\}$ for all $x, y \in X$.

In [3] are given the following definitions:

Definition 2.8. For any $t \in [0,1]$ and a fuzzy set α in a non-empty set X, the set $U(\alpha;t) = \{x \in X/\alpha(x) \ge t\}$ is called an upper t-level cut of α and the set $L(\alpha;t) = \{x \in X/\alpha(x) \le t\}$ is called a lower t-level cut of α .

Definition 2.9. A mapping $f : Y \to Z$ of BCK-algebras is called a homomorphism if f(y * z) = f(y) * f(z) for all $y, z \in Y$.

Note that if $f: Y \to Z$ is a homomorphism of BCK-algebras then f(0) = 0.

Let $f : Y \to Z$ be a homomorphism of BCK-algebras. For any SFS $A = (\alpha_A, \beta_A)$ in Z, we define a new SFS $A^f = (\alpha_A^f, \beta_A^f)$ in Y by, $\alpha_A^f(x) = \alpha_A(f(x))$, $\beta_A^f(x) = \beta_A(f(x))$ for all $y \in Y$.

In [1] is given the definition:

Definition 2.10. A non-empty set X is an object having the form

$$A = \{(x, \alpha_A(x), \beta_A(x)) | x \in X\}$$

where the functions $\alpha : X \to [0,1]$ and $\beta : X \to [0,1]$ denote the degree of membership and the degree of non-membership of each element $x \in X$ to the set A and $0 \le \alpha_A(x) + \beta_A(x) \le 1$ for all $x \in X$.

An intuitionistic fuzzy set $A = \{(x, \alpha_A(x), \beta_A(x)) | x \in X\}$ in X can be identified to an ordered pair (α_A, β_A) in $I^X \times I^X$. For the sake of simplicity, we use the symbol $A = (\alpha_A, \beta_A)$ for the IFS, $A = \{(x, \alpha_A(x), \beta_A(x) | x \in X\}.$

In [4] is given the following definition.

Definition 2.11. A fuzzy subset β in a BCK-algebra X is called fuzzy Bi-ideal if

(a) $\beta_A(0) \ge \beta_A(x)$. (b) $\beta_A(x * y) \ge \min\{\beta_A(x * y * z), \beta_A(z)\}$ for all $x, y, z \in X$.

In [1] is given the following definition.

Definition 2.12. An Intuitionistic fuzzy set $A = (\alpha_A, \beta_A)$ in X is called an intuitionistic fuzzy sub-algebra of X if it satisfies:

- (a) $\alpha_A(x * y) \ge \min\{\alpha_A(x), \alpha_A(y)\}.$
- (b) $\beta_A(x * y) \leq max\{\beta_A(x), \beta_A(y)\}$ for all $x, y \in X$.

Proposition 2.1. Every intuitionistic fuzzy sub-algebra $A = (\alpha_A, \beta_A)$ of X satisfies the inequalities

- (a) $\alpha_A(0) \ge \alpha_A(x)$ and
- (b) $\beta_A(0) \leq \beta_A(x)$ for all $x \in X$.

In [3] is given the following definition.

Definition 2.13. An Intuitionistic fuzzy set $A = (\alpha_A, \beta_A)$ in X is called an intuitionistic fuzzy ideal of X if it satisfies the following inequalities:

- (a) $\alpha_A(0) \ge \alpha_A(x)$ and $\beta_A(0) \le \beta_A(x)$,
- (b) $\alpha_A(x) \ge \min\{\alpha_A(x*y), \alpha_A(y)\},\$
- (c) $\beta_A(x) \leq max\{\beta_A(x * y), \beta_A(y)\}$ for all $x, y \in X$.

In [4] is given the following definition.

Definition 2.14. An intuitionistic fuzzy set $A = (\alpha_A, \beta_A)$ in X is called an intuitionistic fuzzy Bi-ideal of X if it satisfies:

- (a) $\alpha_A(0) \ge \alpha_A(x)$ and $\beta_A(0) \le \beta_A(x)$. (b) $\alpha_A(x * y) \ge \min\{\alpha_A(x * y * z), \alpha_A(z)\}$
- (c) $\beta_A(x * y) \leq max\{\beta_A(x * y * z), \beta_A(z)\}$ for all $x, y, z \in X$.

3. Strong Fuzzy Bi-ideal

Definition 3.1. A Strong fuzzy set $A = (\alpha_A, \beta_A)$ in X is called a Strong fuzzy Bi-ideal of X if it satisfies:

(a) α_A(0) ≥ α_A(x) and β_A(0) ≤ β_A(x).
(b) α_A(y * z) ≥ min{α_A((z * (y * z)) * x), α_A(x)}.
(c) β_A(y * z) ≤ max{β_A((z * (y * z)) * x), β_A(x)} for all x, y, z ∈ X.

Example 1. Consider a BCK-algebra $X = \{0, a, b, c\}$ with the following cayley table.

*	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
C	c	c	c	0

Define an SFS, $A = (\alpha_A, \beta_A)$ in X as follows : $\alpha_A(0) = 0.6, \ \alpha_A(a) = 0.5, \ \alpha_A(b) = 0.8, \ \alpha_A(c) = 0.3$ $\beta_A(0) = 0.1, \ \beta_A(a) = 0.8, \ \beta_A(b) = 0.4, \ \beta_A(c) = 0.6.$ Then $A = (\alpha_A, \beta_A)$ is a Strong fuzzy Bi-ideal of X.

Lemma 3.1. Let a SFS $A = (\alpha_A, \beta_A)$ in X be a Strong fuzzy Bi-ideal of X. If the inequality $y * z \le x$ holds in X, then

(a) $\alpha_A(y * z) \ge \min\{\alpha_A(y), \alpha_A(x)\},\$ (b) $\beta_A(y * z) \le \max\{\beta_A(y), \beta_A(x)\}.$

Proof. Let $x, y, z \in X$ be such that $y * z \leq x$. Then (z * (y * z)) = y. Now

$$(a) \quad \alpha_A(y * z) \geq \min\{\alpha_A((z * (y * z)) * x), \alpha_A(x)\} \\ \geq \min\{\min\{\alpha_A(y), \alpha_A(x)\}, \alpha_A(x)\} \\ = \min\{\alpha_A(y), \alpha_A(x)\}$$

Therefore, $\alpha_A(y * z) \ge \min\{\alpha_A(y), \alpha_A(x)\}$

$$(b) \quad \beta_A(y * z) \leq \max\{\beta_A((z * (y * z)) * x), \beta_A(x)\} \\ \leq \max\{\max\{\beta_A(y), \beta_A(x)\}, \beta_A(x)\} \\ = \max\{\beta_A(y), \beta_A(x)\}$$

Therefore, $\beta_A(y * z) \le max\{\beta_A(y), \beta_A(x)\}$. This completes the proof.

Lemma 3.2. Let $A = (\alpha_A, \beta_A)$ be a Strong fuzzy Bi-ideal of X. If $y * z \le y$ in X, then $\alpha_A(y * z) \ge \alpha_A(y)$, $\beta_A(y * z) \le \beta_A(y)$ that is α_A is order-reserving and β_A is order-preserving.

Proof. Let $x, y, z \in X$ be such that $y * z \le y$.

$$(a) \quad \alpha_A(y * z) \geq \min\{\alpha_A((z * (y * z)) * x), \alpha_A(x)\} \\ \geq \min\{\min\{\alpha_A(y), \alpha_A(x)\}, \alpha_A(x)\} \\ = \min\{\alpha_A(y), \alpha_A(x)\} \\ \geq \min\{\alpha_A(y), \alpha_A(0)\} \\ = \alpha_A(y)$$

Therefore, $\alpha_A(y * z) \ge \alpha_A(y)$.

$$(b) \quad \beta_A(y * z) \leq \max\{\beta_A((z * (y * z)) * x), \beta_A(x)\} \\ \leq \max\{\max\{\beta_A(y), \beta_A(x)\}, \beta_A(x)\} \\ = \max\{\beta_A(y), \beta_A(x)\} \\ \leq \max\{\beta_A(y), \beta_A(0)\} \\ = \beta_A(y)$$

Therefore, $\beta_A(y * z) \leq \beta_A(y)$.

Hence α_A is order-reserving and β_A is order-preserving.

Theorem 3.1. If $A = (\alpha_A, \beta_A)$ is a Strong fuzzy Bi-ideal of X, then for any $y * z, a_1, a_2, ..., a_n \in X, (...((y * z * a_1) * a_2) * ...) * a_n = 0$ implies $\alpha_A(x) \ge min\{\alpha_A(a_1), \alpha_A(a_2), ..., \alpha_A(a_n)\}$ and $\beta_A(x) \le max\{\beta_A(a_1), \beta_A(a_2), ..., \beta_A(a_n)\}$.

Proof. Using induction on *n*, the proof is straightforward.

Theorem 3.2. Every Strong fuzzy Bi-ideal of X is a Strong fuzzy sub-algebra of X.

Proof. Let $A = (\alpha_A, \beta_A)$ be a Strong fuzzy bi-ideal of X.

Since $y * z \le x$ for all $x, y, z \in X$, it follows from Lemma 3.2 that, $\alpha_A(y * z) \ge \alpha_A(x)$, $\beta_A(y * z) \le \beta_A(x)$. Now,

$$\begin{array}{rcl} (a) & \alpha_A(y*z) & \geq & \alpha_A(x) \\ & \geq & \min\{\alpha_A(y*z), \alpha_A(z)\} \\ & \geq & \min\{\alpha_A(x), \alpha_A(z)\} \end{array}$$

Therefore, $\alpha_A(y * z) \ge \min\{\alpha_A(x), \alpha_A(z)\}$

(b)
$$\beta_A(y * z) \leq max\{\beta_A(y * z), \beta_A(z)\}$$

 $\leq max\{\beta_A(x), \beta_A(z)\}$

Therefore, $\beta_A(y * z) \le max\{\beta_A(x), \beta_A(z)\}$ Thus $A = (\alpha_A, \beta_A)$ is a Strong fuzzy sub-algebra of X.

Theorem 3.3. Let $A = (\alpha_A, \beta_A)$ be a Strong fuzzy sub-algebra of X such that $\alpha_A(x) \ge min\{\alpha_A(y), \alpha_A(z)\}, \beta_A(x) \le max\{\beta_A(y), \beta_A(z)\}$ for all $x, y, z \in X$ satisfying the inequality $y * z \le y$. Then $A = (\alpha_A, \beta_A)$ is a Strong fuzzy Bi-ideal of

Proof. Let A = (α_A, β_A) be a Strong fuzzy sub-algebra of X. Recall that V(0) ≥ minα_A(y * z) and β_A(0) ≤ maxβ_A(y * z) for all X.
Since (z * z) * (z * x) ≤ x, it follows from the hypothesis that,
(a) α_A(y * z) ≥ min{α_A((z * (y * z)) * x), α_A(x)}
(b) β_A(y * z) ≤ max{β_A((z(*y * z)) * x), β_A(x)}.
Hence A = (α_A, β_A) is a Strong fuzzy bi-ideal of X.

Lemma 3.3. A SFS $A = (\alpha_A, \beta_A)$ is a Strong fuzzy Bi-ideal of X iff the fuzzy sets α_A and $\overline{\beta_A}$ are fuzzy Bi-ideals of X.

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X.

Proof. Let $A = (\alpha_A, \beta_A)$ be a Strong fuzzy bi-ideal of X.

For every $x, y, z \in X$, we have,

(a)
$$\overline{\beta}_A(0) = 1 - \beta_A(0)$$

 $\geq 1 - \beta_A(y * z)$
 $= \overline{\beta}_A(y * z)$

Therefore, $\bar{\beta}_A(0) \geq \bar{\beta}_A(y * z)$

$$(b) \quad \bar{\beta}_{A}(y * z) = 1 - \beta_{A}(y * z)$$

$$\geq 1 - max\{\beta_{A}((z(*y * z)) * x), \beta_{A}(x)\}$$

$$= min\{1 - \beta_{A}((z * (y * z)) * x), 1 - \beta_{A}(x)\}$$

Therefore, $\bar{\beta}_A(y * z) = min\{\bar{\beta}_A((z * (y * z)) * x), \bar{\beta}_A(x)\}$. Hence $\bar{\beta}_A$ is a fuzzy bi-ideal of *X*.

Conversely assume that α_A and $\overline{\beta_A}$ are fuzzy bi-ideals of X. For every $x, y, z \in X$, we have,

 $\alpha_A(0) \ge \alpha_A(y * z), 1 - \beta_A(0) = \overline{\beta_A}(0) \ge \overline{\beta_A}(y * z) = 1 - \beta_A(y * z)$ That is, $\beta_A(0) \le (y * z); \beta_A(y * z) \ge \min\{\beta_A((z(*y * z)) * x, \beta_A(x))\}$ and

$$\begin{aligned} 1 - \beta_A(y * z) &= \beta_A(y * z) \geq \min\{\beta_A((z * (y * z)) * x), \beta_A(x)\} \\ &= \min\{1 - \beta_A((z * (y * z)) * x), 1 - \beta_A(x)\} \\ &= 1 - \max\{\beta_A((z * (y * z)) * x), \beta_A(x)\} \\ \beta_A(y * z) \leq \max\{\beta_A((z * (y * z)) * x), \beta_A(x)\} \end{aligned}$$

Hence $A = (\alpha_A, \beta_A)$ is a Strong fuzzy Bi-ideal of X.

Theorem 3.4. Let $A = (\alpha_A, \beta_A)$ is a Strong fuzzy set in X. Then $A = (\alpha_A, \beta_A)$ is a Strong fuzzy Bi-ideal of X iff $A = (\alpha_A, \overline{\alpha_A})$ and $A = (\beta_A, \overline{\beta_A})$ are Strong fuzzy Bi-ideals of X.

Proof. If $A = (\alpha_A, \beta_A)$ is a Strong fuzzy bi-ideal of X, then $\alpha_A = \overline{\alpha_A}$ and β_A are strong fuzzy Bi-ideals of X.

Hence $A = (\alpha_A, \bar{\alpha_A})$ and $A = (\beta_A, \bar{\beta_A})$ are Strong fuzzy Bi-ideals of X.

Conversely, if $A = (\alpha_A, \bar{\alpha_A})$ and $A = (\beta_A, \bar{\beta_A})$ are Strong fuzzy Bi-ideals of X, then the fuzzy sets α_A and $\bar{\beta_A}$ are Strong fuzzy Bi-ideals of X.

Hence $A = (\alpha_A, \beta_A)$ is a Strong fuzzy Bi-ideal of X.

Theorem 3.5. Let $f : Y \to Z$ be a homomorphism of BCK-algebra. If a SFS $A = (\alpha_A, \beta_A)$ in Z is a Strong fuzzy Bi-ideal of Z, then a SFS $A^f = (\alpha_A^f, \beta_A^f)$ in Y is a Strong fuzzy Bi-ideal of Y.

Proof. We first have that

(a)
$$\alpha_A^f(y * z) = \alpha_A(f(y * z)) \leq \alpha_A(0)$$

= $\alpha_A(f(0))$

Therefore, $\alpha_A^f(y * z) = \alpha_A^f(0)$.

(b)
$$\beta_A^f(y * z) = \beta_A(f(y * z)) \geq \beta_A(0)$$

= $\beta_A(f(0))$

Therefore, $\beta_A^f(y * z) = \beta_A^f(0)$ for all $y * z \in Y$. Let $x, y, z \in Y$. Then (a)

$$\min\{\alpha_{A}^{f}((z * (y * z)) * x), \alpha_{A}^{f}(x)\}$$

$$= \min\{\alpha_{A}(f((z * (y * z)) * x), \alpha_{A}(f(x)))\}$$

$$= \min\{\alpha_{A}(f((z * (y * z)) * x), \alpha_{A}(f(x)))\}$$

$$\leq \min\{\alpha_{A}(f(z) * f(y * z) * f(x)), \alpha_{A}(f(x))\}$$

$$= \min\{\alpha_{A}(f(z) * f(y * z) * \alpha_{A}f(x)), \alpha_{A}(f(x))\}$$

$$\leq \min\{\alpha_{A}(f(y * z))\}$$

$$= \alpha_{A}^{f}(y * z)$$

Therefore, $\min\{\alpha^f_A((z*(y*z))*x),\alpha^f_A(x)\}\leq \alpha^f_A(y*z)$ (b)

$$max\{\beta_{A}^{f}((z * (y * z)) * x), \beta_{A}^{f}(x)\}$$

$$= max\{\beta_{A}(f((z * (y * z)) * x), \beta_{A}(f(x)))\}$$

$$= max\{\beta_{A}(f((z * (y * z)) * x), \beta_{A}(f(x)))\}$$

$$\geq max\{\beta_{A}(f(z) * f(y * z) * f(x)), \beta_{A}(f(x))\}$$

$$= max\{\beta_{A}(f(z) * f(y * z) * \beta_{A}f(x)), \beta_{A}(f(x))\}$$

$$\leq max\{\beta_{A}(f(y * z))\}$$

$$= \beta_{A}^{f}(y * z)$$

Therefore,
$$max\{\beta_A^F((z*(y*z))*x), \beta_A^f(x)\} \ge \beta_A^f(y*z)$$
.
Hence $A^f = (\alpha_A^f, \beta_A^f)$ in Y is a Strong fuzzy Bi-ideal of Y. \Box

Theorem 3.6. Let $f : Y \to Z$ be an epimorphism of BCK-algebras and let $A = (\alpha_A, \beta_A)$ be a SFS in Z. If $A^f = (\alpha_A^f, \beta_A^f)$ is a Strong fuzzy Bi-ideal of Y, then $A = (\alpha_A, \beta_A)$ is a Strong fuzzy Bi-ideal of Z.

Proof. For any $y, z \in Z$, there exists $a \in Y$ such that f(a) = y * z. Then

(a)
$$\alpha_A(y * z) = \alpha_A(f(a)) = \alpha_A^f(a) \leq \alpha_A^f(0)$$

= $\alpha_A(f(0))$
= $\alpha_A(0)$

Therefore, $\alpha_A(y * z) = \alpha_A(0)$.

(b)
$$\beta_A(y * z) = \beta_A(f(a)) = \beta_A^f(a) \ge \beta_A^f(0)$$

= $\beta_A(f(0))$
= $\beta_A(0)$

Therefore, $\beta_A(y * z) = \beta_A(0)$.

Let $x, y, z \in Z$, then f(a) = x, f(b) = y and f(c) = z for some $a, b, c \in Y$. It follows that

$$\begin{aligned} (a)\alpha_{A}(y * z) &= \alpha_{A}(f(b) * f(c)) = \alpha_{A}^{f}(b * c) \\ &\geq \min\{\alpha_{A}^{f}((c * (b * c)) * a), \alpha_{A}^{f}(a)\} \\ &= \min\{\alpha_{A}((f(c) * (b * c)) * f(a)), \alpha_{A}(f(a))\} \\ &= \min\{\alpha_{A}(f(c) * f(b) * f(a)), \alpha_{A}(f(a))\} \end{aligned}$$

Therefore, $\alpha_A(y * z) \ge \min\{\alpha_A((z * (y * z)) * x), \alpha_A(x)\}$

$$(b) \ \beta_A(y * z) = \beta_A(f(b) * f(c)) = \beta_A^f(b * c) \\ \leq \max\{\beta_A^f((c * (b * c)) * a), \beta_A^f(a)\} \\ = \max\{\beta_A((f(c) * (b * c)) * f(a)), \beta_A(f(a))\} \\ = \max\{\beta_A(f(c) * f(b) * f(a)), \beta_A(f(a))\}$$

Therefore, $\beta_A(y * z) \leq max\{\beta_A((z * (y * z)) * x), \beta_A(x)\}$. Hence $A = (\alpha_A, \beta_A)$ is a Strong fuzzy Bi-ideal of Z.

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