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ON TRIGONOMETRIC TOPOLOGICAL SPACES

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ABSTRACT. In this paper we introduce a new topological space, namely, Trigonometric topological space. A Strong trigonometric topological space is a topological space in which two topologies Sine and Cosine topologies induced from the given topology are coincide. Further, we discuss the properties of Interior and Closure operators in Sine and Cosine topological spaces.

1. INTRODUCTION

In this paper, we introduce Trigonometric topological spaces. These spaces are based on Sine and Cosine topologies. In a bitopological space we have considered two different topologies but in a trigonometric topological space the two topologies are derived from one topology. So, we observe that trigonometric topological space is different from bitopological space. Also, we define interior and closure operators in Sine and Cosine topological spaces and study their basic properties.

Section 2 deals with the preliminary concepts. In section 3, Sine and Cosine topologies are introduced together with their basic properties. The Trigonometric topological spaces are introduced in Section 4.

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2. PRELIMINARIES

Throughout this paper X denotes a set having elements from $[0, \frac{\pi}{2}]$. If (X, τ) is a topological space, then for any subset A of X, $cl(A)$ denotes the closure of A, $int(A)$ denotes the interior of A. Further $X \backslash A$ denotes the complement of A in X. The following definitions are very useful in the subsequent sections.

Definition 2.1. *A topology on a set X is a collection τ of subsets of X having the following properties:*

(i) *$\emptyset$, X are in τ.*
(ii) *The union of elements of any subcollection of τ is in τ.*
(iii) *The intersection of the elements of any finite subcollection of τ is in τ.*

The set X together with the topology τ is called a topological space. The elements of τ are called open sets. The complement of an open set is called a closed set. The set of all closed sets in X is denoted by τ^c.

Definition 2.2. *Let X be a topological space. Let A be a subset of X. Then the intersection of all closed sets containing A is called the closure of A and is denoted by $cl(A)$. Also, the union of all open sets contained in A is called the interior of A and is denoted by $int(A)$.*

3. SINE AND COSINE TOPOLOGICAL SPACES

In this section, we introduce the concepts Sine and Cosine topological spaces and study their basic properties. Also, we discuss the properties of interior and closure operators in Sine and Cosine topological spaces. We begin this section by the construction of Sine topology.

Construction of Sine Topology. Let X be any non-empty set having elements from $[0, \frac{\pi}{2}]$. Let $Sin\ X$ be the set consisting of the Sine values of the corresponding elements of X.

Define a function $f_s : X \to Sin\ X$ by $f_s(x) = Sin(x)$. Then f_s is a bijective function. This implies, $f_s(\emptyset) = \emptyset$ and $f_s(X) = Sin\ X$. That is, $Sin\ \emptyset = \emptyset$.

Result 3.1. *Let X be a set and A, B be subsets of X. Then $A \subseteq B$ if and only if $Sin\ A \subseteq Sin\ B$.*

Proof. The proof is straight forward. □

Result 3.2. *The above result is not true for any subsets of* $[0, 2\pi)$. *That is, A=B implies* $Sin\ A = Sin\ B$ *is true for any subsets of* $[0, 2\pi)$. *But* $Sin\ A = Sin\ B$ *does not imply A=B. For,* $Sin\{0\} = Sin\{\pi\} = \{0\}$, *but* $\{0\} \neq \{\pi\}$. *Hence A=B iff* $SinA = SinB$ *is true only for* $[0, \frac{\pi}{2}]$.

Notation 3.1. *If* τ *is a topology on X, then* τ_s *denotes the set consisting of the images under* f_s *of the corresponding elements of* τ.

Result 3.3. *Let* (X, τ) *be a topological space. Then* τ_s *form a topology on* f_s*(X).*

Proof.

(i) Since $\emptyset,\ X \in \tau$, we have $f_s(\emptyset)$, $f_s(X) \in \tau_s$. That is, $\emptyset$, $f_s(X) \in \tau_s$.

(ii) Let $A_1, A_2, , A_n, \in \tau_s$. Then $A_i = f_s(B_i)$, where $B_i \in \tau$ for $i = 1, 2, 3, .$ Since τ is a topology, we have $\bigcup\limits_{i=1}^{\infty} B_i \in \tau$. This implies, $f_s\left(\bigcup\limits_{i=1}^{\infty} B_i\right) \in \tau_s$. That is, $\bigcup\limits_{i=1}^{\infty} f_s(B_i) \in \tau_s$. Therefore, $\bigcup\limits_{i=1}^{\infty} A_i \in \tau_s$.

(iii) Let $A_1, A_2, , A_n \in \tau_s$. Then $A_i = f_s(B_i)$ where $B_i \in \tau$ for $i = 1, 2, , n$. Since τ is a topology, we have $\bigcap\limits_{i=1}^{n} B_i \in \tau$. This implies, $f_s\left(\bigcap\limits_{i=1}^{n} B_i\right) \in \tau_s$. That is, $\bigcap\limits_{i=1}^{n} f_s(B_i) \in \tau_s$. Therefore, $\bigcap\limits_{i=1}^{n} A_i \in \tau_s$. Hence τ_s is a topology on $f_s(X)$.

□

Definition 3.1. *Let* (X, τ) *be a topological space. Then* τ_s *form a topology on* $f_s(X)$. *This topology is called a Sine topology of X. The space* $(f_s(X), \tau_s)$ *is said to be a Sine topological space corresponding to X.*

That is, τ_s *form a topology on* $SinX$. *The elements of* τ_s *are called Sin-open sets. The complement of Sin-open set is said to be Sin-closed. The set of all Sin-closed subsets of* $SinX$ *is denoted by* τ.

Example 1. *Let* $X = \{0, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{2}\}$ *with topology* $\tau = \{\emptyset, \{0\}, \{\frac{\pi}{3}\}, \{0, \frac{\pi}{3}\}, \{0, \frac{\pi}{4}\}, \{0, \frac{\pi}{3}, \frac{\pi}{4}\}, X\}$. *Then* $SinX = \{0, \frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}, 1\}$,
$\tau_s = \{\emptyset, \{0\}, \{\frac{\sqrt{3}}{2}\}, \{0, \frac{\sqrt{3}}{2}\}, \{0, \frac{1}{\sqrt{2}}\}, \{0, \frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}\}, SinX\}$.

Here the sets $\emptyset, \{0\}, \{\frac{\sqrt{3}}{2}\}, \{0, \frac{\sqrt{3}}{2}\}, \{0, \frac{1}{\sqrt{2}}\}, \{0, \frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}\}, SinX$ *are called Sin-open sets & The Sin-closed sets are* $\emptyset, \{1\}, \{\frac{\sqrt{3}}{2}, 1\}, \{\frac{1}{\sqrt{2}}, 1\}, \{0, \frac{1}{\sqrt{2}}, 1\}, \{\frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}, 1\}$, $SinX$.

Construction of Cosine topology. Let $CosX$ be the set consisting of the Cosine values of the corresponding elements of X. Define a function $f_c : X \to Cos\ X$ by $f_c(x) = Cos\ x$. Then f_c is bijective. Also, $f_c(\emptyset) = \emptyset$ and $f_c(X) = CosX$. This implies, $Cos\emptyset = \emptyset$.

Let τ_{cs} be the set consisting of the images (under f_c) of the corresponding elements ofτ. Then we have,

$$\begin{aligned}
Cos\left(\bigcap_{n=1}^{\infty} A_n\right) &= \bigcap_{n=1}^{\infty} Cos(A_n) \& Cos\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} Cos(A_n),\\
CosX \backslash CosA &= Cos(X \backslash A),\\
A \subseteq B &\Leftrightarrow CosA \subseteq CosB.
\end{aligned}$$

Using above facts, we can easily prove that τ_{cs} form a topology on $CosX$. This topology is called Cosine topology (briefly, Cos-topology) of X. The pair $(CosX, \tau_{cs})$ is called the Cosine topological space corresponding to X. The elements of τ_{cs} are called Cos-open sets. The complement of the Cos-open set is said to be Cos-closed. The set of all Cos-closed subsets of $CosX$ is denoted by τ.

For Example, let $X = \{0, \frac{\pi}{4}, \frac{\pi}{2}\}$ with topology $\tau = \{\emptyset, \{\frac{\pi}{4}\}, \{0, \frac{\pi}{2}\}, X\}$. Then $CosX = \{1, \frac{1}{\sqrt{2}}, 0\}$, $\tau_{cs} = \{\emptyset, \{\frac{1}{\sqrt{2}}\}, \{1, 0\}, CosX\}$. Here the subsets $\emptyset, \{\frac{1}{\sqrt{2}}\}, \{1, 0\}$, $CosX$ are Cos-open sets.

The Cos-closed sets are $\emptyset, \{\frac{1}{\sqrt{2}}\}, \{1, 0\}, CosX$.

That is, $\tau_{cs}^c = \{\emptyset, \{\frac{1}{\sqrt{2}}\}, \{1, 0\}, CosX\}$.

Definition 3.2. *The topology τ on X is said to be a Strong trigonometric topology if its Sine and Cosine topologies are coincide. That is, if $\tau_s = \tau_{cs}$, then τ is said to be a Strong trigonometric topology. The space X together with this τ is called a Strong trigonometric topological space.*

Definition 3.3. *Let X be a set having elements from $[0, \frac{\pi}{2}]$ and τ be the topology on X. If τ_s and τ_{cs} are exist and unequal, then τ is said to be Weak trigonometric topology and the space (X, τ) is called a Weak trigonometric topological space.*

Example 2. *Let $X = \{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}\}$ with topology*
$\tau = \{\emptyset, \{\frac{\pi}{6}\}, \{\frac{\pi}{3}\}, \{\frac{\pi}{6}, \frac{\pi}{3}\}, \{\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{4}\}, X\}$.

Then $SinX = CosX = \{0, \frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}, 1\}$. Also, $\tau_s = \tau_{cs}$. This implies, τ is a Strong trigonometric topology. Hence (X, τ) is a Strong trigonometric topological space.

Let $X = \{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}\}$*. Then clearly,* $\tau = \{\emptyset, \{\frac{\pi}{3}\}, \{\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{2}\}, X\}$ *is a topology on* X*. Here* $SinX = CosX$ *but* τ_s *and* τ_{cs} *are unequal. Therefore,* τ *is a Weak trigonometric topology and the space* (X, τ) *is a Weak trigonometric topological space.*

Remark 3.1. *Every Strong trigonometric topology (resp. Weak trigonometric topology) is a topology but the converse is not true. The above two examples proves this. From this, we observe that the topological space is either a Strong trigonometric topological space or a Weak trigonometric topological space. The following results are true for Strong and Weak trigonometric topological spaces. But both are a topological space and a topological space is either of that. So, we simply write* (X, τ) *is a topological space in the following results.*

Definition 3.4. *Let* (X, τ) *be a topological space and* $A \subseteq SinX$*. The union of all Sin-open sets contained in* A *is called a Sine-interior of* A *and it is denoted by* $Int_{sin}(A)$*. Also, the intersection of all Sin-closed containing* A *is called a Sine-closure of A and it is denoted by* $Cl_{sin}(A)$*. That is,*

$$\begin{aligned} Int_{sin}(A) &= \cup\{B \subseteq SinX : B \subseteq A \;\&\; B \text{ } is \text{ } Sin - open\} \\ Cl_{sin}(A) &= \cap\{B \subseteq SinX : A \subseteq B \;\&\; B \text{ } is \text{ } Sin - closed\}. \end{aligned}$$

Result 3.4. *Let* (X, τ) *be a topological space and* $A \subseteq SinX$*. Then* $Int_{sin}(A)$ *is a Sin-open set.*

Proof. The proof follows from the fact that the union of any collection of Sin-open sets is Sin-open. □

Result 3.5. *Let* (X, τ) *be a topological space and* $A \subseteq SinX$*. Then* $Int_{sin}(A) \subseteq A$*.*

Proof. Let $x \in Int_{sin}(A)$. Then $x \in B$ for some sin-open set $B \subseteq A$. This implies, $x \in A$. Therefore, $Int_{sin}(A) \subseteq A$. □

Result 3.6. *Let* (X, τ) *be a topological space and* $A \subseteq SinX$*. Then* $Int_{sin}(A)$ *is the largest Sin-open set contained in* A *&* A *is Sin-open if and only if* $A = Int_{sin}(A)$*.*

Proof. It follows directly from the definition and Result 3.5. □

Result 3.7. *Let* (X, τ) *be a topological space and A, B be subsets of* $SinX$*. Then* $A \subseteq B \Rightarrow Int_{sin}(A) \subseteq Int_{sin}(B)$*,* $Int_{sin}(A \cap B) = Int_{sin}(A) \cap Int_{sin}(B)$ *&* $Int_{sin}(A) \cup Int_{sin}(B) \subseteq Int_{sin}(A \cup B)$*.*

Proof. It is obvious. □

Remark 3.2. $Int_{sin}(A \cup B)$ *need not be equal to* $Int_{sin}(A) \cup Int_{sin}(B)$.

For example, let $X = \{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}\}$ *with topology* $\tau = \{\emptyset, \{0, \frac{\pi}{6}\}, \{\frac{\pi}{4}, \frac{\pi}{3}\}, \{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}\}, X\}$.

Then $SinX = \{0, \frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}, 1\}$ *and* $\tau_s = \{\emptyset, \{0, \frac{1}{2}\}, \{\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}\}, \{0, \frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}\}, SinX\}$.

Let $A = \{0\}$ *and* $B = \{\frac{1}{2}\}$. *Then* $A \cup B = \{0, \frac{1}{2}\}$. *Now,* $Int_{sin}(A) = \emptyset$, $Int_{sin}(B) = \emptyset$. *This implies,* $Int_{sin}(A) \cup Int_{sin}(B) = \emptyset$. *Also,* $Int_{sin}(A \cup B) = \{0, \frac{1}{2}\}$.

Therefore, $Int_{sin}(A \cup B) \neq Int_{sin}(A) \cup Int_{sin}(B)$.

Result 3.8. *Let* (X, τ) *be a topological space and* $A \subseteq SinX$. *Then* $Cl_{sin}(A)$ *is a Sin-closed set,* $A \subseteq Cl_{sin}(A)$, $Cl_{sin}(A)$ *is the smallest Sin-closed set containing* A *&* A *is Sin-closed if and only if* $A = Cl_{sin}(A)$.

Result 3.9. *Let* (X, τ) *be a topological space and A, B be subsets of* $SinX$. *If* $A \subseteq B$, *then* $Cl_{sin}(A) \subset Cl_{sin}(B)$.

Result 3.10. *Let* (X, τ) *be a topological space and A, B be subsets of* $SinX$. *Then*

$$\begin{aligned} Cl_{sin}(A \cup B) &= Cl_{sin}(A) \cup Cl_{sin}(B), \\ Cl_{sin}(A \cap B) &\subseteq Cl_{sin}(A) \cap Cl_{sin}(B). \end{aligned}$$

Remark 3.3. $Cl_{sin}(A \cap B)$ *need not be equal to* $Cl_{sin}(A) \cap Cl_{sin}(B)$.

For example, let $X = \{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}\}$ *with topology* $\tau = \{\emptyset, \{0, \frac{\pi}{6}\}, \{\frac{\pi}{4}, \frac{\pi}{3}\}, \{\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}\}, X\}$. *Then* $SinX = \{0, \frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}, 1\}$ *and* $\tau_s = \{\emptyset, \{0, \frac{1}{2}\}, \{\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}\}, \{0, \frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}\}, SinX\}$.

This implies, $\tau_s^c = \{\emptyset, \{1\}, \{0, \frac{1}{2}, 1\}, \{\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}, 1\}, SinX\}$.

Let $A = \{0, \frac{1}{2}\}$ *and* $B = \{\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}\}$. *Then* $A \cap B = \emptyset$. *Now,* $Cl_{sin}(A) = \{0, \frac{1}{2}, 1\}$, $Cl_{sin}(B) = \{\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}, 1\}$.

This implies, $Cl_{sin}(A) \cap Cl_{sin}(B) = \{1\}$. *Also,* $Cl_{sin}(A \cap B) = \emptyset$. *Therefore,* $Cl_{sin}(A \cap B) \neq Cl_{sin}(A) \cap Cl_{sin}(B)$.

Result 3.11. *Let* (X, τ) *be a topological space and* $A \subseteq X$.

Then $Sin(int(A)) = Int_{sin}(SinA)$, $Cl_{sin}(SinA) = Sin(cl(A))$ *&* $SinX \backslash (Int_{sin}(SinA)) = Cl_{sin}(SinX \backslash SinA)$.

Definition 3.5. *Let (X, τ) be a topological space and $A \subseteq CosX$. Then we define,*

$$\begin{aligned} Int_{cos}(A) &= \cup\{B \subseteq CosX : B \subseteq A \& \ B \ is \ Cos-open\} \\ Cl_{cos}(A) &= \cap\{B \subseteq CosX : A \subseteq B \& \ B \ is \ Cos-closed\}. \end{aligned}$$

The proof of the following result follows directly from the definition.

Result 3.12. *Let (X, τ) be a topological space and A, B be subsets of $CosX$. Then*

(i) $Int_{cos}(A)$ *is a Cos-open set*
(ii) $Int_{cos}(A) \subseteq A$
(iii) $Int_{cos}(A)$ *is the largest Cos-open set contained in* A
(iv) A *is Cos-open if and only if* $A = int_{cos}(A)$
(v) $A \subseteq B \Rightarrow Int_{cos}(A) \subseteq Int_{cos}(B)$
(vi) $Int_{cos}(A \cap B) = Int_{cos}(A) \cap Int_{cos}(B)$
(vii) $Int_{cos}(A) \cup Int_{cos}(B) \subseteq Int_{cos}(A \cup B)$.

The equality does not hold in (vii).
For example, let $X = \{0, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}\}$ *with topology*
$\tau = \{\emptyset, \{0\}, \{\frac{\pi}{3}\}, \{0, \frac{\pi}{3}\}, \{0, \frac{\pi}{4}\}, \{0, \frac{\pi}{3}, \frac{\pi}{4}\}, X\}$.
Then $CosX = \{1, \frac{1}{\sqrt{2}}, \frac{1}{2}, 0\}$ *and*
$\tau_{cos} = \{\emptyset, \{1\}, \{\frac{1}{2}\}, \{1, \frac{1}{2}\}, \{1, \frac{1}{\sqrt{2}}\}, \{1, \frac{1}{2}, \frac{1}{\sqrt{2}}\}, CosX\}$.
Consider the subsets $A = \{1\}$ *and* $B = \{\frac{1}{\sqrt{2}}\}$. *Then* $A \cup B = \{1, \frac{1}{\sqrt{2}}\}$. *Now,* $Int_{cos}(A) = \{1\}$ *and* $Int_{cos}(B) = \emptyset$. *This implies,* $Int_{cos}(A) \cup Int_{cos}(B) = \{1\}$. *Also,* $Int_{cos}(A \cup B) = \{1, \frac{1}{\sqrt{2}}\}$.
Therefore, $Int_{cos}(A \cup B) \neq Int_{cos}(A) \cup Int_{cos}(B)$.

Result 3.13. *Let (X, τ) be a topological space and A, B be subsets of $CosX$. Then*

(i) $Cl_{cos}(A)$ *is a Cos-closed set*
(ii) $A \subseteq Clcos(A)$
(iii) $Cl_{cos}(A)$ *is the smallest Cos-closed set containing* A
(iv) A *is Cos-closed if and only if* $A = Cl_{cos}(A)$
(v) $A \subseteq B \Rightarrow Cl_{cos}(A) \subseteq Cl_{cos}(B)$
(vi) $Cl_{cos}(A \cup B) = Cl_{cos}(A) \cup Cl_{cos}(B)$
(vii) $Cl_{cos}(A \cap B) \subseteq Cl_{cos}(A) \cap Cl_{cos}(B)$.

Remark 3.4. *The reverse inclusion of (vii) in above result need not be true, which can be verified from the following example.*

Example 3. *Let* $X = \{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}\}$ *with topology* $\tau = \{\emptyset, \{0, \frac{\pi}{6}\}, \{\frac{\pi}{4}, \frac{\pi}{3}\}, \{\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}\}, X\}$*. Then* $CosX = \{0, \frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}, 1\}$ *and* $\tau_{cos} = \{\emptyset, \{1, \frac{\sqrt{3}}{2}\}, \{\frac{1}{\sqrt{2}}, \frac{1}{2}\}, \{1, \frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}\}, CosX\}$.

This implies, $\tau_{cos}^c = \{\emptyset, \{0\}, \{0, \frac{\sqrt{3}}{2}, 1\}, \{0, \frac{1}{2}, \frac{1}{\sqrt{2}}, CosX\}$.

Consider the subsets $A = \{\frac{\sqrt{3}}{2}\}$ *and* $B = \{1, \frac{1}{\sqrt{2}}\}$*. Then* $A \cap B = \emptyset$*. Now,* $Cl_{cos}(A) = \{0, \frac{\sqrt{3}}{2}, 1\}$*,* $Cl_{cos}(B) = CosX$.

This implies, $Cl_{cos}(A) \cap Cl_{cos}(B) = \{0, \frac{\sqrt{3}}{2}, 1\}$*. Also,* $Cl_{cos}(A \cap B) = \emptyset$*. Therefore,* $Cl_{cos}(A \cap B) \neq Cl_{cos}(A) \cap Cl_{cos}(B)$.

Result 3.14. *Let* (X, τ) *be a topological space and* $A \subseteq X$*. Then*

(i) $Cos(int(A)) = Int_{cos}(CosA)$

(ii) $Cl_{cos}(CosA) = Cos(cl(A))$

(iii) $CosX \backslash (Int_{cos}(CosA)) = Cl_{cos}(CosX \backslash CosA)$.

Proof. It is obvious. □

4. Trigonometric topological spaces

In this section, we study about Trigonometric topological spaces. Also, we discuss about some set theory relations.

Definition 4.1. *Let* X *be a set having elements from* $[0, \frac{\pi}{2}]$*. Define* $T_u(X)$ *by* $T_u(X) = SinX \cup CosX$.

Result 4.1. *Let* X *be a set and* A *be a subset of* X*. Then*

(i) $T_u(X) \backslash (SinA \cup CosA) = (T_u(X) \backslash SinA) \cap (T_u(X) \backslash CosA)$,

(ii) $T_u(X) \backslash (SinA \cap CosA) = (T_u(X) \backslash SinA) \cup (T_u(X) \backslash CosA)$.

Proof. The proof follows directly from De-Morgan's law. □

Result 4.2. *Let* X *be a set and* A *be a subset of X. Then*

(i) $T_u(X) \backslash (SinA \cap CosA) = (SinX \backslash SinA) \cup (CosX \backslash CosA)$,

(ii) $Sin(X \backslash A) \subseteq T_u(X) \backslash SinA$,

(iii) $Cos(X \backslash A) \subseteq T_u(X) \backslash CosA$,

(iv) $T_u(X) \backslash SinA = (SinX \backslash SinA) \cup (CosX \backslash SinA)$,

(v) $T_u(X) \backslash CosA = (SinX \backslash CosA) \cup (CosX \backslash CosA)$.

Result 4.3. *From Result 4.1 and 4.2, we observe that* $(T_u(X)\backslash SinA) \cup (T_u(X)\backslash CosA)$ $= (SinX\backslash SinA) \cup (CosX\backslash CosA)$. *But* $(T_u(X)\backslash SinA) \neq (SinX\backslash SinA)$ *and* $(T_u(X)\backslash CosA) \neq (CosX\backslash CosA)$. *Also, from (iv),* $T_u(X)\backslash SinX = CosX\backslash SinX$ *and from (v),* $T_u(X)\backslash CosX = SinX\backslash CosX$.

Result 4.4. *Let* X *be a set and* $A \subseteq X$. *Then* $T_u(X)\backslash(SinA \cup CosA) \subseteq (SinX\backslash SinA) \cup (CosX\backslash CosA)$.

The equality does not hold. For example, let $X = \{0, \frac{\pi}{3}, \frac{\pi}{2}\}$. *Then* $SinX = \{0, \frac{\sqrt{3}}{2}, 1\}$ *&* $CosX = \{1, \frac{1}{2}, 0\}$. *Consider the subset* $A = \{0\}$. *Then* $T_u(X)\backslash(SinA \cup CosA) = \{\frac{1}{2}, \frac{\sqrt{3}}{2}\}$ *and* $(SinX\backslash SinA) \cup (CosX\backslash CosA) = \{0, \frac{1}{2}, \frac{\sqrt{3}}{2}, 1\}$. *Therefore,* $T_u(X)\backslash(SinA \cup CosA) \neq (SinX\backslash SinA) \cup (CosX\backslash CosA)$.

Definition 4.2. *Let* X *be a set having elements from* $[0, \frac{\pi}{2}]$. *Define* $T_i(X)$ *by* $T_i(X) = SinX \cap CosX$.

Result 4.5. *Let* X *be a set and* A *be a subset of* X. *Then*

(i) $T_i(X)\backslash(SinA \cup CosA) = (T_i(X)\backslash SinA) \cap (T_i(X)\backslash CosA)$,
(ii) $T_i(X)\backslash(SinA \cup CosA) = (SinX\backslash SinA) \cap (CosX\backslash CosA)$,
(iii) $T_i(X)\backslash(SinA \cap CosA) = (T_i(X)\backslash SinA) \cup (T_i(X)\backslash CosA)$,
(vi) $T_i(X)\backslash(SinA \cap CosA) \subseteq (SinX\backslash SinA) \cup (CosX\backslash CosA)$,
(v) $T_i(X)\backslash SinA \subseteq Sin(X\backslash A)$,
(vi) $T_i(X)\backslash CosA \subseteq Cos(X\backslash A)$,
(vii) $T_i(X)\backslash SinA = (SinX\backslash SinA) \cap (CosX\backslash SinA)$,
(viii) $T_i(X)\backslash CosA = (SinX\backslash CosA) \cap (CosX\backslash CosA)$.

Remark 4.1. *The reverse inclusion of (iv) need not be true.*

For example, let $X = \{0, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{2}\}$. *Then* $SinX = \{0, \frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}, 1\}$ *&* $CosX = \{0, \frac{1}{2}, \frac{1}{\sqrt{2}}, 1\}$. *Also,* $T_i(X) = \{0, \frac{1}{\sqrt{2}}, 1\}$. *Let* $A = \{\frac{\pi}{4}, \frac{\pi}{2}\}$. *Then* $(SinX\backslash SinA) \cup (CosX\backslash CosA) \nsubseteq T_i(X)\backslash(SinA \cap CosA)$.

Note 1. *Let* X *be a set having elements from* $[0, \frac{\pi}{2}]$. *Then*

$$\begin{aligned} T_i(X)\backslash SinX &= \emptyset \text{ and } T_i(X)\backslash CosX = \emptyset \text{ and} \\ T_u(X)\backslash T_i(X) &= (CosX\backslash SinX) \cup (SinX\backslash CosX). \end{aligned}$$

Definition 4.3. *Let* X *be a set with elements from* $[0, \frac{\pi}{2}]$ *and* τ *be the topology on* X. *We define a set* $\mathcal{J} = \{\emptyset, U \cup V \cup T_i(X) : U \in \tau_s \text{ and } V \in \tau_{cs}\}$. *Then* $\mathcal{J}$ *form a* *topology on $T_u(X)$. This topology is called trigonometric topology on $T_u(X)$. The pair $(T_u(X), \mathcal{J})$ is called a trigonometric topological space.*

The elements of $\mathcal{J}$ are called trigonometric open sets. The complement of a trigonometric open set is said to be a trigonometric closed set. The set of all trigonometric closed sets is denoted by $\mathcal{J}^c$.

Example 4. *Let $X = \{\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{2}\}$ with topology $\tau = \{\emptyset, \{\frac{\pi}{6}\}, \{\frac{\pi}{4}, \frac{\pi}{2}\}, X\}$. Then $\tau_s = \{\emptyset, \{\frac{1}{2}\}, \{1, \frac{1}{\sqrt{2}}\}, SinX\}$ and $\tau_{cs} = \{\emptyset, \{\frac{\sqrt{3}}{2}\}, \{0, \frac{1}{\sqrt{2}}\}, CosX\}$. Now, $\mathcal{J} = \{\emptyset, T_i(X), CosX, \{\frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}\}, \{0, \frac{1}{\sqrt{2}}\}, \{\frac{1}{2}, \frac{1}{\sqrt{2}}, 1, \frac{\sqrt{3}}{2}\}, \{\frac{1}{2}, \frac{1}{\sqrt{2}}, 1, 0\}, \{\frac{1}{2}, \frac{1}{\sqrt{2}}\}, \{\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}, 0\}, \{\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}\}, \{0, \frac{1}{\sqrt{2}}, \frac{1}{2}\}, \{1, \frac{1}{\sqrt{2}}\}, \{1, 0, \frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}\}, \{1, \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}\}, \{1, 0, \frac{1}{\sqrt{2}}\}, SinX, T_u(X)\}$. This implies, $\mathcal{J}$ is a topology on $T_u(X)$.*

Remark 4.2. *If $\mathcal{J}$ is a trigonometric topology, then $\mathcal{J}$ must contain the elements $\emptyset$, $T_i(X)$, $SinX$, $CosX$, $T_u(X)$. For (i) clearly by definition, $\emptyset \in \mathcal{J}$, (ii) Since $\emptyset$ is open in both $SinX$ and $CosX$, we have $\emptyset \cup \emptyset \cup T_i(X) = T_i(X) \in \mathcal{J}$, (iii) Since $\emptyset$ is open in $CosX$ and $SinX$ is open in $SinX$, $SinX \cup \emptyset \cup T_i(X) = SinX \in \mathcal{J}$, (iv) Since $\emptyset$ is open in $SinX$ and $CosX$ is open in $CosX$, $\emptyset \cup CosX \cup T_i(X) = CosX \in \mathcal{J}$ & (v) $SinX \cup CosX \cup T_i(X) = SinX \cup CosX = T_u(X) \in \mathcal{J}$. Also, the smallest non-empty set as an element of $\mathcal{J}$ is $T_i(X)$.*

Definition 4.4. *Let (X, τ) be a topological space with $SinX \neq CosX$. If $\mathcal{J}$ contains only the elements $\emptyset$, $T_i(X)$, $SinX$, $CosX$, $T_u(X)$ then $\mathcal{J}$ is said to be a Standard trigonometric topology. It is denoted by the symbol $\mathcal{J}_s$. The pair $(T_u(X), \mathcal{J}_s)$ is called a Standard trigonometric topological space.*

Note 2. *Trigonometric topology is finer than the Standard trigonometric topology on $T_u(X)$. That is, $\mathcal{J}_s \subseteq \mathcal{J}$.*

Theorem 4.1. *Let (X, τ) be any topological space with $SinX = CosX$. Then $\mathcal{J}$ is an indiscrete topology on $T_u(X)$.*

Proof. Given that $SinX = CosX$. Then $T_i(X) = T_u(X) = SinX = CosX$.

Also, $T_i(X)$ is the smallest non-empty set in $\mathcal{J}$.

This implies, $\mathcal{J} = \{\emptyset, T_u(X)\}$. Therefore, $\mathcal{J}$ is an indiscrete topology on $T_u(X)$. $\square$

The proof of the following Theorem follows from Theorem 4.1

Theorem 4.2. *If τ is a Strong trigonometric topology, then $\mathcal{J}$ is an indiscrete topology.*

Theorem 4.3. *If* $(X, \tau))$ *be an indiscrete topological space with* $SinX \neq CosX$*, then* $\mathcal{J}$ *is a standard trigonometric topology.*

Theorem 4.4. *If* $T_i(X) = SinX \backslash \{p\}$ *where* $p \in SinX$*, then* $\mathcal{J}$ *is a Standard trigonometric topology.*

Theorem 4.5. *If* $T_i(X) = CosX \backslash \{q\}$ *where* $q \in CosX$*, then* $\mathcal{J}$ *is a Standard trigonometric topology.*

Theorem 4.6. *If* $T_i(X) = T_u(X) \backslash \{p, q\}$ *where* $p, q \in T_u(X)$*, then* $\mathcal{J}$ *is a Standard trigonometric topology.*

Proof. Assume that $p, q \notin T_i(X)$.

If $p, q \in SinX$, then $T_i(X) = CosX$. This implies, number of elements of $SinX$ is two more than $CosX$. This is a contradiction. Therefore both p and q does not belongs to $SinX$. Similarly, we can prove both p and q does not belongs to $CosX$.

Hence, either $p \in SinX$ and $q \in CosX$ (or) $q \in SinX$ and $p \in CosX$. Therefore, the proof follows from Theorem 4.4 and Theorem 4.5. $\square$

5. Conclusion

In this paper, we have introduced Sine and Cosine topological spaces and studied their basic properties. Also, we have introduced Trigonometric topological spaces. Further, we have discussed the properties of interior and closure operators in Sine and Cosine topologies.

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