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DECOMPOSITION OF JUMP GRAPH OF CYCLES

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ABSTRACT. The Jump graph $J(G)$ of a graph G is the graph whose vertices are edges of G and two vertices of $J(G)$ are adjacent if and only if they are not adjacent in G. Equivalently complement of line graph $L(G)$ is the Jump graph $J(G)$ of G. In this paper, we give necessary and sufficient condition for the decomposition of Jump graph of cycles into various graphs such as paths, cycles, stars, complete graphs and complete bipartite graphs.

1. INTRODUCTION

Let $G = (V, E)$ be a simple undirected graph without loops or multiple edges. A path on n vertices is denoted by P_n, cycle on n vertices is denoted by C_n and complete graph on n vertices is denoted by K_n. The neighbourhood of a vertex v in G is the set $N(v)$ consisting of all vertices that are adjacent to v. $|N(v)|$ is called the degree of v and is denoted by $d(v)$. A complete bipartite graph with partite sets V_1 and V_2, where $|V_1| = r$ and $|V_2| = s$, is denoted by $K_{r,s}$. The graph $K_{1,r}$ is called a star and is denoted by S_r. Claw is a star with three edges. For any set S of points of G, induced subgraph $< S >$ is the maximal subgraph of G with point set S. The terms not defined here are used in the sense of [2].

A decomposition of a graph G is a family of edge-disjoint subgraphs $\{G_1, G_2, ..., G_k\}$ such that $E(G) = E(G_1) \cup E(G_2) \cup ... \cup E(G_k)$. If each G_i is

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isomorphic to H for some subgraph H of G, then the decomposition is called a H-decomposition of G.

The Jump graph $J(G)$ of a graph G is the graph whose vertices are edges of G and two vertices of $J(G)$ are adjacent if and only if they are not adjacent in G. This concept was introduced by Chartrand in [1]. Let $J(C_n)$ denote the Jump graph of cycles. Then $J(C_n)$ is a connected graph if and only if $n \geq 5$. Let us consider the connected jump graph of cycles. Let the edges of cycle C_n be labelled as $x_1, x_2, ..., x_n$. Since the number of edges of cycle C_n is n, the number of vertices of $J(C_n)$ is n. Let the vertices of jump graph of cycle $J(C_n)$ be labelled as $x_1, x_2, ..., x_n$. The number of edges of Jump graph of cycles $J(C_n)$ is $\frac{n^2-3n}{2}$.

In 2010, Tay - Woei Shyu [3] gave necessary and sufficient condition for the decomposition of complete graph into $P_4's$ and $S_4's$. In this paper, we give necessary and sufficient condition for the decomposition of Jump graph of cycles into various graphs such as paths, cycles, stars, complete graphs and complete bipartite graphs.

Theorem 1.1. *Let n be an odd positive integer. There exists a decomposition of $J(C_n)$ into p copies of C_4 and q copies of C_5 iff $n \geq 5$ and $4p + 5q = \frac{n^2-3n}{2}$, where $p = \frac{(n-5)(n-3)}{8}$ and $q = \frac{(n-3)}{2}$.*

Proof. (Necessity) Let n be an odd positive integer. Suppose that there exists a decomposition of $J(C_n)$ into p copies of C_4 and q copies of C_5 where $p = \frac{(n-5)(n-3)}{8}$ and $q = \frac{(n-3)}{2}$.

Clearly Jump graph of cycle $J(C_n)$ is a connected graph if and only if $n \geq 5$. Since n is odd, $n \geq 5$. Since $|E[J(C_n)]| = \frac{n^2-3n}{2}$, we have $4p + 5q = \frac{n^2-3n}{2}$.

(Sufficiency) Suppose that $4p + 5q = \frac{n^2-3n}{2}$, where $p = \frac{(n-5)(n-3)}{8}$ and $q = \frac{(n-3)}{2}$.

Clearly $\{x_1x_{2k-3}x_2x_{2k-2}x_1/4 \leq k \leq \frac{n+1}{2}\}$, $\{x_3x_{2k-3}x_4x_{2k-2}x_3/5 \leq k \leq \frac{n+1}{2}\}$, $\{x_5x_{2k-3}x_6x_{2k-2}x_5/6 \leq k \leq \frac{n+1}{2}\}$,...,$\{x_{n-6}x_{2k-3}x_{n-5}x_{2k-2}x_{n-6}/k = \frac{n+1}{2}\}$ forms C_4 in $J(C_n)$. Then we get $\frac{(n-5)(n-3)}{8}$ copies of C_4.

Also $x_{2k-3}x_{2k-5}x_{2k-2}x_{2k-4}x_nx_{2k-3}$; $3 \leq k \leq \dfrac{n+1}{2}$ forms C_5 in $J(C_n)$. Then we get $\left(\dfrac{n-3}{2}\right)$ copies of C_5.

Thus $E[J(C_n)] = \underbrace{E(C_4) \cup ... \cup E(C_4)}_{p\ times} \cup \underbrace{E(C_5) \cup ... \cup E(C_5)}_{q\ times}$, where $p = \dfrac{(n-5)(n-3)}{8}$ and $q = \dfrac{(n-3)}{2}$.

So, $J(C_n)$ is decomposable into p copies of C_4 and q copies of C_5. □

Theorem 1.2. *Let n be an even positive integer. There exists a decomposition of $J(C_n)$ into p copies of P_4 and q copies of C_4 iff $n \geq 6$ and $3p+4q = \dfrac{n^2-3n}{2}$, where $p = \dfrac{n}{2}$ and $q = \dfrac{n^2-6n}{8}$.*

Proof. (Necessity) Let n be an even positive integer. Suppose that there exists a decomposition of $J(C_n)$ into p copies of P_4 and q copies of C_4 where $p = \dfrac{n}{2}$ and $q = \dfrac{n^2-6n}{8}$. Since $J(C_n)$ is connected and n is even, $n \geq 6$. Since $|E[J(C_n)]| = \dfrac{n^2-3n}{2}$, we have $3p + 4q = \dfrac{n^2-3n}{2}$.

(Sufficiency) Suppose $3p + 4q = \dfrac{n^2-3n}{2}$ where $p = \dfrac{n}{2}$ and $q = \dfrac{n^2-6n}{8}$.

Clearly $x_{2k-3}x_{2k-5}x_{2k-2}x_{2k-4}$; $3 \leq k \leq \dfrac{n+2}{2}$ and $x_1x_{n-1}x_2x_n$ forms P_4 in $J(C_n)$. Then we get $\dfrac{n}{2}$ copies of P_4.

Also the $x_5x_{2k-3}x_6x_{2k-2}x_5$, $6 \leq k \leq \dfrac{n+2}{2}$,..., $lx_{n-5}x_{2k-3}x_{n-4}x_{2k-2}x_{n-5} = \dfrac{n+2}{2}$ vertices $\{x_1x_{2k-3}x_2x_{2k-2}x_k\}$ forms C_4 in $J(C_n)$. Then we get $\dfrac{n^2-6n}{8}$ copies of C_4.

Thus $E[J(C_n)] = \underbrace{E(P_4) \cup ... \cup E(P_4)}_{p\ times} \cup \underbrace{E(C_4) \cup ... \cup E(C_4)}_{q\ times}$ where $p = \dfrac{n}{2}$ and $q = \dfrac{n^2-6n}{8}$ of C_4. So, $J(C_n)$ is decomposable into p copies of P_4 and q copies. □

Theorem 1.3. *Let n be an odd positive integer. There exists a decomposition of $J(C_n)$ into p copies of C_5, q complete bipartite graphs of the form $K_{2,2l}$;*

$l = 1, 2, ..., \frac{n-5}{2}$ *iff* $n \geq 5$ *and* $5p + 2q(q+1) = \frac{n^2-3n}{2}$ *where* $p = \frac{n-3}{2}$ *and* $q = \frac{n-5}{2}$.

Proof. (Necessity) Suppose that there exists a decomposition of $J(C_n)$ into p copies of C_5 and q complete bipartite graphs of the form $K_{2,2l}$; $l = 1, 2, ..., \frac{n-5}{2}$ where $p = \frac{n-3}{2}$ and $q = \frac{n-5}{2}$.

Clearly $|E[J(C_n)]| = \frac{n^2-3n}{2}$. Thus we have $5p + 2q(q+1) = \frac{n^2-3n}{2}$.

(Sufficiency) Suppose that $5p + 2q(q+1) = \frac{n^2-3n}{2}$ where $p = \frac{n-3}{2}$ and $q = \frac{n-5}{2}$. Let the vertices of $J(C_n)$ be $x_1, x_2, ..., x_n$. Clearly $x_{2k-3}x_{2k-5}x_{2k-2}x_{2k-4}$ x_nx_{2k-3}; $3 \leq k \leq \frac{n+1}{2}$ forms C_5 in $J(C_n)$. Then we get $\binom{\frac{n-3}{2}}{}$ copies of C_5. Thus $p = \binom{\frac{n-3}{2}}{}$. Also, the vertices x_m and x_{m+1} are non adjacent for $m = 1, 3, 5, ..., (n-6)$ and they are adjacent with each of the vertices $x_{m+4}, x_{m+5}, x_{m+6}, ..., x_{n-1}$.

So, we get $\frac{n-5}{2}$ complete bipartite graphs of the form $K_{2,2l}$; $l = 1, 2, ..., \frac{n-5}{2}$. Thus $E[J(C_n)] = \underbrace{E(C_5) \cup ... \cup E(C_5)}_{p\ times} \cup E(K_{2,4}) \cup ... \cup E(K_{2,n-5})$ where $p = \frac{n-3}{2}$.

We conclude that $J(C_n)$ is decomposable into p copies of C_5 and q complete bipartite graphs of the form $K_{2,2l}$; $l = 1, 2, ..., \frac{n-5}{2}$ where $p = \frac{n-3}{2}$ and $q = \frac{n-5}{2}$.

□

Theorem 1.4. *Let n be an even positive integer. There exists a decomposition of $J(C_n)$ into p copies of P_4, q complete bipartite graphs of the form $K_{2,2l}$; $l = 1, 2, ..., \frac{n-8}{2}$ and two copies of $K_{2,r}$ iff $n \geq 6$ and $3p+2q(q+1)+4r = \frac{n^2-3n}{2}$, where $p = \frac{n}{2}$ and $q = \frac{n-8}{2}$ and $r = n-6$.*

Proof. (Necessity) Suppose that there exists a decomposition of $J(C_n)$ into p copies of P_4, q complete bipartite graphs of the form $K_{2,2l}$; $l = 1, 2, ..., \frac{n-8}{2}$ and two copies of $K_{2,r}$, where $p = \frac{n}{2}$ and $q = \frac{n-8}{2}$ and $r = n-6$.

Clearly $|E[J(C_n)]| = \frac{n^2-3n}{2}$. Thus we have $3p + 2q(q+1) + 4r = \frac{n^2-3n}{2}$.

(Sufficiency) Suppose that $3p + 2q(q+1) + 4r = \frac{n^2-3n}{2}$ where $p = \frac{n}{2}$ and $q = \frac{n-8}{2}$ and $r = n-6$.

Let the vertices of $J(C_n)$ be $x_1, x_2, ..., x_n$. Clearly $x_{2k-3}x_{2k-5}x_{2k-2}x_{2k-4}$; $3 \leq k \leq \frac{n+2}{2}$ and $x_1x_{n-1}x_2x_n$ forms P_4 in $J(C_n)$. Then we get $\frac{n}{2}$ copies of P_4. Thus $p = \frac{n}{2}$.

Also, the vertices x_m and x_{m+1} are non adjacent for $m = 5, 7, ..., (n-5)$ and they are adjacent with each of the vertices $x_{m+4}, x_{m+5}, x_{m+6}, ..., x_n$.

Thus we get $\frac{n-8}{2}$ complete bipartite graphs of the form $K_{2,2l}$; $l = 1, 2, ..., \frac{n-8}{2}$. Therefore $q = \frac{n-8}{2}$.

We have that x_1 and x_2 are non adjacent vertices and they are adjacent with each of the vertices $x_5, x_6, ..., x_{n-3}, x_{n-2}$. Thus these vertices forms $K_{2,n-6}$.

Also x_3 and x_4 are non adjacent vertices and they are adjacent with each of the vertices $x_7, x_8, ..., x_{n-1}, x_n$. Thus these vertices forms $K_{2,n-6}$. So we get two copies of $K_{2,n-6}$. Therefore $r = n-6$.

Thus $E[J(C_n)] = \underbrace{E(P_4) \cup ... \cup E(P_4)}_{p\ times} \cup E(K_{2,2}) \cup E(K_{2,4}) \cup ... \cup E(K_{2,n-8}) \cup E(K_{2,n-6}) \cup E(K_{2,n-6})$ where $p = \frac{n}{2}$.

We conclude that $J(C_n)$ is decomposable into p copies of P_4, q complete bipartite graphs of the form $K_{2,2l}$; $l = 1, 2, ..., \frac{n-8}{2}$ and two copies of $K_{2,r}$ where $p = \frac{n}{2}$ and $q = \frac{n-8}{2}$ and $r = n-6$.

□

Theorem 1.5. *Let n be an even positive integer. There exists a decomposition of $J(C_n)$ into two copies of K_p and $\frac{n}{2}$ copies of S_q iff $n \geq 6$ and $p^2 - p + \frac{nq}{2} = \frac{n^2-3n}{2}$, where $p = \frac{n}{2}$ and $q = \frac{n-4}{2}$.*

Proof. (Necessity) Suppose that there exists a decomposition of $J(C_n)$ into two copies of K_p and $\frac{n}{2}$ copies of S_q where $p = \frac{n}{2}$ and $q = \frac{n-4}{2}$. Clearly $|E[J(C_n)]| = \frac{n^2-3n}{2}$. Thus we have $p^2 - p + \frac{nq}{2} = \frac{n^2-3n}{2}$.

(Sufficiency) Let $p^2 - p + \frac{nq}{2} = \frac{n^2-3n}{2}$, where $p = \frac{n}{2}$ and $q = \frac{n-4}{2}$. Let the vertices of $J(C_n)$ be labelled as $x_1, x_2, ..., x_n$. Now, the induced subgraphs are $<\{x_1, x_3, ..., x_{(n-1)}\}>= K_{\frac{n}{2}}$ and $<\{x_2, x_4, ..., x_{(n)}\}>= K_{\frac{n}{2}}$.

Let us make a partition of $V(G)$ into V_1 and V_2, where
$V_1 = \{x_{1+2k}/k = 0, 1, ..., \frac{n-2}{2}\}$ and $V_2 = \{x_{2+2k}/k = 0, 1, ..., \frac{n-2}{2}\}$.

Consider $x_1 \in V_1$. Clearly x_1 is not adjacent with x_2 and x_n in V_2. Also x_1 is adjacent with the remaining $\frac{n-4}{2}$ vertices in V_2. Consider $x_i \in V_1$, where $i = 3, 5, ..., n-1$. Clearly x_i is not adjacent with x_{i+1} and x_{i-1} in V_2 and adjacent with remaining $\frac{n-4}{2}$ vertices in V_2.

Since there are $\frac{n}{2}$ vertices in V_1, we get $\frac{n}{2}$ copies of $S_{\frac{n-4}{2}}$. Therefore $q = \frac{n-4}{2}$.

Thus $E[J(C_n)] = E(K_p) \cup E(K_p) \cup \underbrace{E(S_q) \cup E(S_q) \cup ... \cup E(S_q)}_{(\frac{n}{2})copies}$ where $p = \frac{n}{2}$ and $q = \frac{n-4}{2}$.

So, $J(C_n)$ is decomposable into two copies of K_p and $\frac{n}{2}$ copies of S_q. $\square$

Theorem 1.6. *Let n be an odd positive integer. There exists a decomposition of $J(C_n)$ into two copies of K_p, one copy of S_q, $\frac{n-3}{2}$ copies of S_r and one copy of S_s iff $n \geq 5$ and $p^2 - p + q + \frac{(n-3)r}{2} + s = \frac{n^2-3n}{2}$ where $p = \frac{n-1}{2}$, $q = \frac{n-3}{2}$, $r = \frac{n-5}{2}$ and $s = n-3$.*

Proof. (Necessity) Suppose that there exists a decomposition of $J(C_n)$ into two copies of K_p, one copy of S_q, $\frac{n-3}{2}$ copies of S_r and one copy of S_s where $p = \frac{n-1}{2}$, $q = \frac{n-3}{2}$, $r = \frac{n-5}{2}$ and $s = n-3$.

Clearly $|E[J(C_n)]| = \frac{n^2-3n}{2}$. Thus we have $p^2 - p + q + \frac{(n-3)r}{2} + s = \frac{n^2-3n}{2}$.

(Sufficiency) Suppose that $p^2 - p + q + \frac{(n-3)r}{2} + s = \frac{n^2-3n}{2}$ where $p = \frac{n-1}{2}$, $q = \frac{n-3}{2}$, $r = \frac{n-5}{2}$ and $s = n-3$.

Let the vertices of $J(C_n)$ be labelled as $x_1, x_2, ..., x_n$. Now, the induced subgraphs are $< \{x_1, x_3, ..., x_{(n-2)}\} >= K_{\frac{n-1}{2}}$ and $< \{x_2, x_4, ..., x_{(n-1)}\} >= K_{\frac{n-1}{2}}$. Therefore $p = \dfrac{n-1}{2}$.

Let us make a partition of $V(G)$ into V_1 and V_2 where $V_1 = \{x_{1+2k}/k = 0, 1, ..., \frac{n-3}{2}\}$ and $V_2 = \{x_{2+2k}/k = 0, 1, ..., \frac{n-3}{2}\}$.

Let $x_1 \in V_1$. x_1 is not adjacent with x_2 alone in V_2. Thus x_1 is adjacent with the remaining $x_4, x_6, ..., x_{n-1}$ in V_2.

So, we get one copy of $S_{\frac{n-3}{2}}$. Therefore $q = \dfrac{n-3}{2}$.

Consider the remaining vertices $x_i \in V_1 - \{x_1\}$; $i = 3, 5, ..., n-2$. Clearly x_i is not adjacent with x_{i+1} and x_{i-1} in V_2.

Also x_i is adjacent with each of the remaining $\left(\dfrac{n-5}{2}\right)$ vertices in V_2. Thus we get $\dfrac{n-3}{2}$ copies of $S_{\frac{n-5}{2}}$.

Therefore $r = \left(\dfrac{n-5}{2}\right)$. Now the vertex x_n is adjacent with $x_2, x_3, ..., x_{n-2}$ and is not adjacent with both x_1 and x_{n-1}. Thus we get S_{n-3}. Therefore $s = n-3$.

So, $E[J(C_n)] = E(K_p) \cup E(K_p) \cup E(S_q) \cup \underbrace{E(S_r) \cup ... \cup E(S_r)}_{\left(\frac{n-3}{2}\right) copies} \cup E(S_s)$, where $p = \dfrac{n-1}{2}$, $q = \dfrac{n-3}{2}$, $r = \dfrac{n-5}{2}$ and $s = n-3$.

We conclude that $J(C_n)$ is decomposable into two copies of K_p, one copy of S_q, $\frac{n-3}{2}$ copies of S_r and one copy of S_s. □

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